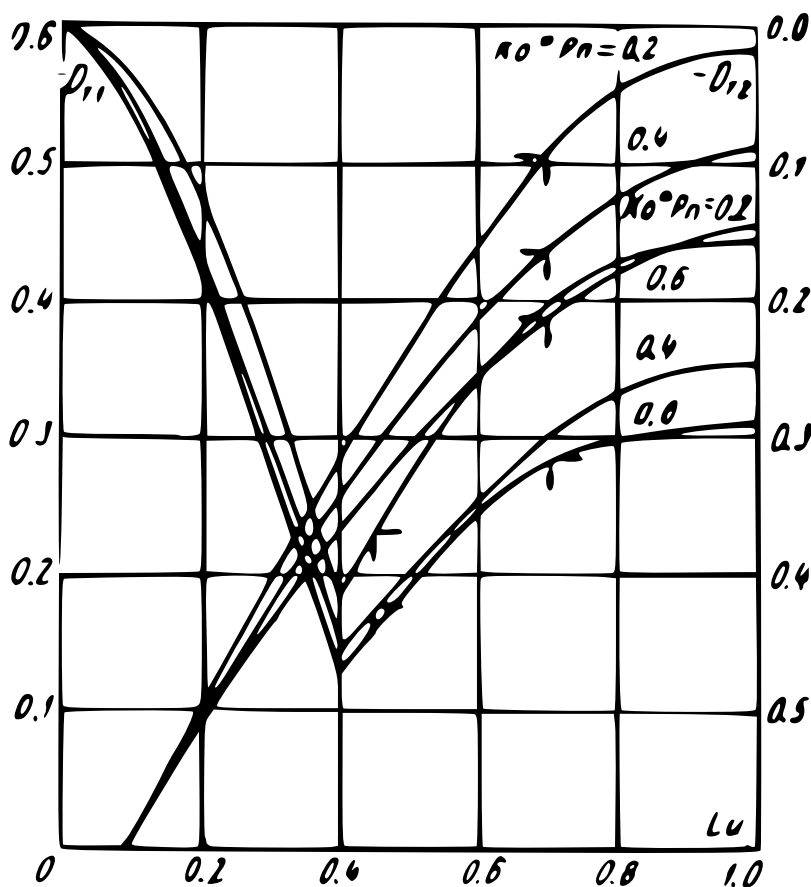


A.V. Lykov, Yu.A. Mikhailov

# Theory of Heat and Mass Transfer











A. V. LYKOV and Yu. A. MIKHAILOV

# THEORY OF HEAT AND MASS TRANSFER

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## PREFACE

The study of the transfer of energy (heat) and mass represents one of the most vital branches of modern science. It is of great practical importance in power-station and industrial engineering, and in the technological processes related to the chemical industry and light industry. Problems of heat and mass exchange become particularly important in jet and rocket engineering, since the flight velocity and the height of a jet device depend to a great extent on the conditions of heat and mass exchange. These processes are equally critical in atomic-power engineering, during the design and construction of nuclear reactors. A characteristic feature of heat and mass transfer in the above fields of modern engineering is the interrelation between these processes, where heat transfer and mass transfer constitute one single complex process.

A similar situation exists with respect to heat-power-engineering processes when they are intensified considerably due to a transition to high-rate parameters. It is important to note too that the laws governing heat and mass exchange are the very same and also that the regularities observed in one engineering field may be successfully applied to another. A characteristic feature of engineering development at the present time is the transference of methods and design solutions from one branch of industry to another. This makes possible a radical modification of industrial production processes, and also leads to new methods for the production of materials and manufactured articles. The theory of heat and mass transfer constitutes the scientific basis for many heat-power-engineering processes and incorporates scientific knowledge from the hydrodynamics of continuous media, molecular physics, thermodynamics, and the physical chemistry of dispersed media. The molecular-kinetic theory describing heat-transfer and mass-transfer phenomena is very complicated and has yet not been worked out satisfactorily. Therefore the modern theory of heat and mass exchange is mainly a phenomenological theory, based on the hydrodynamics and thermodynamics of continuous media. Thanks to the studies of Dutch and Belgian physicists, especially of de Groot, a powerful new method for the phenomenological investigation of transfer phenomena has recently been created. This method, known as the thermodynamics of irreversible processes (or the thermodynamics of nonequilibrium states), makes it possible to study the transfer of heat and mass as one unified subject, including within its scope the hydrodynamics of viscous fluids, heat conduction, diffusion, and internal friction. As a result, instead of separate differential equations of motion (Navier-Stokes), heat-transfer equations (Fourier-Kirchhoff), and diffusion equations (Fick), we obtain a system of interrelated differential equations of mass and energy transfer.

The solution of this system of differential equations is extremely difficult mathematically, so that in most cases numerical methods of solution by computers are used. However, in certain particular cases of heat and mass transfer (in molecular solutions, in binary mixtures at rest, in dispersed media and porous bodies, in the absorbing materials of nuclear reactors, etc.) this system of differential equations can be solved completely. The data of the solution are of definite interest not only for the calculation of heat-transfer and mass-transfer processes but also for the study of the basic laws governing heat and mass exchange and, in particular, for the development of new methods for determining thermophysical parameters.

This text deals with the derivation of an analytical theory describing heat and mass transfer in stationary media and dispersed systems. In order for the solutions of the system of differential equations of heat and mass transfer to be applicable to other transfer processes, these solutions are all given in dimensionless form, using the methods of similarity theory (the theory of generalized variables). Therefore, this monograph constitutes essentially an analytical theory of the thermodynamics of nonequilibrium states. Since references on the thermodynamics of irreversible processes are extremely meager, Chapter I will give a basic description of the thermodynamics of heat-transfer and mass-transfer phenomena.

In Chapter II we shall examine in detail heat and mass transfer in gaseous mixtures, liquid solutions, and dispersed media, as well as several other cases of heat and mass transfer. The basic methods for solving nonstationary [unsteady-state] problems in heat and mass transfer will also be considered in Chapter II. The fundamentals of similarity theory are discussed separately in Chapter III, since it is the authors' opinion that similarity theory is an important means of generalizing not only experimental but also analytical investigations. Finally, Chapters IV to XI are devoted to analytical investigations of heat and mass transfer (solutions of unsteady-state problems and analyses of the results obtained).

A large number of original tables of basic solutions and parametric coefficients are included in the text, as well as special calculation graphs. Together they make it possible to carry out rapidly the calculations required in heat engineering, a factor which will undoubtedly facilitate the introduction of these solutions into engineering practice.

The authors are indebted to Prof. S. R. de Groot for his useful discussions of certain fundamental problems related to the thermodynamics of transfer phenomena.

The Authors

This text presents an analytical theory describing the transfer of heat and matter in gaseous mixtures, dispersed systems, and porous bodies. The methods of the thermodynamics of irreversible processes are used to obtain a system of differential equations for heat and mass transfer in the presence of phase and chemical transformations, and then the solutions of this system for problems of unsteady heat and mass conduction with various boundary conditions are found. The data of these solutions may be useful in the calculation of drying, gasification, and combustion processes, as well as in the calculation of the separation of gaseous mixtures and molecular solutions.

This text will be of interest to a large number of scientists and engineers in research and in industry and may also be used as a textbook for students.

## TRANSLATION EDITOR'S NOTE

Some obvious errors in the formulas and text have been corrected or indicated by footnotes, and the corrections listed in the Russian errata have been incorporated. Numbers in slashes pertain to the reference lists for the given chapter, located at the end of the book. Additions made by the translation editor are enclosed in brackets.

The Russian term "molyarnyi" has been translated as "macroscopic," since it refers to processes which are contrasted with molecular or microscopic processes and since the English term "molar" is not generally used in this way. In section 2-7 the author defines a class of boundary conditions which includes both conditions at the boundaries of the system (true boundary conditions) and initial conditions. This larger class of conditions is referred to throughout the translation as "limiting conditions."

The vector and tensor notation employed by the author in the first three chapters is summarized in Appendix I. The Russian "ch" and "sh" are equivalent to the English "cosh" and "sinh." The Russian term "kriterii podobiya" is sometimes translated literally as "similarity criterion," but usually the more common terms "dimensionless number" or "dimensionless parameter" have been preferred. A list of the dimensionless numbers used in the text has been added (Appendix II). A subject index has also been compiled.

For states which are time-independent, corresponding to the Russian term "statsionarnyi," the term "steady state" has been used rather than "stationary state," and the latter term has been reserved for systems which are at rest in space. The subscript "c", used throughout the text, generally pertains to conditions in the surroundings. All units refer to the metric system.

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## Chapter I

### THERMODYNAMICS OF HEAT-TRANSFER AND MASS-TRANSFER PHENOMENA

#### 1-1. Onsager's Theory

In its present form the thermodynamics of nonequilibrium states includes phenomenological theories of the molecular transfer of heat (heat conduction), mass (diffusion), and fluid momentum (internal friction); in addition, it includes the hydrodynamics of viscous fluids during phase and chemical transformations. All these fields may now be studied as one unified subject.

Two fundamental natural laws constitute the real basis of the thermodynamics of transfer phenomena. These are the law of mass conservation and the law of energy conservation and transformation, together with the principle of entropy increase (the second law of classical thermodynamics), the latter being the basis of Onsager's theorem.

Let us consider an adiabatically isolated system, whose state is characterized by the parameters  $\Pi_1, \Pi_2, \Pi_3, \dots, \Pi_n$  (pressure, temperature, concentration, etc.). The values of the parameters in the equilibrium state are  $\Pi_1^0, \Pi_2^0, \Pi_3^0, \dots, \Pi_n^0$ , while their variations from the equilibrium-state values are  $\vartheta_i = \Pi_i - \Pi_i^0$  (where  $i=1, 2, 3, \dots, n$ ). In the equilibrium state the entropy is a maximum, and the variable parameters  $\vartheta_i$  are zero. In a nonequilibrium state the variation  $\Delta S$  of the entropy from the equilibrium value can be written approximately as

$$\Delta S = -\frac{1}{2} \sum_{i,k} A_{ik} \vartheta_i \vartheta_k. \quad (1-1-1)$$

The time-derivative of the parameter  $\vartheta_i$  is called, according to Onsager, the flux  $J_i$ , and the derivative  $\partial(\Delta S)/\partial \vartheta_i$  is called the thermodynamic motive force  $X_i$ . \* Thus, in equation form we have

$$J_i = \frac{d\vartheta_i}{d\tau}; \quad X_i = \frac{\partial(\Delta S)}{\partial \vartheta_i} = -\sum_{k=1}^n A_{ik} \vartheta_k. \quad (1-1-2)$$

We may now write the entropy increase per unit time (the entropy-production rate) as

$$\frac{dS}{d\tau} = -\sum_{i,k} A_{ik} \frac{d\vartheta_i}{d\tau} \vartheta_k = \sum_i J_i X_i. \quad (1-1-3)$$

\* [This quantity is also known as the "kinetic force" (cf. Veinik, A. I. Thermodynamics, A Generalized Approach, p. 108, — Israel Program for Scientific Translations, Cat. No. 1021, Jerusalem, 1964), the "affinity," or simply the "thermodynamic force" (cf. de Groot, S. R. and P. Mazur. Non-Equilibrium Thermodynamics, p. 25 — North Holland Publishing Co. Amsterdam, 1962). In the following the term "thermodynamic force" will usually be used.]

Consequently, the rate of entropy increase is the sum of the products of the fluxes times their corresponding thermodynamic motive forces. Relation (1-1-3) therefore may serve as a basis for the choice of the thermodynamic forces. It should be noted that these forces have nothing in common with forces in the Newtonian sense of the word; they cause such irreversible phenomena as the transfer of energy, heat, and mass. As an example, let us consider a one-component system in which very simple molecular processes of heat and mass transfer (heat conduction and self-diffusion) take place. The system as a whole is divided into two subsystems, between which energy exchange occurs by means of heat conduction and mass exchange (self-diffusion). Now, the entropy variation in one of the subsystems can be obtained from Gibbs's equation:

$$TdS = dU + pdV - \sum_{k=1}^n \mu_k dM_k. \quad (1-1-4)$$

For a one-component system at constant volume ( $V$  constant) this equation becomes

$$TdS = dU - \mu dM. \quad (1-1-5)$$

In this case the thermodynamic motive forces for the transfer of energy and mass are

$$X_u = \left( \frac{\partial S}{\partial U} \right)_M = \nabla \left( \frac{1}{T} \right) = -\frac{1}{T^2} \nabla T, \quad (1-1-6)$$

$$X_M = -\nabla \left( \frac{\mu}{T} \right). \quad (1-1-7)$$

Consequently, the thermodynamic force for the energy (heat) transfer is directly proportional to the temperature gradient, while the force for the mass transfer is proportional to the gradient of the ratio of the chemical potential to the absolute temperature.

Gibbs's equation (1-1-4), together with Onsager's theorem (1-1-3), represents a basis for choosing the fluxes and the thermodynamic forces. For convenience in using these relations to describe various transfer phenomena, let us carry out certain transformations. Gibbs's equation, representing the second law of thermodynamics, may be written in terms of the specific values of the entropy, internal energy, volume and concentration ( $s = S/M$ ,  $u = U/M$ ,  $v = V/M$ ,  $\rho_{k0} = M_k/M = \rho_k/\rho$ ) as follows:

$$Tds = du + pdv - \sum_{k=1}^n \mu_k d\rho_{k0}. \quad (1-1-8)$$

Differentiation with respect to time gives

$$T \frac{ds}{d\tau} = \frac{du}{d\tau} + p \frac{dv}{d\tau} - \sum_{k=1}^n \mu_k \frac{d\rho_{k0}}{d\tau}. \quad (1-1-9)$$

The left side of (1-1-9) is the product of the absolute temperature times the rate of entropy increase. It will be convenient to write Onsager's basic relation (1-1-3) in a similar form, choosing the product  $X_i T$  as the thermodynamic motive force and denoting this quantity by the same letter  $X_i$ ,\*

\* The fluxes and thermodynamic forces can be chosen in various ways; details will be given later.

so that

$$T \frac{ds}{d\tau} = \sum_i J_i X_i. \quad (1-1-10)$$

*Consequently, the product of the rate of entropy production times the absolute temperature is equal to the sum of the products of the fluxes times their corresponding thermodynamic motive forces.*

According to this definition, the thermodynamic forces for the transfer of energy (heat) and mass are

$$X_u = -\frac{1}{T} \nabla T \quad \text{and} \quad X_M = -T \nabla (\mu/T). \quad (1-1-11)$$

In contrast to the classical theory of molecular transfer, the [individual] transfers of heat, matter, and electrical charge are determined not only by the action of the single corresponding forces but by the action of all the thermodynamic forces involved:

$$J_i = \sum_{k=1}^n L_{ik} X_k \quad (i=1, 2, \dots, n). \quad (1-1-12)$$

Equation (1-1-12) is known as Onsager's system of linear equations and it constitutes the basic relation of the thermodynamics of irreversible processes.

The quantities  $L_{ik}$  are called kinetic coefficients and a reciprocal relation (the principle of symmetry of the kinetic coefficients) exists between them:

$$L_{ik} = L_{ki}. \quad (1-1-13)$$

From the macroscopic point of view the reciprocal relation constitutes an axiom, but it may be proved statistically using the time-reversal invariance of the laws of mechanics (invariance for  $\tau \rightarrow -\tau$ ).

As an example let us consider molecular heat transfer under nonisothermal conditions in a molecular solution, in which diffusion of the solute takes place. In this case the transfer of energy (heat) and mass is described by Onsager's system of equations:

$$J_1 = L_{11} X_1 + L_{12} X_2 = -\frac{L_{11}}{T} \nabla T - L_{12} T \nabla \left( \frac{\mu}{T} \right), \quad (1-1-14)$$

$$J_2 = L_{21} X_1 + L_{22} X_2 = -\frac{L_{21}}{T} \nabla T - L_{22} T \nabla \left( \frac{\mu}{T} \right). \quad (1-1-15)$$

It follows from equation (1-1-15) that the mass flux  $J_m$  ( $J_2 = J_m$ ) is determined not only by the action of the direct thermodynamic force  $X_2$  (concentration diffusion), but also by the action of force  $X_1$  (thermal diffusion, or the Soret effect). Similarly, energy (heat) transfer takes place by means of heat conduction (direct effect) and also as a result of diffusion of the solute (Dufour effect). The coefficient  $L_{11}$  is proportional to the thermal conductivity, while the coefficient  $L_{22}$  is proportional to the diffusion coefficient. According to the reciprocity principle, the cross coefficients, which establish the mutual relation between the energy flux and the material flux, are equal to one another:

$$L_{21} = L_{12}. \quad (1-1-16)$$

This indicates that there is symmetry between the effect which the diffusion force has on the energy flux and the effect which the energy (thermal) force

has on the material flux. Coefficients  $L_{12}$  and  $L_{21}$  are called entrainment coefficients and are proportional to the coefficient of thermal diffusion. The Dufour effect is slight in molecular solutions, but it can be observed experimentally in gaseous mixtures. The diffusion coefficient  $D$  for solutions is of the order of  $10^{-5}$  cm<sup>2</sup>/sec, whereas in gaseous mixtures it is of the order of  $10^{-1}$  cm<sup>2</sup>/sec; the coefficient of thermal diffusion ranges approximately from  $10^{-8}$  to  $10^{-10}$  cm<sup>2</sup>/sec · degree for solutions and from  $10^{-4}$  to  $10^{-6}$  cm<sup>2</sup>/sec · degree for gaseous mixtures. The reciprocity principle ( $L_{12}=L_{21}$ ) has been confirmed by experiments on thermal diffusion and diffusion heat conduction in binary gaseous mixtures. In the thermodynamics of irreversible processes the transfer energy  $u^*$  and the transfer heat  $Q^*$  ( $Q^*=u^*-h$ ) are extremely important. These may be determined from equations (1-1-14) and (1-1-15). For an isothermal process ( $T$  constant and  $X_1=0$ ) equations (1-1-14) and (1-1-15) give

$$J_1 = \frac{L_{12}}{L_{22}} J_2 = u^* J_2. \quad (1-1-17)$$

It follows from this relation that the quantity  $u^*$ , which we have called the transfer energy, is the energy transported by a unit mass under isothermal conditions.

## 1-2. Methods for Defining the Fluxes and Thermodynamic Forces

Onsager's reciprocal relations remain valid for a linear transformation of the fluxes and forces, due to the invariance of the symmetry of the  $L_{ik}$  matrices. This makes it possible to choose the thermodynamic forces and the corresponding fluxes in various ways.

Let us consider some examples. If we introduce new fluxes  $J'_u$  and  $J'_m$  according to the relations

$$J'_u = J_u - \mu J_m, \quad (1-2-1)$$

$$J'_m = J_m, \quad (1-2-2)$$

then the basic relation (1-1-10) may be written as

$$T \frac{ds}{d\tau} = J_u X_u + J_m X_m,$$

$$T \frac{ds}{d\tau} = J'_u X'_u + J'_m X'_m = J_u X'_u - \mu J_m X'_u + J_m X'_m,$$

and so

$$X'_u = X_u, \quad (1-2-3)$$

$$X'_m = X_m + \mu X_u. \quad (1-2-4)$$

Consequently the thermodynamic forces are

$$X'_u = -\frac{1}{T} \nabla T \text{ and } X'_m = -[\nabla \mu]_T. \quad (1-2-5)$$

The subscript  $T$  means that the gradient of the chemical potential is taken at constant temperature. Onsager's linear equations and the reciprocity principle are thus valid:

$$J'_u = L'_{11} X'_u + L'_{12} X'_m, \quad (1-2-6)$$

$$J'_m = L'_{11}X'_u + L'_{12}X'_m, \quad (1-2-7)$$

$$L'_{12} = L'_{21}. \quad (1-2-8)$$

For an isothermal transfer process we obtain the relations\*

$$J_u = \frac{L'_{12}}{L'_{22}} J'_m = TS^* J'_m = (u^* - \mu) J'_m, \quad (1-2-9)$$

where  $S^*$  is the entropy transferred by a unit mass ( $J_s/J_m$ ).

The second method for determining the thermodynamic forces is based on the following relations for the heat (energy) flux  $J'_u$  and the mass flux  $J'_m$ :

$$J'_u = J_u - hJ_m, \quad (1-2-10)$$

$$J'_m = J_m. \quad (1-2-11)$$

In this case we have

$$X''_u = X_u, \quad (1-2-12)$$

$$X''_m = X_m + hX_u, \quad (1-2-13)$$

from which the following expressions for the forces are obtained:

$$X''_u = -\frac{1}{T} \nabla T \quad \text{and} \quad X''_m = -(\nabla \mu)_T. \quad (1-2-14)$$

Onsager's system of linear equations and the reciprocal relations hold true ( $L'_{12} = L'_{21}$ ). For isothermal conditions ( $X''_u = 0$ ) the heat transferred  $Q^*$  will be determined by the relation

$$J'_u = \frac{L'_{12}}{L'_{22}} J'_m = Q^* J'_m, \quad (1-2-15)$$

since in accordance with (1-2-13) and (1-1-17)

$$\frac{L'_{12}}{L'_{22}} = u^* - h = Q^*.$$

As will be shown below, it will be convenient to call the magnitude of the flux  $J'_u$  the heat flux  $J_q$  ( $J_q = J'_u$ ).

The internal energy  $u$  is determined except for a constant factor. For some state of the system the internal energy is taken as zero, and this state is considered to be the initial one.

Let us assume that the internal energy  $U'''$  is

$$U''' = U + C. \quad (1-2-16)$$

Now, in terms of specific quantities, with the constant written as  $C = cM$ , we have

$$u''' = u'' + c, \quad h''' = h + c \quad \text{and} \quad \mu''' = \mu + c. \quad (1-2-17)$$

The energy and mass fluxes will then be

$$J'''_u = J_u + cJ_m \quad \text{and} \quad J'''_m = J_m. \quad (1-2-18)$$

By a similar method the thermodynamic forces are found to be

$$X'''_u = X_u; \\ X'''_m = X_m - cX_u = -T \nabla \left( \frac{\mu}{T} \right) - cT \nabla \left( \frac{1}{T} \right); \quad (1-2-19)$$

\* According to equation (1-1-5) we have

$$TJ_s = J_u - \mu J_m = J'_u, \quad \text{where} \quad J_s = \frac{dS}{dt}.$$

\*\* [To be consistent, this should be  $u''' = u + c$ .]

$$X'''_m = -T \nabla \left( \frac{\mu'''}{T} \right). \quad (1-2-20)$$

Consequently, the thermodynamic forces are expressed by the same relations as  $X_u$  and  $X_m$ . In exactly the same way it is possible to specify the following relations:

$$L'''_{11} = L_{11}; \quad L'''_{12} = L'''_{21} = L_{12} + cL_{11} = L_{21} + cL_{11}, \quad (1-2-21)$$

$$L'''_{22} = L_{22} + c(L_{12} + L_{21}) + cL_{11}. \quad (1-2-22)$$

These relations determine the effect of the initial state of the system (the effect of a shift in the reference zero of the internal energy) on the kinetic transfer coefficients.

The transfer energy is

$$u^{***} = \frac{L'''_{12}}{L'''_{22}} = \frac{L_{12}}{L_{22}} + c = u^* + c. \quad (1-2-23)$$

The transfer heat  $Q^{***}$  remains unchanged:

$$Q^{***} = u^{***} - h'' = u^* - h = Q^*. \quad (1-2-24)$$

Table 1-1 gives the basic thermodynamic characteristics describing the transfer of energy and mass.

TABLE 1-1  
Basic thermodynamic characteristics of energy and mass transfer

Fluxes		Thermodynamic forces		Transfer under isothermal conditions	Transfer quantity
$J_1$	$J_2$	$X_1$	$X_2$		
$J_u$	$J_m$	$X_u = -\frac{\nabla T}{T}$	$X_m = -T \nabla \left( \frac{\mu}{T} \right)$	$J_u = u^* J_m$	$u^*$
$J'_u = J_u - \mu J_m$	$J'_m = J_m$	$X'_u = -\frac{\nabla T}{T}$	$X'_m = -(\nabla \mu)_T$	$J'_u = TS^* J_m$	$TS^* = u^* - \mu$
$J''_u = J_u - h J_m$	$J''_m = J_m$	$X''_u = -\frac{\nabla T}{T}$	$X''_m = -(\nabla \mu)_T$	$J''_u = Q^* J_m$	$Q^* = u^* - h$
$J'''_u = J_u + c J_m$	$J'''_m = J_m$	$X'''_u = -\frac{\nabla T}{T}$	$X'''_m = -T \nabla \left( \frac{\mu}{T} \right)$	$J'''_u = u^{***} J_m$	$u^{***} = u^* + c$

### 1-3. The Curie Principle

The basic assumption of the thermodynamics of nonequilibrium states, as expressed in the form of Onsager's system of linear equations  $J_i = \sum_k L_{ik} X_k$ , has a restriction known as Curie's theorem or the Curie principle. A flux  $J_i$  is determined by the effects of a number of forces  $X_k$ , if these have the same tensorial character. Thermodynamic motive forces may be scalars (zero-rank tensors), vectors (first-rank tensors), and tensors (second-rank tensors). For example, thermodynamic forces  $X_u$  and  $X_m$  are tensors of the first rank, since the gradients of the scalar quantities  $T$  and  $\mu/T$  are vectors. The thermodynamic motive force for chemical and



phase transformations is the chemical affinity  $A_i$ , which is proportional to the difference  $(\mu'_i - \mu_i)$  between scalar quantities; thus it is a zero-rank tensor. The transfer of fluid momentum or impulse transfer is described by a second-rank tensor.

According to the Curie principle, for an isotropic system a coupling  $\sum_k L_{ik} X_k$  is possible only if the thermodynamic forces  $X_k$  are tensors of the same rank or if the difference in ranks is even.

The thermodynamic forces  $X_u$  and  $X_m$  are tensors of the first rank (vectors) and so coupling between them is possible. This coupling is manifested by concurrent transfer phenomena, namely the Soret effect during molecular heat transfer and the Dufour effect during the diffusion of matter. On the other hand, heat conduction or diffusion cannot be coupled with chemical and phase transformations, since the difference between the ranks of the forces  $X_u$  and  $A_i$  or the forces  $X_m$  and  $A_i$  is one (an odd number). Similarly, the molecular transfer of heat cannot couple with the transfer of momentum and diffusion cannot couple with internal friction, since the thermodynamic forces for the molecular transfer of heat and mass are tensors of the first rank, while the thermodynamic forces for the molecular transfer of momentum are tensors of the second rank (the difference between the tensor ranks is odd). However, in certain particular cases the internal friction may be considered as a molecular transfer of the kinetic energy of a fluid flow, which takes place under the action of a thermodynamic force, namely the kinetic energy (the gradient of a scalar). In this case coupling between the molecular transfer of heat, mass, and the energy of the fluid motion is possible, since all these correspond to the action of thermodynamic forces which are tensors of the same rank (vectors). On the basis of the Curie principle coupling between the molecular transfer of momentum (volume viscosity) and chemical and phase transformation processes is possible, since in the first case the forces  $A_i$  are zero-rank tensors, and in the second case the forces are second-rank tensors. Since the difference between the tensor ranks is two (an even number), coupling between them is possible.

The above examples indicate the importance of the Curie principle in the investigation of transfer phenomena. This principle, which was formulated by Curie, has been verified in detail by de Groot. A short proof of Curie's theorem will now be given.

Consider a molecular transfer taking place under the action of a thermodynamic force  $X_c$  which is a scalar and another molecular transfer taking place under the action of a vector force  $\vec{X}_b$ . According to the basic relation of the thermodynamics of nonequilibrium states we may write

$$T \frac{dS}{dt} = J_c X_c + \vec{J}_b \vec{X}_b. \quad * \quad (1-3-1)$$

Consequently, for an unsymmetric system

$$J_c = L_{cc} X_c + \vec{L}_{cb} \vec{X}_b, \quad (1-3-2)$$

$$\vec{J}_b = \vec{L}_{bc} X_c + \mathcal{L}_{bb} \vec{X}_b, \quad (1-3-3)$$

\* [See Appendix for a brief review of vector and tensor notation.]

where the kinetic coefficients are:  $L_{cc}$ , a scalar;  $\vec{L}_{cb}$  and  $\vec{L}_{bc}$ , vectors; and  $\mathcal{L}_{bb}$ , a tensor. Altogether there are 16 transfer coefficients.

If the system under consideration is isotropic, then a tensor  $\mathcal{L}$  of rank  $n$  can be expressed by means of an orthogonal transformation  $\mathcal{A}$  (the determinant of  $\mathcal{A}$  is  $|\mathcal{A}| = \pm 1$ ) as follows:

$$L_{i_1, i_2, \dots, i_n} = \sum_{j_1, j_2, \dots, j_n} \mathcal{A}_{i_1}^{j_1} \mathcal{A}_{i_2}^{j_2} \dots \mathcal{A}_{i_n}^{j_n} L_{j_1, j_2, \dots, j_n}, \quad (1-3-4)$$

where the subscripts of  $i_k$  and  $j_k$  denote the  $x$ ,  $y$ , and  $z$  components. Relation (1-3-4) is true for polar tensors; for axial tensors it is necessary to add the quantity  $|\mathcal{A}|$ .

The transformation (1-3-4) may be written in the abbreviated notation

$$\mathcal{L}' = \mathcal{A}^n(\cdot) \mathcal{L}, \quad (1-3-5)$$

where  $\mathcal{A}^n$  denotes the scalar  $(\cdot)$  product of  $\mathcal{A}_{i_k}^{j_k}$  and  $L_{j_k}$  in an  $n$ th-degree expansion.

In the general case, in the absence of symmetry,

$$\mathcal{L}' \neq \mathcal{L}. \quad (1-3-6)$$

In the case of spatial symmetry, the transformation  $\mathcal{A}$  is invariant with respect to  $\mathcal{L}$ :

$$\mathcal{L}' = \mathcal{L}. \quad (1-3-7)$$

Consequently,

$$\mathcal{A}^n(\cdot) \mathcal{L} = \mathcal{L}. \quad (1-3-8)$$

One of the symmetry properties of an isotropic system is invariance with respect to the inversion operation  $|$  (where  $|/| = -1$ ), in which case

$$\begin{aligned} x' &= x, \\ y' &= y, \\ z' &= z', * \end{aligned} \quad (1-3-9)$$

or in tensor form

$$\vec{r}' = | \cdot \vec{r}, \quad (1-3-10)$$

where

$$| = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (1-3-11)$$

If  $\mathcal{A} = |$ , then relation (1-3-8) becomes

$$(-1)^n \mathcal{L} = \mathcal{L}. \quad (1-3-12)$$

It follows from (1-3-12) that when  $n$  is odd  $-\mathcal{L} = \mathcal{L}$ , so that  $\mathcal{L} = 0$ , and consequently all the kinetic coefficients  $\mathcal{L}$  are zero. If  $n$  is even or zero, then  $\mathcal{L} = \mathcal{L}$  and the kinetic coefficients are not zero. For the system represented by the linear equations (1-3-2) and (1-3-3) the vector coefficients ( $n=1$ )  $\vec{L}_{cb}$  and  $\vec{L}_{bc}$  are zero:

$$\vec{L}_{cb} = 0 \quad \text{and} \quad \vec{L}_{bc} = 0. \quad (1-3-13)$$

Consequently, instead of the system of linear equations (1-3-2) and (1-3-3) we have two independent equations

$$J_c = L_{cc} X_c, \quad (1-3-14)$$

\* [This should apparently be  $z' = -z$ .]

$$\vec{J}_b = \mathcal{L}_{bb} \vec{X}_b. \quad (1-3-15)$$

Thus, in an isotropic system thermodynamic forces of different tensorial character do not interact during the production of a molecular-transfer flux. This theorem, or the Curie principle, follows directly from the symmetry properties of an isotropic system.

#### 1-4. Onsager's Basic Relation

The exclusion, according to Curie's theorem, of coupling between thermodynamic motive forces of different tensorial character in the equations for the fluxes does not apply to the basic relation for the rate of entropy increase.

Let us consider this in detail. Once again an adiabatically insulated system is considered, characterized by parameters of state  $\Pi_i (i=1, 2, 3, \dots, n)$  and  $R_i (i=1, 2, 3, \dots, m)$ . The variable parameters  $\Pi_i$ , which characterize the molecular transfer, are assumed to be even functions of the molecular velocities (these may be the temperature, energy, concentration, etc.), while variables  $R_i$  are odd functions (momentum). The entropy increase  $\Delta S$  is then

$$\Delta S = -\frac{1}{2} \sum_{i,k=1}^n A_{ik} \beta_i \beta_k - \frac{1}{2} \sum_{i,k=1}^m B_{ik} \gamma_i \gamma_k, \quad (1-4-1)$$

where  $\beta_i = \Pi_i - \Pi_i^0$  and  $\gamma_i = R_i - R_i^0$ , the superscript 0 denoting the equilibrium state.

In tensor notation equation (1-4-1) becomes

$$\Delta S = -\frac{1}{2} \mathcal{E} : \vec{\beta} \vec{\beta} - \frac{1}{2} \mathcal{D} : \vec{\gamma} \vec{\gamma}, \quad (1-4-2)$$

where  $\mathcal{E}$  and  $\mathcal{D}$  are symmetric and positive matrices. No cross terms between  $\vec{\beta}$  and  $\vec{\gamma}$  appear, since the entropy is an even function of the velocity. The time variations of  $\vec{\beta}$  and  $\vec{\gamma}$  are

$$\frac{d\vec{\beta}}{dt} = -\mathcal{M}^{(\beta\beta)} \vec{\beta} - \mathcal{M}^{(\beta\gamma)} \vec{\gamma}, \quad (1-4-3)$$

$$\frac{d\vec{\gamma}}{dt} = -\mathcal{M}^{(\gamma\beta)} \vec{\beta} - \mathcal{M}^{(\gamma\gamma)} \vec{\gamma}, \quad (1-4-4)$$

where the matrices  $\mathcal{M}$  are time-independent kinetic coefficients.

Onsager's reciprocal relations are represented by the following formulas:

$$\mathcal{M}^{(\beta\beta)} \mathcal{E}^{-1} = \mathcal{E}^{-1} \tilde{\mathcal{M}}^{(\beta\beta)}, \quad (1-4-5)$$

$$\mathcal{M}^{(\beta\gamma)} \mathcal{D}^{-1} = -\mathcal{E}^{-1} \tilde{\mathcal{M}}^{(\gamma\beta)}, \quad (1-4-6)$$

$$\mathcal{M}^{(\gamma\gamma)} \mathcal{D}^{-1} = \mathcal{D}^{-1} \tilde{\mathcal{M}}^{(\gamma\gamma)}, \quad (1-4-7)$$

where  $\mathcal{E}^{-1}$  and  $\mathcal{D}^{-1}$  are the reciprocals of matrices  $\mathcal{E}$  and  $\mathcal{D}$ , and  $\tilde{\mathcal{M}}$  is the transposed or symmetric matrix corresponding to  $\mathcal{M}$  (the matrix  $\mathcal{M}$  with the columns interchanged with the rows).

Let us write the thermodynamic forces  $X'_i$  and  $Y'_i$  as

$$X'_i = -\sum_k A_{ik} \beta_k, \text{ or } \vec{X}' = -\mathcal{E} \vec{\beta}, \quad (1-4-8)$$

$$Y'_i = - \sum_k B_{ik} \gamma_k, \text{ or } \vec{Y}' = - \mathcal{B} \vec{\gamma}. \quad (1-4-9)$$

The system of linear equations may now be written as

$$\frac{d\vec{\beta}}{d\tau} = \mathcal{L}^{(\beta\beta)} \vec{X}' + \mathcal{L}^{(\beta\gamma)} \vec{\gamma}, \quad (1-4-10)$$

$$\frac{d\vec{\gamma}}{d\tau} = \mathcal{L}^{(\gamma\beta)} \vec{X}' + \mathcal{L}^{(\gamma\gamma)} \vec{\gamma}, \quad (1-4-11)$$

where

$$\mathcal{L}^{(\beta\beta)} = \mathcal{M}^{(\beta\beta)} \mathcal{C}^{-1}. \quad (1-4-12)$$

Kinetic coefficients  $\mathcal{L}^{(\beta\gamma)}$ ,  $\mathcal{L}^{(\gamma\beta)}$ , and  $\mathcal{L}^{(\gamma\gamma)}$  may be expressed by similar relations.

The reciprocal relations now become

$$\mathcal{L}^{(\beta\beta)} = \tilde{\mathcal{L}}^{(\beta\beta)}; \mathcal{L}^{(\beta\gamma)} = -\tilde{\mathcal{L}}^{(\gamma\beta)}; \mathcal{L}^{(\gamma\gamma)} = \tilde{\mathcal{L}}^{(\gamma\gamma)}. \quad (1-4-13)$$

If the fluxes  $J_i$  and  $I_i$  are

$$J_i = \frac{d\beta_i}{d\tau} \text{ and } I_i = \frac{d\gamma_i}{d\tau}, \quad (1-4-14)$$

then the basic relation of the thermodynamics of a nonequilibrium state will have the form:

$$\frac{dS}{d\tau} = \vec{J} \vec{X}' + \vec{I} \vec{\gamma}. \quad (1-4-15)$$

If the thermodynamic forces are defined as  $\vec{X} = T \vec{X}'$  and  $\vec{Y} = T \vec{\gamma}'$ , then (1-4-15) may be written as

$$T \frac{dS}{d\tau} = \vec{J} \vec{X} + \vec{I} \vec{Y}, \quad (1-4-16)$$

that is, the product of the absolute temperature times the rate of entropy increase equals the sum of the products of the fluxes times their corresponding motive forces, independently of their tensorial character. Using Gibbs's equation, which expresses the second law of thermodynamics, together with the differential equations describing energy and mass transfer and relation (1-4-16), it is possible to determine the thermodynamic motive forces. Equation (1-4-16) together with Onsager's system of linear equations thus constitute the basic relations of the thermodynamics of nonequilibrium states.

## 1-5. The Differential Transfer Equations

Before going on to derive the differential transfer equations, let us discuss some mathematical operations which are very important in an analytical theory of transfer phenomena. The divergence of the vector flux  $\vec{j}$  of the transferred substance (mass, energy, concentration of some component, entropy, etc.) has the following physical meaning. If the substance enters through some part  $A_1$  of a closed control surface and leaves through the remaining part  $A_2$  of the surface, then the ratio of the difference between the outgoing and incoming fluxes to the volume bounded by the closed surface, as the latter approaches zero (the surface contracts to a point), is equal

to the divergence of the flux  $\vec{J}$ :

$$\lim_{\Delta V \rightarrow 0} \frac{\int_{(A_2)} \vec{1}_n \cdot \vec{J} dA_2 - \int_{(A_1)} \vec{1}_n \cdot \vec{J} dA_1}{\Delta V} = \lim_{\Delta V \rightarrow 0} \left\{ \frac{\int_A \vec{1}_n \cdot \vec{J} dA}{\Delta V} \right\} = \text{div } \vec{J}, \quad (1-5-1)$$

where  $\vec{1}_n$  is the unit vector along the normal  $n$  to the surface  $A$ . Integration is carried out over the entire closed surface  $A$ .

It follows that for a stationary [steady] state, when the thermodynamic characteristics do not vary with time, the divergence of the flux of the transferred substance equals the sum of the sources (sinks) of the given substance. If there are no sources or sinks  $\text{div } \vec{J}$  is zero. For an unsteady state the divergence of the flux of the transferred substance is equal to the sum of its sources (sinks) minus the amount of accumulated substance, which is equal to the partial derivative with respect to time of the volume concentration of this substance.

Let us consider some examples. If masses of a fluid (either a liquid or a gas) flow through a control surface, then the fluid flux  $\vec{J}_m$  equals the product of the density  $\rho$  times the linear velocity  $\vec{w}$  (that is,  $\vec{J}_m = \rho \vec{w}$ ). Consequently, the divergence of the fluid flux equals the partial derivative with respect to time of the density, since according to the law of conservation of matter no sources or sinks for the whole mass of fluid can exist. The corresponding equation,

$$\text{div } \rho \vec{w} = -\frac{\partial \rho}{\partial \tau}, \quad (1-5-2)$$

is known as the mass-conservation equation or the continuity equation.

The energy flux through the system is denoted as  $\vec{J}_e$ , and the specific energy is  $e$  (here  $e = E/M$ , where  $E$  is the energy of the entire system of mass  $M$ ). According to the law of energy conservation, energy sources and sinks do not exist, so that we may also write

$$\text{div } \vec{J}_e = -\frac{\partial(\rho e)}{\partial \tau}, \quad (1-5-3)$$

where  $\rho$  is the density ( $\rho = M/V$ ).

In an irreversible process the entropy increases. Consequently, in the differential equation for the entropy flux  $\vec{J}_s$  an entropy source  $I_s$ \* must be introduced, giving

$$\text{div } \vec{J}_s = -\frac{\partial(\rho s)}{\partial \tau} + I_s, \quad (1-5-4)$$

where  $I_s$  is the volume strength of the entropy source.

Let us recall the expression for the divergence of the product of a vector times a scalar:

$$\text{div } \rho \vec{w} = \nabla \cdot (\rho \vec{w}) = \vec{w} \nabla \rho + \rho \nabla \cdot \vec{w} = \vec{w} \text{grad } \rho + \rho \text{div } \vec{w}, \quad (1-5-5)$$

where  $\nabla$  is the Hamiltonian. Taking into account (1-5-5), we may write the continuity equation as

$$\rho \text{div } \vec{w} = -\frac{\partial \rho}{\partial \tau} - \vec{w} \nabla \rho = -\frac{d\rho}{d\tau}, \quad (1-5-6)$$

where  $d\rho/d\tau$ , the total (substantial) derivative of the density with respect to time, is

$$\frac{d\rho}{d\tau} = \frac{\partial \rho}{\partial \tau} + \vec{w} \nabla \rho. \quad (1-5-7)$$

\* For more details see below.

This quantity is related to the following physical considerations. The total variation of any physical quantity in a fluid element moving with a velocity  $\vec{w}$  (the velocity with respect to the center of gravity) is the result of both a variation in time and a displacement of the element from one point of space to another.

The total derivative is

$$\frac{db}{d\tau} = \frac{\partial b}{\partial \tau} + \frac{\partial b}{\partial x} \frac{dx}{d\tau} + \frac{\partial b}{\partial y} \frac{dy}{d\tau} + \frac{\partial b}{\partial z} \frac{dz}{d\tau} = \frac{\partial b}{\partial \tau} + w_x \frac{\partial b}{\partial x} + w_y \frac{\partial b}{\partial y} + w_z \frac{\partial b}{\partial z} = \frac{\partial b}{\partial \tau} + \vec{w} \nabla b, \quad (1-5-8)$$

where  $\frac{dx}{d\tau}$ ,  $\frac{dy}{d\tau}$  and  $\frac{dz}{d\tau}$  represent the velocity components  $w_x$ ,  $w_y$ , and  $w_z$ . Consequently, the total derivative

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + \vec{w} \text{grad} = \frac{\partial}{\partial \tau} + \vec{w} \nabla, \quad (1-5-9)$$

when related to a moving substance, is called the substantial derivative.

We may now use the concept of the total derivative (1-5-9) and the continuity equation (1-5-2) to obtain the following relation, which will be used in the derivation of the transfer equations:

$$\rho \frac{db}{d\tau} = \frac{\partial(b\rho)}{\partial \tau} + \text{div } b\rho \vec{w}, \quad (1-5-10)$$

where  $b$  is any finite quantity.

The mass-conservation equation. The differential equation expressing the law of conservation of matter for any  $k$ th component differs from the continuity equation (1-5-6) for the whole fluid mass. The former equation includes a source or sink for the  $k$ th component, corresponding to the mass of the  $k$ th component which is released or absorbed per unit time per unit volume as a result of chemical reactions or phase transformations:

$$\frac{\partial \rho_k}{\partial \tau} = -\text{div } \rho_k \vec{w}_k + I_k, \quad (1-5-11)$$

where  $\rho_k$  is the concentration of the  $k$ th component, equal to the ratio of the mass  $M_k$  of the component to the mixture volume  $V$  (so that  $\rho_k = \frac{M_k}{V}$ ), and  $\vec{w}_k$  is the velocity of the  $k$ th component, which is related as follows to the velocity  $\vec{w}$  of the center of gravity of the mixture:

$$\vec{w} = \frac{1}{\rho} \sum_k \rho_k \vec{w}_k. \quad (1-5-12)$$

Here  $\rho$  is the overall concentration or density of the mixture:

$$\rho = \sum_k \rho_k = \frac{1}{V} \sum_k M_k = \frac{M}{V} = \frac{1}{v}, \quad (1-5-13)$$

where  $v$  is the specific volume of the mixture.

For a chemical reaction the source  $I_k$  of material is proportional to the chemical reaction rate  $I_j$ :

$$I_k = \sum_{j=1}^r \nu_{kj} I_j = \sum_{j=1}^r \frac{1}{V} \frac{d_i M_{kj}}{d\tau} = \sum_{j=1}^r \nu_{kj} \rho \frac{d\xi_j}{d\tau}, \quad (1-5-14)$$

where  $d_i M_{kj}$  is the amount of material of the  $k$ th component which takes part in the chemical reaction  $j$ ,  $\nu_{kj}$  is the product of the molecular weight

of the  $k$ th component times the stoichiometric coefficient in the equation of chemical reaction  $j$ , and  $\xi_j$  is a quantity characterizing the probability that a given chemical reaction will take place (the completeness parameter for the chemical reaction).

When equations (1-5-11) for each of the components in the mixture are added together, we obtain the usual continuity equation for a homogeneous fluid:

$$\frac{\partial \rho}{\partial \tau} = -\operatorname{div} \rho \vec{w}, \quad (1-5-15)$$

since the sum of all the mass sources and sinks in a given volume of mixture is zero:

$$\sum_k I_k = 0. \quad (1-5-16)$$

Differential equation (1-5-11), representing the law of mass conservation, is a starting relation in the derivation of the mass-transfer equation.

During the flow of a gaseous mixture in which the concentration  $\rho_k$  is uneven [nonuniform], diffusion occurs. Diffusion represents a molecular transfer of material, since it is due to molecular (random) motion.

The amount of substance diffusing per unit time through a unit area of an isopotential surface (a surface of equal mass-transfer potential) is called the density  $\vec{j}_k$  of the molecular flux of the  $k$ th component, and is measured in  $\text{kg}/\text{m}^2 \cdot \text{hour}$ . This quantity is a vector directed along the normal  $n$  to the isopotential surface:

$$\vec{j}_k = \vec{i}_n \frac{dM_k}{Ad\tau}. \quad (1-5-17)$$

If this density  $\vec{j}_k$  of the molecular flux of matter is divided by the concentration  $\rho_k$  of the  $k$ th component, then we obtain a quantity which has the units of velocity ( $\text{m}/\text{hour}$ ), and which is known as the linear diffusion velocity  $\vec{w}_{\text{diff}}$ :

$$\vec{w}_{\text{diff}} = \frac{1}{\rho_k} \vec{j}_k. \quad (1-5-18)$$

However, this quantity is not a velocity in the usual sense of the word, since it does not characterize the macroscopic displacement of a body per unit time.

Diffusion is a mixing of a substance as a result of molecular (random) motion. A greater number of molecules pass in the direction of the material diffusion flux  $\vec{j}_k$  than in the reverse direction. This is the reason for the diffusive transfer, which is described by the molecular-transfer flux or by the arbitrary linear diffusion velocity  $\vec{w}_{\text{diff}}$ . Therefore the diffusion velocity  $\vec{w}_{\text{diff}}$  cannot enter into relation (1-5-9) in place of the velocity of motion  $\vec{w}$  of the gas, when the total derivative  $\frac{d}{d\tau}$  is obtained as the sum of the partial derivative  $\frac{\partial}{\partial \tau}$  and the convective transfer  $\vec{w} \nabla$ .

If the material diffusion flux in a moving gaseous mixture is determined as the difference between the velocity of the whole mixture and that of the given component relative to the center of gravity, then we have

$$\vec{j}_k = \rho_k (\vec{w}_k - \vec{w}). \quad (1-5-19)$$

The sum of the fluxes for all components is

$$\sum_k \vec{j}_k = 0. \quad (1-5-20)$$

For a binary mixture ( $k = 1$  and  $2$ ) we have  $\vec{J}_1 = -\vec{J}_2$ , indicating that the mass flows of the interdiffusing substances are equal and opposite to one another.

If  $\rho_k \vec{w}_k$  is determined from (1-5-19) and then substituted into differential equation (1-5-11), then we obtain

$$\frac{\partial \rho_k}{\partial \tau} + \text{div}(\rho_k \vec{w}) = -\text{div} \vec{J}_k + I_k. \quad (1-5-21)$$

Now let us return to equation (1-5-10), with  $b = \rho_{k0}$ , where  $\rho_{k0}$  is the relative concentration of the  $k$ th component or the specific mass content of the  $k$ th component:

$$\rho_{k0} = \frac{\rho_k}{\rho} = \frac{M_k}{M}. \quad (1-5-22)$$

Equations (1-5-10) and (1-5-21) now combine to give

$$\rho \frac{d\rho_{k0}}{d\tau} = -\text{div} \vec{J}_k + I_k, \quad (1-5-23)$$

since  $b\rho = \rho_{k0}\rho = \rho_k$ .

Differential equation (1-5-23) constitutes a mass-conservation equation for the  $k$ th component in the presence of phase and chemical transformations.

The momentum-conservation equation. According to classical mechanics the total [rate of] variation of the momentum of a system is the sum of all external driving forces plus the divergence of the pressure tensor. If the Cartesian coordinates  $x$ ,  $y$  and  $z$  are denoted as  $\alpha$  and  $\beta$  (where  $\alpha, \beta = 1, 2, 3$ ), then we may write

$$\rho \frac{dw_\alpha}{d\tau} = -\sum_{\beta=1}^3 \frac{\partial}{\partial \xi_\beta} P_{\beta\alpha} + \sum_{k=1}^n \rho_k \vec{F}_{k\alpha}, \quad (1-5-24)$$

where  $\vec{F}_{k\alpha}$  is the external force exerted on the  $k$ th component of the substance, and  $P_{\alpha\beta}$  are the components of the pressure tensor along the  $x$ ,  $y$ ,  $z$  coordinates ( $\alpha, \beta = 1, 2, 3$ ).

If we assume that the pressure tensor is a symmetric tensor ( $P_{\alpha\beta} = P_{\beta\alpha}$ ), then equation (1-5-24) may be written as

$$\rho \frac{d\vec{w}}{d\tau} = -\text{div} \mathcal{P} + \sum_{k=1}^n \rho_k \vec{F}_k. \quad (1-5-25)$$

Differential equation (1-5-25) is similar to equation (1-5-23) for mass conservation of the  $k$ th component. In this equation the external forces  $\rho_k \vec{F}_k$  may be interpreted as a momentum source. Equation (1-5-25) may now be combined with (1-5-10) to give

$$\frac{\partial(\rho \vec{w})}{\partial \tau} = -\text{div}(\rho \vec{w} \vec{w} + \mathcal{P}) + \sum_{k=1}^n \rho_k \vec{F}_k, \quad (1-5-26)$$

where  $\vec{w} \vec{w}$  is a dyad (a tensor with  $\alpha$  and  $\beta$  components  $w_\alpha w_\beta$ ). \* Equation (1-5-26) represents the law of momentum conservation; the partial derivative of the momentum with respect to time, referred to unit volume, equals the divergence of the momentum flux plus the sum of the [momentum] sources, considered to be due to the action of the external forces. The momentum flux consists of a convective component  $\rho \vec{w} \vec{w}$  and a conductive component  $\mathcal{P}$ .

\* Note that the dyad  $\vec{w} \vec{w}$  is different from the scalar product  $\vec{w} \cdot \vec{w}$ .



Similarly, it is possible to write the differential equation for the mass transfer of the  $k$ th component as

$$\frac{\partial \rho_k}{\partial \tau} = -\operatorname{div}(\rho_k \vec{w} + \vec{j}_k) + I_k = -\operatorname{div}(\rho_k \vec{w} + \rho_k \vec{w}_{diff}) + I_k, \quad (1-5-27)$$

where  $\rho_k \vec{w}$  is the convective component of the mass flux of the  $k$ th component and  $\vec{j}_k = \rho_k \vec{w}_{diff}$  is the conductive (molecular) component of the mass flux of the  $k$ th component.

If we carry out a scalar multiplication of equation (1-5-26) by  $\vec{w}$ , we will obtain a differential equation for the kinetic energy ( $\vec{w}^2/2$ ) of a unit fluid mass. First, it will be convenient to perform the following transformation: the scalar product of (1-5-26) times  $\vec{w}$  is taken, with  $\vec{F}_k$  equal to zero:

$$\rho \frac{d\left(\frac{1}{2} \vec{w}^2\right)}{d\tau} = -\sum_{\alpha, \beta} w_\alpha \frac{\partial}{\partial x_\beta} P_{\beta\alpha} = -\sum_{\alpha, \beta} \frac{\partial}{\partial x_\beta} (P_{\beta\alpha} w_\alpha) + \sum_{\alpha, \beta} P_{\beta\alpha} \frac{\partial}{\partial x_\beta} w_\alpha. \quad (1-5-28)$$

In vector notation this relation is

$$\rho \frac{d\left(\frac{1}{2} \vec{w}^2\right)}{d\tau} = -\vec{w} \cdot \nabla \cdot \mathcal{P} = -\nabla \cdot \mathcal{P} \cdot \vec{w} + \tilde{\mathcal{P}} : \nabla \vec{w}, \quad (1-5-29)$$

where  $\tilde{\mathcal{P}}$  is the transpose of the matrix  $\mathcal{P}$  (i.e.,  $\tilde{\mathcal{P}}_{\alpha\beta} = \mathcal{P}_{\beta\alpha}$ ).

Now let us carry out the scalar multiplication of equation (1-5-26) by  $\vec{w}$  to obtain

$$\frac{\partial \left(\frac{1}{2} \rho \vec{w}^2\right)}{\partial \tau} = -\nabla \cdot \left(\frac{1}{2} \rho \vec{w}^2 \vec{w} + \mathcal{P} \cdot \vec{w}\right) + \tilde{\mathcal{P}} : \nabla \vec{w} + \sum_{k=1}^n \rho_k \vec{F}_k \vec{w}. \quad (1-5-30)$$

Differential equation (1-5-30) for the kinetic energy of the fluid contains two sources, one associated with the work  $\rho_k \vec{F}_k \vec{w}$  of the external forces and another, equal to  $\tilde{\mathcal{P}} : \nabla \vec{w}$ , associated with viscous deformation during the fluid flow (energy dissipation due to internal friction). If the external forces are conservative and constant in time, then they may be replaced by their corresponding potentials, while the work of the external forces may be replaced by the potential energy:

$$\vec{F}_k = -\nabla \psi_k \text{ for } \partial \psi_k / \partial \tau = 0, \quad (1-5-31)$$

where  $\psi_k$  is the potential [energy] of a unit mass of the  $k$ th component of the substance in an external force field.

If we denote the potential energy of a unit mass of mixture as

$$\psi = \frac{1}{\rho} \sum_k \rho_k \psi_k,$$

then the differential equation for the transfer of potential energy is

$$\frac{\partial(\rho\psi)}{\partial \tau} = -\operatorname{div}\left(\rho\psi\vec{w} + \sum_{k=1}^n \vec{j}_k \psi_k\right) - \sum_{k=1}^n \rho_k \vec{F}_k \vec{w} - \sum_{k=1}^n \vec{j}_k \vec{F}_k. \quad (1-5-32)$$

\* [The first term in this equation should be  $\rho \frac{d\left(\frac{1}{2} \vec{w}^2\right)}{d\tau}$ ]

Thus, the partial derivative with respect to time of the volume density of the potential energy equals the divergence of the energy flux plus the sum of the potential-energy sinks. The potential-energy flux consists of a convective component  $\rho \vec{\psi} \vec{w}$  (macroscopic transfer) and a conductive component  $\sum_{k=1}^n \vec{j}_k \psi_k$  (molecular transfer). The potential-energy sink is due to the work  $\sum_{k=1}^n \rho_k \vec{F}_k \vec{w}$  performed by all the external forces when the fluid moves and to the work  $\sum_{k=1}^n \vec{j}_k \vec{F}_k$  performed by the external forces during diffusion of all the components of the mixture.

The equation for the conservation of angular momentum. The equation for the conservation of angular momentum (amount of rotation)  $K$  per unit mass may be written as

$$\rho \frac{dK_{\alpha\beta}}{d\tau} = - \sum_{\gamma=1}^3 \frac{\partial}{\partial r_\gamma} (r_\alpha P_{\gamma\beta} - r_\beta P_{\gamma\alpha}), \quad (1-5-33)$$

where the index  $\gamma$  denotes summation over  $x, y$ , and  $z$  ( $\alpha, \beta = x, y, z$ ), and  $r$  is the radius vector.

The total axial vector of the angular momentum can be represented as the sum of an external and an internal axial vector:

$$K_{\alpha\beta} = D_{\alpha\beta} + \Pi_{\alpha\beta} \text{ or } \vec{K} = \vec{D} + \vec{\Pi}, \quad (1-5-34)$$

where the vector  $\vec{D}$  of the external angular momentum is

$$D_{\alpha\beta} = r_\alpha w_\beta - r_\beta w_\alpha \text{ or } \vec{D} = \vec{r} \times \vec{w},$$

while the vector of the internal angular momentum is

$$\Pi_{\alpha\beta} = I \omega_{\alpha\beta} \text{ or } \vec{\Pi} = I \vec{\omega},$$

$\vec{\omega}$  being the angular-velocity vector.

The differential equation for the conservation of [external] angular momentum  $\vec{D}$  is obtained by multiplying equation (1-5-33) by the vector  $\vec{r}$ :

$$\rho \frac{dD_{\alpha\beta}}{d\tau} = - \sum_{\gamma=1}^3 \frac{\partial}{\partial r_\gamma} (r_\alpha P_{\gamma\beta} - r_\beta P_{\gamma\alpha}) - P_{\alpha\beta} - P_{\beta\alpha}. \quad (1-5-35)$$

The equation for the conservation of the quantity  $\vec{\Pi}$  is obtained from equations (1-5-33) and (1-5-35):

$$\rho \frac{d\Pi_{\alpha\beta}}{d\tau} = -2P_{\alpha\beta}^a \text{ or } \rho \frac{d\vec{\Pi}}{d\tau} = -2\vec{P}^a, \quad (1-5-36)$$

where  $P_{\alpha\beta}^a$  is the antisymmetric part,  $\frac{1}{2}(P_{\alpha\beta} - P_{\beta\alpha})$ , of the total pressure tensor  $P_{\alpha\beta}$ , represented as an axial vector.

From equation (1-5-35) it is possible to obtain an equation for the kinetic energy of rotation if we perform a scalar multiplication by the angular-velocity vector:

$$\rho \frac{d\left(\frac{1}{2} I \omega^2\right)}{d\tau} = \rho \vec{\omega} \cdot \frac{d\vec{\Pi}}{d\tau} = -2\vec{\omega} \vec{P}^a. \quad (1-5-37)$$

The energy-conservation equation. The differential equation representing the law of energy conservation is equation (1-5-3), in which  $e$ , the total specific energy, is

$$e = \frac{1}{2} \vec{w}^2 + \frac{1}{2} I \omega^2 + \psi + u, \quad (1-5-38)$$

where  $u$  is the specific internal energy. The total energy is here given as the sum of the kinetic energies of translation and rotation plus the potential and internal energies of the fluid.

The energy-flux density is

$$\vec{J}_e = \rho e \vec{w} + \mathcal{P} \cdot \vec{w} + \sum_k \vec{J}_k \psi_k + \vec{J}_u, \quad (1-5-39)$$

where  $\vec{J}_u$  is the density of the molecular energy flux, often called the "heat flux"  $\vec{J}_q$ . However, this energy flux includes not only the specific heat flux caused by a temperature differential but also the heat transfer due to the diffusion of matter. Therefore it is preferable to call  $\vec{J}_u$  the molecular energy flux. Consequently, the total energy flux  $\rho e \vec{w}$  consists of: 1) a convective (macroscopic) energy flux caused by the motion of the fluid; 2) a conductive (molecular) flux of the energy of motion, the latter being transferred by the random motion of the molecules (the phenomenon of internal friction); 3) a molecular flux of potential energy caused by the diffusion of material in an external-force field; and 4) a molecular energy flux due to heat conduction and diffusion.

Equation (1-5-3) can therefore be written as

$$\frac{\partial}{\partial \tau} \left[ \rho \left( \frac{1}{2} \vec{w}^2 + \frac{1}{2} I \vec{\omega}^2 + \psi + u \right) \right] = -\operatorname{div} \left( \rho e \vec{w} + \mathcal{P} \cdot \vec{w} + \sum_k \vec{J}_k \psi_k + \vec{J}_u \right). \quad (1-5-40)$$

Now, by subtracting from (1-5-40) equations (1-5-29), (1-5-32), and (1-5-37), where the latter equation can be written as

$$\frac{\partial \left( \frac{1}{2} \rho I \vec{\omega}^2 \right)}{\partial \tau} = -\operatorname{div} \left( \frac{1}{2} \rho I \vec{\omega}^2 \vec{w} \right) - 2 \vec{\omega} P^a, \quad (1-5-41)$$

we obtain the differential equation [balance equation] for the internal energy:

$$\frac{\partial (\rho u)}{\partial \tau} = -\operatorname{div} (\rho u \vec{w} + \vec{J}_u) + 2 \vec{\omega} P^a - \mathcal{P} : \nabla \vec{w} + \sum_k \vec{J}_k \vec{F}_k. \quad (1-5-42)$$

The pressure tensor can be represented as

$$\mathcal{P} = p \mathbf{1}_m + \mathcal{J}, \quad (1-5-43)$$

where  $p$  is the hydrostatic pressure,  $\mathbf{1}_m$  is the unit matrix, and  $\mathcal{J}$  is the tensor of the pressure due to the viscous forces. The latter tensor can be written as the sum of three components:

$$\mathcal{J} = \Gamma \mathbf{1}_m + \Gamma^s + \Gamma^a, \quad (1-5-44)$$

where  $\Gamma$  is one third of the trace of the tensor,  $\Gamma^s$  is the symmetric part of the tensor with zero trace, and  $\Gamma^a$  is the antisymmetric part of the tensor.

The antisymmetric part of the tensor can be represented by an axial vector. The velocity-gradient tensor can be represented similarly as

$$\nabla \vec{w} = \frac{1}{3} (\operatorname{div} \vec{w}) \mathbf{1}_m + (\nabla \vec{w})^s + (\nabla \vec{w})^a, \quad (1-5-45)$$

where the antisymmetric part  $(\nabla \vec{w})^a$  corresponds to the axial vector  $\frac{1}{2} \operatorname{curl} \vec{w}$ .

We may use these relations, plus the continuity equation, which may be written as

$$\frac{d\rho}{d\tau} = \rho^{-1} \nabla \cdot \vec{w}, \quad (1-5-46)$$

where  $v$  is the specific volume ( $v = \rho^{-1}$ ), to obtain the differential equation describing the transfer of internal energy:

$$\rho \frac{du}{d\tau} + \rho p \frac{dv}{d\tau} = -\Gamma \nabla \cdot \vec{w} - \dot{\Gamma}^s : (\nabla \vec{w})^s - \Gamma^a \cdot (\nabla \times \vec{w} - 2\vec{\omega}) - \nabla \cdot \vec{j}_u + \sum_k \vec{j}_k \vec{F}_k. \quad (1-5-47)$$

When the pressure is constant the left side of the differential equation can be represented as the total derivative of the specific enthalpy:

$$\rho \frac{dh}{d\tau} = \frac{\partial(\rho h)}{\partial \tau} + \nabla \cdot (\rho h \vec{w}).$$

Consequently, differential equation (1-5-47) becomes

$$\frac{\partial(\rho h)}{\partial \tau} = -\nabla \cdot (\rho h \vec{w} + \vec{j}_u) - \Gamma \nabla \cdot \vec{w} - \dot{\Gamma}^s : (\nabla \vec{w})^s - \Gamma^a \cdot (\nabla \times \vec{w} - 2\vec{\omega}) + \sum_k \vec{j}_k \vec{F}_k. \quad (1-5-48)$$

The differential equation for entropy transfer. The total variation  $dS$  in the entropy of a system is due to: a) the influx of an amount of entropy  $d_e S$  from the surroundings (for example, by the supply of heat); and b) the production of an amount of entropy  $d_i S$  by irreversible processes inside the system:

$$dS = d_e S + d_i S. \quad (1-5-49)$$

The quantity  $d_e S$  may be either positive or negative ( $d_e S \geq 0$ ), but according to the second law of thermodynamics  $d_i S$  must always be positive ( $d_i S > 0$ ). For a closed system (a system which exchanges heat but not mass), for example,

$$d_e S = \frac{dQ}{T}, \quad (1-5-50)$$

where  $dQ$  is the amount of heat supplied to the system at a temperature  $T$ . The total variation in entropy will be

$$dS \geq \frac{dQ}{T}. \quad (1-5-51)$$

For an adiabatically insulated system  $d_e S = 0$ , and so entropy variation is always positive:

$$dS = d_i S \geq 0. \quad (1-5-52)$$

This second part,  $d_i S$ , of the total entropy variation is the most interesting, since its variation rate determines the magnitude of the thermodynamic motive forces (see relation (1-4-16)).

If the specific entropy is denoted by  $s$ , then we obtain the relations

$$S = \int_{(V)} \rho s dV \quad \text{and} \quad \frac{dS}{d\tau} = \int_{(V)} \frac{\partial(\rho s)}{\partial \tau} dV, \quad (1-5-53)$$

where the integration is carried out over the entire volume  $V$ . Now, the Gauss-Ostrogradskii theorem may be applied to obtain

$$\frac{dS}{d\tau} = - \int_{(A)} \vec{j}_{s,t} \cdot \vec{l}_n dA = - \int_{(V)} \text{div} \vec{j}_{s,t} dV, \quad (1-5-54)$$

where  $\vec{j}_{s,t}$  is the total entropy flux for the macroscopic and molecular transfers. The entropy increment  $d_i S$  due to irreversible processes may be related to an entropy source of volume strength  $I_s$ :

$$\frac{d_i S}{d\tau} = \int_{(V)} I_s dV. \quad (1-5-55)$$

The total entropy variation per unit time is, on the basis of (1-5-49), (1-5-53), and (1-5-55), equal to

$$\frac{dS}{d\tau} = \frac{d_e S}{d\tau} + \frac{d_i S}{d\tau} = \int_{(V)} \frac{\partial(\rho s)}{\partial\tau} dV = - \int_{(V)} \nabla \cdot \vec{J}_{s,t} dV + \int_{(V)} I_s dV, \quad (1-5-56)$$

from which we obtain the differential equation for entropy transfer:

$$\frac{\partial(\rho s)}{\partial\tau} = - \nabla \cdot \vec{J}_{s,t} + I_s, \quad (1-5-57)$$

where  $I_s \geq 0$ .

On the basis of (1-5-10) this equation may also be written as

$$\rho \frac{ds}{d\tau} = - \nabla \cdot \vec{J}_s + I_s, \quad (1-5-58)$$

where  $\vec{J}_s$  is the molecular entropy flux

$$\vec{J}_s = \vec{J}_{s,t} - \rho s \vec{w}. \quad (1-5-59)$$

## 1-6. Calculation of the Transfer Thermodynamic Forces and Fluxes

The entropy source  $I_s$  determines the magnitudes of the thermodynamic motive forces. This is evident from relation (1-4-16), which can be written in terms of specific quantities as

$$T I_s = T \frac{d_i S_V}{d\tau} = \vec{J} \vec{X} + \vec{I} \vec{I}, \quad (1-6-1)$$

where  $S_V$  is the volume concentration of the entropy produced due to irreversible processes.

To find an expression for the entropy source  $I_s$  in the entropy-transfer equation (1-5-58) it is necessary to use Gibbs's equation, which in terms of specific values of the thermodynamic parameters is

$$T \frac{ds}{d\tau} = \frac{du}{d\tau} + p \frac{dv}{d\tau} - \sum_{k=1}^n \mu_k \frac{d\rho_{k0}}{d\tau}. \quad (1-6-2)$$

Multiplication by  $\rho$  and division by  $T$  gives

$$\rho \frac{ds}{d\tau} = \frac{\rho}{T} \left( \frac{du}{d\tau} + p \frac{dv}{d\tau} \right) - \sum_{k=1}^n \rho \frac{\mu_k}{T} \frac{d\rho_{k0}}{d\tau}. \quad (1-6-3)$$

Now, the corresponding expressions from equations (1-5-47), (1-5-23), and (1-5-11), also taking into account (1-5-14), may be introduced to give

$$\begin{aligned} \rho \frac{ds}{d\tau} = & - \frac{1}{T} \nabla \cdot \vec{J}_u + \sum_{k=1}^n \left[ \frac{1}{T} \vec{J}_k \cdot \vec{F}_k + \frac{\mu_k}{T} \left( \nabla \cdot \vec{J}_k + \sum_{j=1}^r \nu_{kj} I_j \right) \right] - \\ & - \frac{1}{T} [\Gamma \nabla \cdot \vec{w} - \vec{I}^s : (\vec{\nabla} \vec{w})^s - \Gamma^a (\nabla \times \vec{w} - 2\vec{\omega})]. \end{aligned} \quad (1-6-4)$$

This equation may also be written as

$$\begin{aligned} \rho \frac{ds}{d\tau} = & -\nabla \cdot \left( \frac{\vec{J}_u}{T} - \sum_{k=1}^n \frac{\mu_k}{T} \vec{J}_k \right) - \vec{J}_u \cdot \frac{\nabla T}{T^2} - \\ & - \sum_{k=1}^n \vec{J}_k \cdot \left( \nabla \frac{\mu_k}{T} - \frac{\vec{F}_k}{T} \right) - \frac{1}{T} \sum_{k=1}^n \sum_{j=1}^n v_{kj} I_j - \\ & - \frac{1}{T} [\Gamma \nabla \cdot \vec{w} - \Gamma^s : (\vec{\nabla} \vec{w})^s - \Gamma^a (\nabla \times \vec{w} - 2\vec{\omega})], \end{aligned} \quad (1-6-5)$$

since

$$\begin{aligned} \nabla \cdot \frac{\vec{J}_u}{T} &= -\vec{J}_u \cdot \frac{\nabla T}{T^2} + \frac{1}{T} \nabla \cdot \vec{J}_u; \\ \nabla \cdot \sum_{k=1}^n \frac{\mu_k}{T} \vec{J}_k &= \frac{\mu_k}{T} \nabla \cdot \vec{J}_k + \vec{J}_k \nabla \frac{\mu_k}{T} \end{aligned}$$

A comparison of equations (1-6-5) and (1-5-58) indicates that

$$\vec{J}_s = \frac{\vec{J}_u}{T} - \sum_{k=1}^n \frac{\mu_k}{T} \vec{J}_k; \quad (1-6-6)$$

$$\begin{aligned} I_s = & -\vec{J}_u \cdot \frac{1}{T^2} \nabla T - \sum_{k=1}^n \vec{J}_k \cdot \left( \nabla \frac{\mu_k}{T} - \frac{\vec{F}_k}{T} \right) - \frac{1}{T} [\Gamma \nabla \cdot \vec{w} - \\ & - \Gamma^s : (\vec{\nabla} \vec{w})^s - \Gamma^a (\nabla \times \vec{w} - 2\vec{\omega})] - \frac{1}{T} \sum_{j=1}^r A_j I_j, \end{aligned} \quad (1-6-7)$$

where  $A_j$ , the chemical affinity, is defined as

$$A_j = \sum_{k=1}^n v_{kj} \mu_k. \quad (1-6-8)$$

If both sides of (1-6-7) are multiplied by  $T$  and the result is compared with (1-6-1), then we obtain the following thermodynamic motive forces:

$$\begin{aligned} \vec{X}_u &= -\frac{\nabla T}{T}, \quad \vec{X}_{mk} = \vec{F}_k - T \nabla \frac{\mu_k}{T}, \quad X_{mj} = A_j, \\ Y_1 &= \nabla \cdot \vec{w}, \quad Y_2 = \nabla \cdot \vec{w}, \quad Y_3 = (\nabla \times \vec{w} - 2\vec{\omega}), \end{aligned} \quad (1-6-9)$$

since the corresponding fluxes are

$$\left. \begin{aligned} j_u &= j_u, \quad j_{mk} = j_k, \quad j_{mj} = I_j, \\ i_1 &= \Gamma, \quad i_2 = \Gamma^s, \quad i_3 = \Gamma^a. \end{aligned} \right\} \quad (1-6-10)$$

The expressions for the thermodynamic forces  $\vec{X}_u$  and  $\vec{X}_{mk}$  for energy and mass transfer are the same as those obtained previously for a one-component system.

If we take into account that  $\sum_{k=1}^n \vec{J}_k = 0$ , we may write the thermodynamic force  $\vec{X}_{mk}$  as

$$\vec{X}_{mk} = \vec{F}_k - \vec{F}_n - T \nabla \frac{\mu_k - \mu_n}{T}, \quad (1-6-11)$$

since in the entropy-production equation, instead of

$$\sum_{k=1}^n \vec{J}_k \cdot \left( \nabla \frac{\mu_k}{T} - \frac{\vec{F}_k}{T} \right)$$

we will have

$$\sum_{k=1}^n \vec{j}_k \cdot \left( \nabla \frac{\mu_k - \mu_n}{T} - \frac{\vec{F}_k - \vec{F}_n}{T} \right).$$

Once the corresponding expressions for the thermodynamic motive forces are obtained in terms of the parameters of state, it is possible to write Onsager's system of linear equations taking into account Curie's theorem.

Since  $\vec{X}_u$  and  $\vec{X}_{m_k}$  are both vectors (first-rank tensors), their coupling is possible, whereas coupling with  $X_{mj}$ ,  $Y_1$ ,  $Y_2$ , or  $Y_3$  is impossible according to the Curie principle.

The thermodynamic force  $A_j$  for phase or chemical transformations is a scalar (zero-rank tensor), and so it may be coupled with the force  $Y_1$  (second-rank tensor):

$$\vec{j}_u = -L_{uu} \frac{\nabla T}{T} - \sum_{k=1}^{n-1} L_{uk} \left[ T \nabla \frac{\mu_k - \mu_n}{T} - (\vec{F}_k - \vec{F}_n) \right]; \quad (1-6-12)$$

$$\vec{j}_{mi} = -L_{iu} \frac{\nabla T}{T} - \sum_{k=1}^{n-1} L_{ik} \left[ T \nabla \frac{\mu_k - \mu_n}{T} - (\vec{F}_k - \vec{F}_n) \right]; \quad (1-6-13)$$

( $i = 1, 2, \dots, n$ )

$$j_j = -L_{jV} \operatorname{div} \vec{w} - \sum_{m=1}^r L_{jm} A_m; \quad (1-6-14)$$

$$\Gamma = -L_{VV} \operatorname{div} \vec{w} - \sum_{m=1}^r L_{Vm} A_m; \quad (1-6-15)$$

$$\hat{\Gamma}^s = -L(\vec{\nabla} \vec{w})^s; \quad (1-6-16)$$

$$\Gamma^a = -L_r (\vec{\nabla} \times \vec{w} - 2\vec{\omega}). \quad (1-6-17)$$

Now, on the basis of Onsager's reciprocity principle, we know that

$$L_{iu} = L_{ui}, \quad i = 1, 2, 3, \dots, (n-1); \quad (1-6-18)$$

$$L_{ik} = L_{ki}, \quad i, k = 1, 2, 3, \dots, (n-1); \quad (1-6-19)$$

$$L_{Vj} = -L_{jV}; \quad (1-6-20)$$

$$L_{mj} = L_{jm}. \quad (1-6-21)$$

Equation (1-6-18) represents the coupling between thermal diffusion and the Dufour effect, while equation (1-6-19) represents the coupling between the diffusion coefficients. For binary mixtures ( $n=2$ ) there is only one coefficient  $L_{1u}$ , and so equation (1-6-19) is absent. Relation (1-6-20) takes into account the coupling between the volume viscosity and chemical transformations. The minus sign appears because the force  $A_j$  is an even variable, while the force  $\vec{\nabla} \cdot \vec{w}$  is odd with respect to changes [reversals] in the particle velocity. Equation (1-6-21) shows the equality of the chemical cross coefficients for damping.

If instead of the molecular energy flux  $\vec{j}_u$  we introduce the heat flux  $\vec{j}_q$ , defined as

$$\vec{j}_q = \vec{j}_u - \sum_{k=1}^n h_k \vec{j}_k, \quad (1-6-22)$$

then the thermodynamic motive force  $\vec{X}_u = \vec{X}_q$  remains the same, namely

$\vec{X}_q = -\frac{1}{T} \nabla T$ . The motive force for diffusion, on the other hand, becomes  $\vec{X}_m = \vec{F}_k - (\nabla \mu_k)_T$ , where the subscript  $T$  indicates that the gradient of the chemical potential is taken at constant temperature. The same result is obtained if we substitute the relation (1-6-22) into equation (1-6-4) to eliminate  $\vec{J}_u$  and then compare the result with equation (1-6-1). To do this it is necessary to carry out similar transformations. Thus, the first two equations in the system of linear equations (1-6-12) to (1-6-17) become:

$$\vec{J}_q = -L_{qq} \frac{\nabla T}{T} - \sum_{k=1}^{n-1} L_{qk} \{[\nabla (\mu_k - \mu_n)]_T - (\vec{F}_k - \vec{F}_n)\}, \quad (1-6-23)$$

$$\vec{J}_{mi} = -L_{iq} \frac{\nabla T}{T} - \sum_{k=1}^{n-1} L_{ik} \{[\nabla (\mu_k - \mu_n)]_T - (\vec{F}_k - \vec{F}_n)\}. \quad (1-6-24)$$

The reciprocity relation  $L_{iq} = L_{qi}$  is also valid.

The kinetic coefficients  $L_{qq}$ ,  $L_{iq}$ ,  $L_{ik}$ ,  $L_{vj}$ ,  $L_{mj}$ ,  $L$ ,  $L_{vv}$ , and  $L_r$  are all related to the thermophysical characteristics describing molecular transfer. As an example let us consider a binary mixture (a two-component system with  $n=1, 2$ ), with constant pressure in the absence of an external-force field ( $\vec{F}_k=0$ ), and in which a single chemical reaction or phase transformation ( $j=1$ ) takes place. The gradient of the chemical potential is then

$$[\nabla (\mu_1 - \mu_2)]_{T, p} = \left(1 + \frac{p_{10}}{p_{20}}\right) (\nabla \mu_1)_{T, p} = \frac{1}{p_{20}} \left(\frac{\partial \mu_1}{\partial p_{10}}\right)_{T, p} \nabla p_{10}, \quad (1-6-25)$$

since according to the Gibbs-Duhem equation, with  $p$  and  $T$  constant,

$$p_{10} d\mu_1 + p_{20} d\mu_2 = 0. \quad (1-6-26)$$

If we denote  $\left(\frac{\partial \mu_1}{\partial p_{10}}\right)_{T, p}$  as  $\mu'_1$ , then we have

$$\vec{J}_q = -L_{qq} \frac{\nabla T}{T} - L_{q1} \frac{\mu'_1}{p_{20}} \nabla p_{10} = -\lambda \nabla T - p p_{10} \mu'_1 T D'_T \nabla p_{10}; \quad (1-6-27)$$

$$\vec{J}_1 = -L_{1q} \frac{\nabla T}{T} - L_{11} \frac{\mu'_1}{p_{20}} \nabla p_{10} = -D_T p p_{10} p_{20} \nabla T - D_{12} p \nabla p_{10}; \quad (1-6-28)$$

$$j_j = -L_{ev} \nabla \cdot \vec{w} - L_e A_e; \quad (1-6-29)$$

$$\Gamma = -L_{vv} \nabla \cdot \vec{w} - L_v A_v = -\eta_v \nabla \cdot \vec{w} - L_v A_v; \quad (1-6-30)$$

$$\vec{\Gamma}^s = -L(\vec{\nabla} \vec{w})^s = -2\eta(\vec{\nabla} \vec{w})^s; \quad (1-6-31)$$

$$\vec{\Gamma} = -L_r (\nabla \times \vec{w} - 2\vec{\omega}) = -\eta_r (\nabla \times \vec{w} - 2\vec{\omega}), * \quad (1-6-32)$$

where  $\lambda$  is the thermal conductivity;  $D_{12}$  and  $D_T$  are, respectively, the coefficients of interdiffusion and thermal diffusion;  $D'_T$  is the coefficient of diffusion thermal conductivity [Dufour coefficient]; and  $\eta$ ,  $\eta_v$ , and  $\eta_r$  are, respectively, the viscosity coefficients for shear, volume, and rotational deformations. The above equations indicate the relations between Onsager's kinetic coefficients and the coefficients of molecular transfer:

$$\lambda = L_{qq}/T, \quad D_{12} = L_{11} \mu'_1 / p_{20}; \quad (1-6-33)$$

\* [The left side of this equation should be  $\Gamma^a$ , not  $\vec{\Gamma}$ .]



$$D'_T = L_{q1}/\rho_1\rho_{20}T_1; \quad (1-6-34)$$

$$D_T = L_{1q}/\rho_1\rho_{20}T; \quad (1-6-35)$$

$$L_{VV} = \eta_V, \quad L = 2\eta, \quad L_r = \eta_r. \quad (1-6-36)$$

Onsager's relation  $L_{q1} = L_{1q}$  shows that the coefficients of thermal diffusion and diffusion thermal conductivity are equal ( $D_T = D'_T$ ).

The kinetic coefficient  $L_c$  characterizing the retardation of the chemical reaction is related to the relaxation coefficient, since relaxation processes are similar to the processes of mass formation during a chemical reaction.

If relaxation processes are present, then the entropy source will contain an additional term related to these processes. The expression for this additional term is obtained from Gibbs's equation, which in the given case is

$$T \frac{ds}{d\tau} = \frac{du}{d\tau} + p \frac{dv}{d\tau} - \sum_k \mu_k \frac{d\rho_k}{d\tau} - A_c \frac{d\xi}{d\tau}, \quad (1-6-37)$$

where  $A_c$  and  $\xi$  are, respectively, the affinity and thermodynamic parameter for the relaxation process. The additional term in the entropy-source equation is, accordingly,

$$I_{s\text{rel}} = \frac{p}{T} A_c \frac{d\xi}{d\tau}. \quad (1-6-38)$$

Equation (1-6-38) is analogous to the expression for entropy production during chemical reaction. If  $A_c$  is taken as the motive force, and if  $\rho d\xi/d\tau$  is the mass flux during the relaxation process, then we obtain relations which are identical to equations (1-6-29) and (1-6-30).

From equation (1-6-38) we obtain

$$\frac{d\xi_c}{d\tau} = -\frac{L_c}{p} A_c = -\frac{L_c}{p} \frac{\partial A_c}{\partial \xi} (\xi - \xi_0), \quad (1-6-39)$$

where  $\xi_0$  is the equilibrium value of  $\xi$ . The relaxation time will then be

$$\tau_{\text{rel}} = \frac{p}{L_c} \frac{\partial \xi}{\partial A_c}. \quad (1-6-40)$$

If the complete system of differential transfer equations is known, as well as the equation of state (which represents the relation between  $A_c$  and the parameters of state, including  $\xi$ ), then it is possible to solve problems concerning the dispersion and adsorption of sound in a viscous medium under nonisothermal conditions. For audio frequencies  $\nu$ , for which  $\nu\tau_{\text{rel}} \ll 1$ , the relaxation process is formally represented as a volume viscosity.

**Fields of nonconservative forces.** Let us consider what happens when the external forces are not conservative, as in the case of electromagnetic forces. The Lorentz force  $\vec{F}_k$  exerted on an electrical charge  $q_k$  moving with a velocity  $\vec{w}_k$  is

$$\vec{F}_k = q_k \left( \vec{E} + \frac{1}{c} \vec{w}_k \times \vec{B} \right), \quad (1-6-41)$$

where  $\vec{E}$  and  $\vec{B}$  are the electric-field intensity and the magnetic induction [or magnetic flux-density], and  $c$  is the propagation velocity of electromagnetic waves (the velocity of light).

Kinetic-energy equation (1-5-29) now becomes

$$\frac{\partial \left( \frac{1}{2} \rho \vec{w}^2 \right)}{\partial \tau} = -\nabla \cdot \left( \frac{1}{2} \rho \vec{w}^2 \vec{w} + \mathcal{P} \cdot \vec{w} \right) + \tilde{\mathcal{P}} \cdot \nabla \vec{w} + \left( \rho q \vec{E} + \frac{1}{c} \vec{I} \times \vec{B} \right) \cdot \vec{w}, \quad (1-6-42)$$

where  $q$  is the total electrical charge per unit mass and  $\vec{I}$  is the density of the total electrical current:

$$q = \sum_{k=1}^n \rho_k q_k / \rho, \quad \text{and} \quad \vec{I} = \sum_{k=1}^n \rho_k q_k \vec{w}_k. \quad (1-6-43)$$

In the absence of polarization, when the electric-field intensity  $\vec{E}$  equals the electric induction [displacement]  $\vec{D}$ , and when the magnetic-field intensity  $\vec{H}$  equals the magnetic induction  $\vec{B}$ , the Umov-Poynting equation is

$$\frac{\partial}{\partial \tau} \left[ \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \right] = -\text{div} (c \vec{E} \times \vec{B}) - \vec{E} \cdot \vec{I}, \quad (1-6-44)$$

where  $\frac{1}{2}(\vec{E}^2 + \vec{B}^2)$  is the energy density of the electromagnetic field,  $(c \vec{E} \times \vec{B})$  is the Umov-Poynting vector and  $\vec{E} \cdot \vec{I}$  is the work performed by the electric field. Equation (1-6-44) thus states that the partial derivative of the electromagnetic energy density equals the divergence of the electromagnetic-energy flux minus the work done by the current in the electric field, the latter constituting a sink of electromagnetic energy.

The differential equation for the total specific energy  $\rho e$  remains, as before, (1-6-44), its magnitude being

$$\rho e = \frac{1}{2} \rho w^2 + \frac{1}{2} (E^2 + B^2) + \rho u, \quad (1-6-45)$$

while the density of the total-energy flux is

$$\vec{J}_e = \frac{1}{2} \rho \vec{w}^2 \vec{w} + \rho u \vec{w} + \mathcal{P} \cdot \vec{w} + (c \vec{E} \times \vec{B}) + \vec{J}_u. \quad (1-6-46)$$

The total-current density  $\vec{I}$  is the sum of a convective component of the current  $\rho q \vec{w}$  plus a conductive transfer  $\vec{I}$ :

$$\vec{I} = \sum_k \rho_k q_k \vec{w} + \sum_k \rho_k q_k (\vec{w}_k - \vec{w}) = \rho q \vec{w} + \vec{I}. \quad (1-6-47)$$

Using these relations it is possible to obtain a differential transfer equation for the internal energy. This equation will be identical to equation (1-5-47), except that instead of  $\sum_k \vec{J}_k \vec{F}_k$  the quantity  $\vec{I} \cdot \vec{E}'$  must be introduced, where  $\vec{E}'$  is the electric-field intensity for an observer moving with a velocity  $\vec{w}$ :

$$\vec{E}' = \vec{E} + \frac{1}{c} (\vec{w} \times \vec{B}).$$

In the expression for the entropy source an additional term,

$$\begin{aligned} \frac{1}{T} \vec{I} \cdot \vec{E}' &= \frac{1}{T} \sum_{k=1}^n q_k \vec{J}_k \cdot \vec{E}' = \frac{1}{T} \sum_{k=1}^n q_k \vec{J}_k \cdot \left( \vec{E} + \frac{1}{c} \vec{w} \times \vec{B} \right) = \\ &= \frac{1}{T} \vec{I} \cdot \left[ \vec{E} + \frac{1}{c} (\vec{w} \times \vec{B}) \right], \end{aligned} \quad (1-6-48)$$

appears, where  $\vec{i}$  is the electrical-conduction flux  $\left(\vec{i} = \sum_{k=1}^n q_k \vec{j}_k\right)$ .

In the presence of polarization, which is characterized by the vectors  $\vec{P}$  and  $\vec{M}$  (referred to unit mass), two new terms must be added to (1-6-48). These terms are obtained from the following form of Gibbs's equation:

$$T \frac{ds}{d\tau} = \frac{du}{d\tau} + p \frac{dv}{d\tau} - \sum_k \mu_k \frac{d\rho_k}{d\tau} - \vec{E}' \cdot \frac{d\vec{P}}{d\tau} - \vec{B}' \cdot \frac{d\vec{M}}{d\tau}. \quad (1-6-49)$$

The additional terms are

$$I_{s\text{ pol}} = \frac{\rho}{T} \Delta \vec{E}' \cdot \frac{d\vec{P}}{d\tau} + \frac{\rho}{T} \Delta \vec{B}' \cdot \frac{d\vec{M}}{d\tau}, \quad (1-6-50)$$

where  $\Delta \vec{E}'$  and  $\Delta \vec{B}'$  are the deviations of  $\vec{E}$  and  $\vec{B}$  from their equilibrium values, which correspond to the polarization vectors  $\vec{P}$  and  $\vec{M}$ . As previously, the prime indicates that the given quantity is measured by an observer moving with a velocity  $\vec{w}$ . Equation (1-6-50) for the entropy production in an electric [electromagnetic] field in the presence of polarization describes the phenomenon of electrical and magnetic relaxation (compare equations (1-6-37) and (1-6-38)).

## 1-7. The Differential Transfer Equations for a One-Component System

The differential equations describing mass transfer of the  $n$ th component of a system and internal-energy transfer constitute the basic differential equations of heat and mass transfer. If we insert into these equations the expressions for the corresponding fluxes (equations (1-6-12) to (1-6-17)), then we obtain a system of  $(n+7)$  partial differential equations for  $(n+7)$  independent variables (the concentrations  $\rho_{10}, \rho_{20}, \dots, \rho_{(n-1)0}$ ; the coordinates  $x, y$ , and  $z$ ; the velocities  $\vec{w}$  and  $\vec{\omega}$ ; and the temperature  $T$ ). The equation of state establishes a relation between the pressure  $p$ , the internal energy  $u$ , and the chemical potential  $\mu_k$  as a function of these independent variables.

For a one-component system (an isotropic liquid or gas) the system of differential transfer equations has the form:

$$\frac{\partial \rho}{\partial \tau} = -\nabla \cdot \rho \vec{w}; \quad (1-7-1)$$

$$\rho \frac{\partial \vec{w}}{\partial \tau} = -\nabla p + \eta \nabla^2 \vec{w} + \left(\frac{1}{3} \eta + \eta_v\right) \nabla \nabla \cdot \vec{w} + \eta_r \nabla \times (\vec{\omega} - \nabla \times \vec{w}); \quad (1-7-2)$$

$$\frac{\partial \vec{\omega}}{\partial \tau} = -\frac{4\eta_r}{\rho l} \left(\vec{\omega} - \frac{1}{2} \nabla \times \vec{w}\right); \quad (1-7-3)$$

$$\begin{aligned} \rho \left( \frac{\partial u}{\partial \tau} + p \frac{dv}{d\tau} \right) &= \lambda \nabla^2 T + \eta_v (\nabla \cdot \vec{w})^2 + \\ &+ 2\eta (\nabla \vec{w})^s : (\nabla \vec{w})^s + \eta_r (\nabla \times \vec{w} - 2\vec{\omega})^2. \end{aligned} \quad (1-7-4)$$

It is assumed here that the viscosity coefficients  $\eta$ ,  $\eta_v$  and  $\eta_r$  and the thermal conductivity are constant.

In order for the system of differential equations describing the transfer of mass (1-7-1), momentum (1-7-2), angular momentum (1-7-3), and internal energy (1-7-4) to be complete it is necessary to add to it the equations of state

$$p = f(\rho, T), \quad u = F(\rho, T). \quad (1-7-5)$$

System of equations (1-7-1) to (1-7-4) describes the thermohydrodynamic properties of an isotropic fluid. The familiar equations of hydrodynamics may be obtained from it as a particular case, provided we assume isothermal or isentropic conditions of the fluid motion.

Equation (1-7-2) is the Navier-Stokes equation, with an extra term characterizing the rotation. If the viscosities  $\eta$  and  $\eta_v$  are variable (with  $\eta$  constant), then equation (1-7-2) becomes

$$\begin{aligned} \rho \frac{d\vec{w}}{d\tau} = & -\nabla p + \eta \nabla^2 \vec{w} + \left(\frac{1}{3}\eta + \eta_v\right) \nabla \nabla \cdot \vec{w} + 2(\nabla \vec{w})^s \cdot \nabla \eta + \\ & + (\nabla \cdot \vec{w}) \nabla \left(\eta_v - \frac{2}{3}\eta\right) + \eta_v \nabla \times (2\vec{\omega} - \nabla \times \vec{w}). \end{aligned} \quad (1-7-6)$$

Equation (1-7-4) describes the transfer of internal energy, the last two terms on the right representing the Rayleigh dissipation function.

If the dissipation terms are neglected equation (1-7-4) becomes the familiar Fourier-Kirchhoff equation. At constant pressure this is (compare equation (1-5-48))

$$\rho \frac{dh}{d\tau} = -\nabla \cdot (\lambda \nabla T) \quad (1-7-7)$$

or

$$c_p \rho \frac{dt}{d\tau} = -\nabla \cdot (\lambda \nabla t), \quad (1-7-8)$$

where  $c_p$  is the specific heat at constant pressure for the fluid,  $c_p = \frac{dh}{dT}$ , and  $t$  is the Celsius temperature,  $t = T - 273$ .

If the thermal conductivity  $\lambda$  is constant, we obtain

$$\frac{dt}{d\tau} = \frac{\partial t}{\partial \tau} + \vec{w} \nabla t = a_p \nabla^2 t, \quad (1-7-9)$$

where  $a_p$  is the thermal diffusivity, defined as

$$a_p = \frac{\lambda}{c_p \rho}. \quad (1-7-10)$$

For solid bodies the partial derivative  $\frac{\partial t}{\partial \tau}$  is equal to the total derivative  $\frac{dt}{d\tau}$ , since the heat transfer is molecular only ( $\vec{w} = 0$ ). The variation in the volume of the body is negligible ( $v$  is constant and  $dv/d\tau = 0$ ). Thus, if  $c_v$  is the specific heat at constant volume,

$$c_v = \frac{du}{dT}, \quad (1-7-11)$$

we obtain Fourier's classical differential heat-conduction equation

$$c_v \rho \frac{\partial t}{\partial \tau} = -\text{div} (\lambda \nabla t), \quad (1-7-12)$$

or

$$\frac{\partial t}{\partial \tau} = a_v \nabla^2 t, \quad (1-7-13)$$

where  $a_v$  is the thermal diffusivity, defined as

$$a_v = \lambda / c_v \rho.$$

## Chapter II

### MASS-TRANSFER AND HEAT-TRANSFER EQUATIONS. BASIC METHODS FOR THEIR SOLUTION

#### 2-1. The Mass-Transfer and Heat-Transfer Equations for a Binary Gaseous Mixture

The transfer of heat and mass may be described by a system of differential equations derived from the equations for the transfer of mass and energy. The latter are usually replaced by the equations for the transfer of internal energy and fluid momentum. Together with the equations of state, the differential heat-transfer and mass-transfer equations form a complete system.

To solve this system of equations certain uniqueness conditions are required. In most cases, however, it is impossible to solve this system of differential transfer equations; only in some special cases (binary gaseous mixtures, molecular solutions, porous bodies, and dispersed media) is it possible to solve the system of equations rigorously by analytical methods.

The differential equations for mass and energy transfer were derived in the previous chapter using the methods of the thermodynamics of irreversible processes. In this chapter differential heat-transfer and mass-transfer equations will be derived for specific systems and basic methods for the solution of these will be discussed.

First let us derive the basic equations for mass and heat transfer in a binary gaseous mixture. The differential equation representing the law of mass conservation for the  $k$ th component is an initial relation in the derivation of the differential mass-transfer equation. In a gaseous mixture with an uneven [nonuniform] concentration  $\rho_k$  diffusion occurs as a result of molecular (random) motion. The amount of material diffusing per unit time through a unit area of an isopotential surface (a surface of equal mass-transfer potential) is called the molecular-flux density  $\vec{j}_k$  ( $\text{kg}/\text{m}^2 \cdot \text{h}$ ) for the  $k$ th component. This quantity is a vector, the direction of which is along the normal to the isopotential surface. During diffusion, which is a mixing of matter through molecular (random) motion, a larger number of molecules pass in the direction of the diffusion flow than in the reverse direction. This is the reason for the diffusion transfer, which is characterized by a molecular-transfer flux  $\vec{j}_k$  or a linear diffusion velocity

$$\vec{w}_{\text{diff}} = \vec{j}_k / \rho_k.$$

If the diffusion flow of matter in a moving gaseous mixture is expressed in terms of the difference in the velocities relative to the center of gravity, then we have

$$\vec{j}_k = \rho_k (\vec{w}_k - \vec{w}), \quad (2-1-1)$$

where  $\vec{w}$  and  $\vec{w}_k$  are the velocities of the whole mixture and of the  $k$ th component of the mixture, relative to the center of gravity. Now, equation (2-1-1) may be summed up over all the components, and since  $w\rho = \sum_{k=1}^n \rho_k w_k$  we obtain

$$\sum_{k=1}^n \vec{J}_k = 0. \quad (2-1-2)$$

For a binary mixture ( $k=1, 2$ ) this means that

$$\vec{J}_1 = -\vec{J}_2, \quad (2-1-3)$$

that is, the mass fluxes (in  $\text{kg/m}^2 \cdot \text{h}$ ) for the interdiffusing substances are equal and opposite to one another. According to Fick's diffusion law the density of the material flow is directly proportional to the gradient of the concentration  $\rho_{i0}$ . Consequently, for isothermal conditions and a constant mixture density we have

$$\vec{J}_1 = -\vec{J}_2 = -D\rho\nabla\rho_{i0}, \quad (2-1-4)$$

where  $D$  is the coefficient of concentration interdiffusion.

In a molecular solution the diffusion flux of matter is defined as the number of moles transferred per unit time through unit area, whereas in the molecular-kinetic theory of gases this flux is defined as the number of molecules diffusing per unit time through unit area.

In order to establish an interrelationship between the fluxes obtained using different methods it is necessary to apply Prigogine's theorem, which states that for mechanical equilibrium ( $p$  constant) the entropy production is independent of the choice of the average velocity of the mixture.

The average velocity of a gaseous mixture may be defined in terms of a weight function  $G_k$  as follows:

$$\vec{w}_{av} = \frac{\sum_{k=1}^n G_k \vec{w}_k}{\sum_{k=1}^n G_k}. \quad (2-1-5)$$

The molecular flux of matter relative to the velocity  $\vec{w}_{av}$  will then be

$$\vec{J}_k = \rho_k (\vec{w}_k - \vec{w}). \quad (2-1-6)$$

Now, when both sides of (2-1-6) are multiplied by  $G_k$  and the sum over all the components is taken, we obtain, taking into account (2-1-5),

$$\sum_{k=1}^n \frac{G_k}{\rho_k} \vec{J}_k = 0. \quad (2-1-7)$$

Equation (2-1-2) follows from the more general relation (2-1-7) provided we write  $G_k = \rho_k$  and  $w_{av} = w$ . If the function  $G_k$  is assumed to be equal to the molecular concentration  $v_k$ , then we obtain\*\*

$$\sum_{k=1}^n \frac{\vec{J}_k}{M_k^0} = 0. \quad (2-1-8)$$

\* [To be consistent this equation should read  $\vec{J}_k = \rho_k (\vec{w}_k - \vec{w}_{av})$ .]

\*\* The molar concentration  $v_k$  is the number of moles in a unit volume of the mixture ( $v_k = \rho_k / M_k^0$ ), where  $M_k^0$  is the molecular weight of the  $k$ th component. Since  $M$  is used to denote the mass of a body or system, the molecular weight is written as  $M^0$ .

As a result, for isothermal interdiffusion in a binary gaseous mixture or a molecular solution we have

$$\frac{\vec{j}_1}{M_1^0} = -\frac{\vec{j}_2}{M_2^0} = -vD_{12}\nabla v_{10}, \quad (2-1-9)$$

where  $v_{10}$  is the relative molar concentration ( $v_{10}=v_1/v$ ), and  $D_{12}$  is the coefficient of interdiffusion. In this case the material fluxes (in moles/m<sup>2</sup>·h) are equal and the interdiffusion is called molecular interdiffusion. If the function  $G_k$  is assumed to be equal to the number of molecules of the  $k$ th component per unit volume of mixture ( $G_k=n_k$ ), then for isothermal interdiffusion the molecular fluxes (in moles/m<sup>2</sup>·h) will be equal:

$$\frac{j_1}{m_1} = -\frac{j_2}{m_2} = -nD_{12}\nabla n_{10}, \quad (2-1-10)$$

where  $m_1$  and  $m_2$  are the molecular masses of the components of the binary mixture,  $n_{10}$  is the relative content of molecules of the first component ( $n_{10}=n_1/n$ ), and  $D_{12}$  is the coefficient of molecular interdiffusion ( $D_{12}=D_{21}$ ), since we know that

$$n_{10}=v_{10}=p_{10}, \quad (2-1-11)$$

where  $p_{10}$  is the relative partial pressure of the first component ( $p_{10}=p_1/p$ ).

Equations (2-1-9) and (2-1-10) combine to give for molecular isothermal diffusion the single relation\*:

$$\vec{j}_1 = -D_{12}p \frac{M_1^0}{M^0} \nabla p_{10}. \quad (2-1-12)$$

The reduced molecular weight  $M^0$  of the binary mixture is

$$M^0 = M_1^0 p_{10} + M_2^0 p_{20}, \quad (2-1-13)$$

where  $M_1^0$  and  $M_2^0$  are the molecular weights of the mixture components. Consequently,  $M^0$  is a function of  $p_{10}$  (where  $p_{10}+p_{20}=1$ ). If  $p_{10}$  is small in comparison with  $p_{20}$  (i.e.,  $p_{10} \ll p_{20}$ ), then  $M^0$  may be assumed to be approximately constant, and so from (2-1-12) we obtain the following expression for the concentration diffusion\*\*:

$$\vec{j}_1 = -D_{12}p \nabla p_{10}, \quad (2-1-14)$$

since according to the equation of state for an ideal gas  $p_k M_k^0 = \rho_k RT$ ,  $T$  being the absolute temperature.

Thus, on condition that  $M^0$  is constant the coefficient  $D$  of concentration diffusion is equal to the coefficient of molecular interdiffusion when the weight fluxes are equal ( $j_1=-j_2$ ). Relation (2-1-14) is often used in engineering calculations.

In order to clarify the physical meaning of the parameters entering into equation (2-1-12), let us point out the analogy between processes of molecular

\* Some investigators define the thermal velocity of the molecules with respect to the center of gravity of the gas mixture, with the result that the expression for molecular binary isothermal diffusion is  $\vec{j}_1 = -D'_{12}p \frac{M_1^0 M_2^0}{M^{02}} \nabla n_{10}$ . Consequently, the diffusion coefficients  $D_{12}$  and  $D'_{12}$  are related by the equation  $D_{12} = D'_{12} M_1^0 M_2^0 / M^{02}$ , a relation which may be obtained on the basis of Prigogine's theorem. The difference between the two above methods of calculation is that in our case the molar interdiffusion fluxes are equal, whereas in the first case the weight fluxes are equal.

\*\* Calculations for moist air (a binary mixture of dry air ( $M_2^0=29$ ) and water vapor ( $M_1^0=18$ )) show that with an accuracy up to 1% it is possible to take  $M^0$  constant, provided  $p_{10} \leq 0.026$ . For a total mixture pressure of  $p=760$  mm Hg the partial pressure of the vapor must be  $p_1 \leq 20$  mm Hg.

heat transfer (heat conduction) and mass transfer (diffusion). Fourier's heat-conduction law may be written as

$$\vec{j}_q = -\lambda \nabla t = -a_p \rho \nabla h, \quad (2-1-15)$$

since

$$\nabla t = \left( \frac{dt}{dh} \right)_p \nabla h, \quad (2-1-16)$$

where  $h$  is the specific enthalpy, and  $a_p$  is the thermal diffusivity, defined as  $a_p = \lambda / c_p \rho$ , where  $c_p$  is the isobaric specific heat ( $c_p = dh/dt$ ). It is evident that the thermal diffusivity  $a_p$  here represents the thermophysical characteristic describing the molecular transfer of enthalpy (heat content) in the system. This quantity is numerically equal to the amount of heat transferred per unit time through a unit area of an isothermal (isenthalpic) surface, in the presence of a gradient of the volume enthalpy concentration  $\rho h$  equal to unity, with dimensions of  $\frac{\text{kcal}}{\text{m}^2 \cdot \text{h}} / \frac{\text{kcal}}{\text{m}^3 \cdot \text{m}} = \text{m}^2/\text{h}$ .

For a solid the isochoric specific heat (specific heat at constant volume) is usually taken instead of the isobaric specific heat. Fourier's heat-conduction law then becomes

$$\vec{j}_q = -\lambda \nabla t = a_v \rho \nabla u; \quad (2-1-17)$$

where  $u$  is the specific internal energy of the solid ( $u = c_v t$ );  $c_v$  is the isochoric specific heat, and  $a_v$  is the thermal diffusivity calculated in terms of the isochoric specific heat ( $a_v = \lambda / c_v \rho$ ).

It is evident from (2-1-17) that the quantity  $a_v$  characterizes the molecular transfer of internal energy, that is, the physical meaning of the thermal diffusivity is that it constitutes a coefficient of energy conduction. Coefficients  $a_v$  and  $D$ , which have the same units ( $\text{m}^2/\text{h}$ ), thus describe the molecular transfer of energy and mass. Some investigators have therefore called the thermal diffusivity the thermal-diffusion coefficient.\* However, a different physical meaning is usually attributed to the thermal diffusivity, and it is thought of as a quantity characterizing the amount of variation of the temperature of a body during an unsteady-state process. This follows from the law describing the development of the temperature field in a solid when it is heated or cooled with its surface temperature maintained constant. During the regular-regime stage for bodies of very simple geometrical shape we have

$$\frac{\partial (\ln \Delta t)}{\partial \tau} = \text{const} \cdot a_v, \quad (2-1-18)$$

where  $\Delta t$  is the excess temperature at some arbitrary point in the body.

If the equation for the isothermal surface is  $t(x, y, z, \tau) = \text{constant}$ , then the total differential will be

$$\frac{\partial t}{\partial \tau} d\tau + \frac{\partial t}{\partial n} dn = 0. \quad (2-1-19)$$

The time derivative  $dn/d\tau$  of the normal to the isothermal surface is the rate of displacement or the propagation velocity  $w_t$  of the isothermal surface ( $w_t = dn/d\tau$ ). Consequently, equation (2-1-19) may be written as

$$\frac{\partial t}{\partial \tau} + w_t \frac{\partial t}{\partial n} = 0. \quad (2-1-20)$$

\* [Actually, this quantity is so-called in English, since the term "thermal diffusivity" implies a thermal-diffusion coefficient. The Russian term for the thermal diffusivity is "temperature conductivity." ]



From differential heat-conduction equation (1-7-13) it follows that

$$a_v = -w_t = \frac{\nabla t}{\nabla^2 t} = A w_t. \quad (2-1-21)$$

The quantity  $\nabla t / \nabla^2 t$  is the ratio of the first differential parameter to the second, and it is invariant with respect to a linear transformation group. This ratio thus has units of  $M^{-1}$  and is denoted as  $A$ . Consequently, the thermal diffusivity is proportional to the velocity of propagation of the isothermal surface.

This physical explanation of the quantity  $a_v$  does not contradict the one given previously, and such double explanations of physical quantities are often encountered in physics. For example, the surface tension may be defined either as the force exerted per unit length of an arbitrary contour drawn along the interface at the liquid surface (the force aspect) or else as the free potential energy per unit area of the liquid surface (the energy aspect).

Comparisons of (2-1-17), (2-1-15), (2-1-14), and (2-1-12) lead us to conclude that the relative mass concentration (specific mass content)  $\rho_{h0}$  is analogous to the specific enthalpy (heat content)  $h$  or the specific internal energy. It is known, however, that the enthalpy and internal energy do not constitute potentials describing the molecular heat transfer, since at the contact boundary between two bodies with different specific heat capacities [specific heats] the enthalpy and internal energy are discontinuous functions.

Consequently, the relative concentrations  $\rho_{i0}$  or  $v_{i0}$  do not represent potentials for the molecular transfer of matter. If as an approximation we take the relative partial pressure  $p_{i0}$  as the potential for the diffusion transfer of material, then a comparison of (2-1-12) with Fourier's law  $\vec{J}_q = -a_v c_v \rho \nabla t$  indicates that the ratio  $M_i^0 / M$  represents the analog of the specific heat capacity. This ratio may thus be called the specific mass capacity

$$c_{m1} = \frac{M_1^0}{M^0} = \frac{M_1^0}{M_1^0 p_{10} + M_2^0 p_{20}}. \quad (2-1-22)$$

For moist air ( $M_1^0 = 18$  and  $M_2^0 = 29$ ) the specific mass capacity, or vapor capacity,  $c_{m1}$  is  $18 / (29 - 11 p_0)$ . If  $p_{10} = 1$  (the mixture is transformed into vapor), then  $c_{m1} = 1$ ; if  $p_{10} = 0$  (dry air), then  $c_{m1} = 0.622$ . Consequently, the specific vapor capacity for air varies from 1 to 0.622 ( $1 \leq c_{m1} \leq 0.622$ ).

Equation (2-1-12) may now be rewritten as

$$\vec{J}_1 = -\rho c_{m1} D_{12} \nabla p_{10} = -\lambda_{m1} \nabla p_{10}, \quad (2-1-23)$$

where  $\lambda_{m1}$  is the mass conductivity, a quantity analogous to the thermal conductivity and defined as

$$\lambda_{m1} = \rho c_{m1} D_{12}. \quad (2-1-24)$$

Consequently, the coefficients describing the molecular transfer of mass and internal energy are

$$D_{12} = \frac{\lambda_{m1}}{c_{m1} \rho} \text{ and } a_v = \frac{\lambda}{c_v \rho}. \quad (2-1-25)$$

This analogy is also observed when the chemical potential is taken as the potential for the diffusion transfer. For isothermal molecular mass transfer,

$$\vec{J}_1 = -L_{11} \nabla (\mu_1 - \mu_2)_{T,p} = -\frac{L_{11}}{\rho_{10}} \left( \frac{\partial \mu_1}{\partial p_{10}} \right)_{T,p} \nabla p_{10} = -D_p \nabla p_{10}. \quad (2-1-26)$$

\* [Obviously an erratum; the equation should read:  $a_v = -w_t \frac{\nabla t}{\nabla^2 t} = A w_t$ .]

This relation indicates that the quantities  $\lambda_{m1}$ ,  $D$ , and the specific mass capacity  $c_{m1}$  have the values:

$$\lambda_{m1} = L_{11}; \quad c_{m1} = \rho_{20}/\rho \left( \frac{\partial \mu_1}{\partial \rho_{10}} \right)_{T, p}, \quad (2-1-27)$$

$$D = \frac{\lambda_{m1}}{c_{m1}\rho} = \frac{L_{11}}{\rho_{20}} \left( \frac{\partial \rho_1}{\partial \rho_{10}} \right)_{T, p}. \quad (2-1-28)$$

As defined above, the specific mass capacity is measured in kg/kcal (or kg/joule in the MKS system), and the quantity  $\lambda_{m1}$  has units of  $\text{kg}^2/\text{kcal} \cdot \text{m} \cdot \text{h}$  (or  $\text{kg}^2/\text{joule} \cdot \text{m} \cdot \text{sec}$ ).

Under nonisothermal conditions the mass transfer is coupled with the energy transfer or heat transfer. The system of mass-transfer and energy-transfer equations is then

$$\vec{J}_1 = -\vec{J}_2 = -L_{11}T\nabla \left( \frac{\mu_1 - \mu_2}{T} \right) - \frac{L_{1u}}{T} \nabla T, \quad (2-1-29)$$

$$\vec{J}_u = -\frac{L_{uu}}{T} \nabla T - L_{u1}T\nabla \left( \frac{\mu_1 - \mu_2}{T} \right). \quad (2-1-30)$$

Let us now define  $u^*$ , the transfer energy of the binary mixture, as the energy transfer per unit mass under isothermal conditions. Thus

$$\vec{J}_u = -L_{u1}T\nabla \left( \frac{\mu_1 - \mu_2}{T} \right) = \frac{L_{u1}}{L_{11}} \vec{J}_1 = u^* \vec{J}_1, \quad (2-1-31)$$

where  $u^* = \frac{L_{u1}}{L_{11}}$  is the isothermal-transfer energy.

System of equations (2-1-29) and (2-1-30) may now be written as

$$\vec{J}_1 = -L_{11}T\nabla \left( \frac{\mu_1 - \mu_2}{T} \right) - L_{1u}u^* \frac{\nabla T}{T}; \quad (2-1-32)$$

$$\vec{J}_u = -\frac{L_{uu}}{T} \nabla T - L_{1u}u^*T\nabla \left( \frac{\mu_1 - \mu_2}{T} \right), \quad (2-1-33)$$

since the reciprocity relation states that  $L_{u1} = L_{1u}$ . We may also carry out the expansion

$$T\nabla \frac{\mu_k}{T} = \nabla \mu_k - \frac{\mu_k}{T} \nabla T = -s_k \nabla T + v_k \nabla p + \sum_{h=1}^{n-1} \left( \frac{\partial \mu_k}{\partial \rho_{h0}} \right)_{T, p} \nabla \rho_{h0} - \frac{\mu_k}{T} \nabla T, \quad (2-1-34)$$

since the chemical potential  $\mu_k$  is a function of the temperature  $T$ , the pressure  $p$ , and the  $(n-1)$  concentrations  $\rho_{h0}$  (where  $k=1, 2, 3, \dots, n-1$ ).

To obtain (2-1-34) the following relations were used:

$$\left( \frac{\partial \mu_k}{\partial p} \right)_{\rho_{h0}, T} = v_k \quad \text{and} \quad \left( \frac{\partial \mu_k}{\partial T} \right)_{p, \rho_{h0}} = -s_k,$$

where  $v_k$  and  $s_k$  are the specific volume and specific entropy of the  $k$ th component of the mixture. Let us now introduce once again the Gibbs-Duhem equation

$$\rho_{10}d\mu_1 + \rho_{20}d\mu_2 = 0 \quad (2-1-35)$$

and the relations

$$h_k = \mu_k + Ts_k \quad \text{and} \quad \rho_{10} + \rho_{20} = 1. \quad (2-1-36)$$

After some algebraic transformations we then obtain the fluxes at constant pressure ( $\nabla p = 0$ ):

$$\vec{J}_1 = -L_{11} \frac{1}{\rho_{10}} \left( \frac{\partial \mu_1}{\partial \rho_{10}} \right)_{p,T} \nabla \rho_{10} - \frac{L_{11}}{T} [u^* - (h_1 - h_2)] \nabla T; \quad (2-1-37)$$

$$\vec{J}_u = -[L_{uu} + L_{11} u^* (h_2 - h_1)] \frac{\nabla T}{T} - \frac{L_{11} u^*}{\rho_{10}} \left( \frac{\partial \mu_1}{\partial \rho_{10}} \right)_{p,T} \nabla \rho_{10}. \quad (2-1-38)$$

The quantity in brackets in the first equation is the transfer heat  $Q^*$ , defined as the amount of heat transferred by unit mass under isothermal conditions. This quantity is determined by the relation

$$\vec{J}_q = Q^* \vec{J}_1 = \frac{L_{u1}}{L_{11}} \vec{J}_1 \quad \text{for } T \text{ constant}, \quad (2-1-39)$$

where the heat flux  $j_q$  is

$$\vec{J}_q = \vec{J}_u - \sum_{k=1}^n h_k \vec{J}_k = \vec{J}_u - (h_1 - h_2) \vec{J}_1, \quad (2-1-40)$$

and  $L_{q1}$  and  $L_{1q}$  are the kinetic coefficients of the system of equations

$$\vec{J}_1 = -L_{11} \nabla (\mu_1 - \mu_2)_T - L_{1q} \frac{\nabla T}{T}; \quad (2-1-41)$$

$$\vec{J}_q = -L_{qq} \frac{\nabla T}{T} - L_{q1} \nabla (\mu_1 - \mu_2)_T. \quad (2-1-42)$$

From equations (2-1-31), (2-1-39), and (2-1-40) we obtain

$$u^* = \frac{j_u}{j_1} = \frac{j_q}{j_1} + (h_1 - h_2) = Q^* + (h_1 - h_2) \quad \text{for } T \text{ constant}, \quad (2-1-43)$$

where  $u^* = L_{u1}/L_{11}$  and  $Q^* = L_{q1}/L_{11}$ .

Consequently, system of equations (2-1-37) and (2-1-38) may be written as

$$\vec{J}_1 = -\frac{L_{11}}{\rho_{10}} (\mu'_1)_{p,T} \nabla \rho_{10} - \frac{L_{11}}{T} Q^* \nabla T; \quad (2-1-44)$$

$$\vec{J}_u = -[L_{uu} + L_{11} u^* (h_2 - h_1)] \frac{\nabla T}{T} - \frac{L_{11} u^*}{\rho_{10}} (\mu'_1)_{p,T} \nabla \rho_{10}, \quad (2-1-45)$$

where  $(\mu'_1)_{p,T} = (\partial \mu_1 / \partial \rho_{10})_{p,T}$ . From (2-1-41) we obtain an equation identical to (2-1-44), since  $\nabla (\mu_1 - \mu_2)_T$  equals  $\frac{1}{\rho_{20}} (\mu'_1)_{p,T} \nabla \rho_{10}$  and since the coefficient  $L_{1q}$  is shown by (2-1-39) to be equal to  $L_{11} Q^*$ . If the expressions for  $\vec{J}_u$  and  $\vec{J}_1$  obtained in (2-1-44) and (2-1-45) are now substituted into (2-1-40), we obtain

$$\vec{J}_q = -[L_{uu} + (u^{**} - Q^*) L_{11}] \frac{\nabla T}{T} - [L_{u1} - (u^* - Q^*) L_{11}] \nabla (\mu_1 - \mu_2)_T. \quad (2-1-46)$$

A comparison of this equation with (2-1-42) shows that

$$L_{uu} + (u^{**} - Q^*) L_{11} = L_{qq} \quad \text{and} \quad L_{u1} - (u^* - Q^*) L_{11} = L_{q1}. \quad (2-1-47)$$

Let us return to equation (2-1-41). The first term of this equation represents the molecular transfer of material under isothermal conditions (isothermal diffusion), while the second term indicates the effect of the temperature gradient on the mass transfer (thermal diffusion). Equation (2-1-44) for nonisothermal diffusion may also be written as

$$\vec{J}_{m1} = -D_p \left( \nabla \rho_{10} + \frac{k_T}{T} \nabla T \right), \quad (2-1-48)$$

where  $k_T$  is the thermal-diffusion ratio, defined as the product of the isobaric-isochoric mass capacity times the transfer heat ( $k_T = c_{mi} Q^*$ ). The coefficient  $k_T$  may be determined experimentally from the relation

$$k_T = \frac{p''_{10} - p'_{10}}{\ln(T''/T')} \text{ for } j_{m1} = 0. \quad (2-1-49)$$

If a pressure difference exists during the flow of a binary mixture, then barodiffusion occurs, which is a molecular transfer of the component of greater molecular weight into the region of increased pressure. The mass flux associated with the barodiffusion is

$$\vec{j}_{mb} = -\rho D p_{10} \rho_{20} \left( \frac{M_2^0 - M_1^0}{M^0} \right) \nabla (\ln p). \quad (2-1-50)$$

If the molecular weights of the mixture components are equal ( $M_2^0 = M_1^0$ ), then the barodiffusion flow is zero.

In the presence of a plane-parallel flow of a gaseous mixture about the surface of a body there occurs, in addition to diffusion and thermal diffusion,

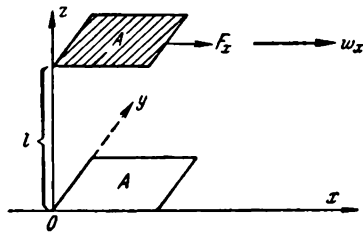


FIGURE 2-1. Transfer of kinetic energy of translational motion in a viscous medium, as a result of internal friction

separation of the mixture as a result of internal friction. This new phenomenon of molecular transfer of matter is similar to thermal diffusion, in which the mixture component with a greater molecular weight is displaced toward the region of lower temperature (in the direction of heat flow). In the new effect of separation, molecules of greater mass are shifted toward the region of lower flux rate, that is, they move in the direction of molecular transfer of the kinetic energy corresponding to the apparent motion.

Let us now consider this in more detail.

Suppose that a [plate of] area  $A$  situated at a distance  $l$  from the surface of a solid body moves through a viscous medium with a velocity  $w_x$  as the result of a driving force  $F_x$  (see Figure 2-1). The fluid (a gaseous mixture) contained between the plate and the surface of the body is set into motion accordingly due to internal friction. At the surface of the body the velocity is zero, whereas at the surface of the moving plate the fluid velocity equals the velocity of the plate. The velocity distribution  $w_x(z)$  in the  $z$  direction is linear;  $\partial w_x(z)/\partial z = w/l$  is constant. Thus the pressure tensor  $\mathcal{P}$  for this case is

$$\mathcal{P} = \begin{pmatrix} p & 0 & -\eta \frac{\partial w_x}{\partial z} \\ 0 & p & 0 \\ -\eta \frac{\partial w_x}{\partial z} & 0 & p \end{pmatrix}. \quad (2-1-51)$$

Element  $p_{xx}$  of the pressure tensor equals the  $x$  component of the force exerted on a unit area perpendicular to the  $z$  axis. Consequently, from matrix (2-1-51) we obtain

$$\frac{F_x}{A} = -\eta \frac{\partial w_x}{\partial z} = \eta \frac{w}{l}, \quad (2-1-52)$$

indicating that the frictional stress is directly proportional to the velocity gradient.

A similar relation can be written for the plane-parallel flow about the surface of the body:

$$p_{zx} = -\eta \frac{\partial w_x}{\partial z}. \quad (2-1-53)$$

Here it is assumed that  $\partial w_x / \partial x \ll \partial w_x / \partial z$ . If we multiply both sides by  $w_x$ , we have

$$p_{zx} w_x = -\eta \frac{\partial}{\partial z} (w_x^2 / 2). \quad (2-1-54)$$

The right side of (2-1-54) is the work done by the internal-friction forces. This work is performed during the transfer of the kinetic energy of apparent motion of a unit fluid mass ( $w_x^2/2$ ) from one layer to another, in the negative  $z$  direction, by means of molecular (random) motion. Numerically this quantity equals the loss of kinetic energy of apparent motion required to overcome the molecular-friction forces. Thus, it may be called the molecular

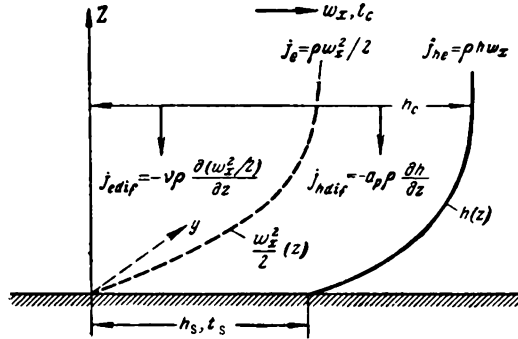


FIGURE 2-2. Diffusion transfer of enthalpy  $h(z)$  and kinetic energy of translational motion  $w_x^2(z)/2$  for fluid flow about a plane surface

flux (with units of energy/ $m^2 \cdot h$ ) or the diffusion flux  $j_{e, \text{diff}}$  of kinetic energy of apparent motion. The right side of (2-1-54) is the product of the viscosity times the kinetic-energy gradient per unit mass of moving fluid. Consequently, (2-1-39) may be written as

$$j_{e, \text{diff}} = -\nu \rho \frac{\partial}{\partial z} \left( \frac{w_x^2}{2} \right), \quad (2-1-55)$$

where  $\nu$  is the kinematic viscosity ( $\nu = \eta/\rho$ ).

The diffusion flow of kinetic energy of apparent fluid motion is analogous to the conductive heat flow (heat conduction) in a moving fluid. When a fluid flows over the plane surface of a solid body (Figure 2-2), the [macroscopic] enthalpy flux is  $\rho h w_x$ . Due to the temperature difference  $\Delta t$  between the fluid and the wall surface (where  $\Delta t = t_c - t_s$ ), enthalpy is transferred by means of molecular motion (heat being conducted from the flow to the wall of the body at the boundary layer).

As was shown above, this conductive heat flux constitutes a differential enthalpy flux:

$$j_q = j_{h, \text{diff}} = -\lambda \frac{\partial t}{\partial z} = -a_p \rho \frac{\partial h}{\partial z}. \quad (2-1-56)$$

Similarly, the mass flux of the  $k$ th component of the mixture, for molecular isothermal diffusion, represents a diffusion flux:

$$j_{mk} = j_{mk, \text{diff}} = -D\rho \frac{\partial p_{k0}}{\partial z}. \quad (2-1-57)$$

According to the elementary kinetic theory of gases all the molecular-transfer coefficients are equal to one another ( $\alpha = D = \nu$ ). This indicates that to a first approximation the diffusion mechanisms for the transfer of energy of apparent motion of the flow, internal energy, and mass are the same. In this case transfer coefficients  $\alpha$ ,  $D$ , and  $\nu$  reduce to a single diffusion-transfer coefficient. For a real gas these coefficients are not equal, due to the interaction of molecules and due to collision events.

In all three types of molecular transfer the motive forces  $\left[ \frac{\partial}{\partial z} \left( \frac{w_x^2}{2} \right), \partial h / \partial z, \partial p_{10} / \partial z \right]$  are vectors (first-rank tensors). Consequently, according to the Curie principle they may be mutually coupled.

When a gaseous mixture seeps through a porous body at constant temperature the gas undergoes separation due to friction between the mixture and the walls of the capillary passage. In the general case the mass flux of a binary mixture is

$$\vec{j}_1 = L_{1T} X_T + L_{1m} X_m + L_{1e} X_e, \quad (2-1-58)$$

where  $X_e$  is the motive force, equal to the friction force between the gas and the wall of the capillary passage or else to the gradient of the kinetic energy of the seepage flow of the gas.

The differential equation describing mass transfer in a binary mixture, in the presence of phase or chemical transformations, is equation (1-5-23), which may be written as

$$\rho \frac{dp_{10}}{dz} = -\text{div } \vec{j}_1 + I_1, \quad (2-1-59)$$

where  $I_1$  is a source term for the  $k$ th component of the substance ( $\text{kg}/\text{m}^2 \cdot \text{h}$ ). Now, if we neglect barodiffusion and the effect of separation during diffusion of the apparent-motion energy, that is, if we use the expression for  $\vec{j}_1$  given in (2-1-48), then we obtain

$$\rho \frac{dp_{10}}{dz} = \text{div} \left[ D\rho \left( \nabla p_{10} + \frac{k_T}{T} \nabla T \right) \right] + I_1. \quad (2-1-60)$$

According to the zonal-calculation method, in which the transfer process is divided up into consecutive zones (sections), in each of which the thermophysical parameters are assumed constant, equation (2-1-60) may be written as

$$\frac{dp_{10}}{dz} = D \nabla^2 p_{10} + D \frac{k_T}{T} \nabla^2 T + \frac{I_1}{\rho}, \quad (2-1-61)$$

where  $\nabla^2 = \text{div grad}$  stands for the Laplace operator. The mass source strength  $I_k$  for the  $k$ th component can be determined from the concentration variation.

The total variation  $dM_k$  in the mass of the  $k$ th component equals the variation  $d_e M_k$  due to mass received from outside by transfer plus the variation  $d_i M_k$  due to chemical reaction:

$$dM_k = d_e M_k + d_i M_k. \quad (2-1-62)$$

The quantity  $d_i M_k$  is determined by the rate of the chemical reactions (compare equation (1-5-14)).

If we define

$$\beta_k = \frac{d_i M_k}{d_e M_k}, \quad (2-1-63)$$

then we obtain a dimensionless ratio which indicates the mass variation due to chemical reaction relative to the mass variation due to transfer. If no chemical reaction is present, then coefficient  $\beta_k$  is of course zero, since  $d_i M_k = 0$ . In this case the total variation in the mass of the  $k$ th component is due to transfer only and  $dM_k = d_e M_k$ . If no mass transfer occurs, then  $d_e M_k = 0$  and  $\beta_k$  is infinity, the mass variation being due to chemical reactions only.

Consequently, coefficient  $\beta_k$  varies over a wide range:  $0 < \beta_k < \infty$ . Therefore it is convenient to introduce another coefficient  $\epsilon_k$  for chemical transformations of the  $k$ th component, defined as:

$$\epsilon_k = \frac{\beta_k}{1 + \beta_k} = \frac{1}{1 + \frac{1}{\beta_k}}. \quad (2-1-64)$$

The coefficient  $\epsilon_k$  varies only from zero to unity:  $0 < \epsilon_k < 1$ . The absence of chemical reaction corresponds to  $\epsilon_k = 0$  (and  $\beta_k = 0$ ), and  $\epsilon_k = 1$  means that the variation in the mass of the  $k$ th component is due to chemical transformation alone ( $\beta_k = \infty$ ).

The total variation in the mass of the  $k$ th component may now be written as

$$dM_k = d_e M_k + \beta_k d_e M_k = \frac{1}{1 - \epsilon_k} d_e M_k. \quad (2-1-65)$$

The differential mass-transfer equation

$$\rho \frac{d\rho_{k0}}{d\tau} = -\operatorname{div} j_k + \sum_j v_{kj} J_j \quad (2-1-66)$$

becomes, accordingly,

$$\rho \frac{d\rho_{k0}}{d\tau} = -\operatorname{div} j_k + \epsilon_k \rho \frac{d\rho_{k0}}{d\tau}. \quad (2-1-67)$$

Consequently, the mass production due to chemical transformations is

$$\rho \frac{d_i \rho_{k0}}{d\tau} = \sum_j v_{kj} J_j = \epsilon_k \rho \frac{d\rho_{k0}}{d\tau}. \quad (2-1-68)$$

For a binary gaseous mixture the differential mass-transfer equation becomes

$$\frac{d\rho_{10}}{d\tau} = D\nabla^2 \rho_{10} + D \frac{k_T}{T} \nabla^2 T + \epsilon_1 \frac{d\rho_{10}}{d\tau}. \quad (2-1-69)$$

The differential heat-transfer equation. This equation is obtained from the equation describing internal-energy transfer (equation (1-5-47)):

$$\begin{aligned} \rho \frac{du}{d\tau} + \rho p \frac{dv}{d\tau} = & -\operatorname{div} \vec{J}_u + \sum_{k=1}^n \vec{J}_k \cdot \vec{F}_k - \Gamma \vec{\nabla} \cdot \vec{\omega} - \dot{\Gamma}^s : (\vec{\nabla} \vec{\omega}) - \\ & - \Gamma^a (\vec{\nabla} \times \vec{\omega} - \vec{2}\omega). \end{aligned} \quad (2-1-70)$$

For constant pressure the left side of equation (2-1-70) becomes  $\rho \frac{dh}{d\tau}$ . If we assume that  $F_k = 0$  (no external-force field) and also that the energy dissipation (represented by the last three terms on the right side of the

equation) is small in comparison with  $\text{div } \vec{J}_u$ , then we have

$$\rho \frac{dh}{d\tau} = -\text{div } \vec{J}_u. \quad (2-1-71)$$

The molecular energy flux  $\vec{J}_u$  for a binary gaseous mixture is

$$\vec{J}_u = \vec{J}_q + (h_1 - h_2) \vec{J}_1, \quad (2-1-72)$$

and the specific enthalpy  $h$  of the mixture is

$$h = \sum_k h_k \rho_{k0} = h_1 \rho_{10} + h_2 \rho_{20}. \quad (2-1-73)$$

Now, when  $h$  and  $\vec{J}_u$  from (2-1-72) and (2-1-73) are substituted into differential equation (2-1-71), we obtain

$$\rho \frac{d}{d\tau} (h_1 \rho_{10} + h_2 \rho_{20}) = -\text{div } \vec{J}_q - \text{div} (h_1 - h_2) \vec{J}_1. \quad (2-1-74)$$

If the isobaric specific heats  $c_k$  of the mixture components are defined as

$$c_k = \frac{dh_k}{dt} \quad (k=1, 2), \quad (2-1-75)$$

then we have

$$\begin{aligned} \rho (h_1 - h_2) \frac{d\rho_{10}}{d\tau} + (c_1 \rho_1 + c_2 \rho_2) \frac{dt}{d\tau} = -\text{div } \vec{J}_q - \\ - (h_1 - h_2) \text{div } \vec{J}_1 - (c_1 - c_2) \vec{J}_1 \text{grad } t. \end{aligned} \quad (2-1-76)$$

Next, differential mass-transfer equation (2-1-59) may be introduced, denoting the volume isobaric specific heat of the mixture as  $c_p \rho$ , where

$$c_p \rho = c_1 \rho_1 + c_2 \rho_2, \quad (2-1-77)$$

to obtain

$$c_p \rho \frac{dt}{d\tau} = -\text{div } \vec{J}_q + (h_1 - h_2) I_1 - (c_1 - c_2) \vec{J}_1 \text{grad } t. \quad (2-1-78)$$

For moist air (a binary mixture of vapor and dry air) the difference  $(h_1 - h_2)$  in specific enthalpies is  $595 + 0.23t$ , since  $c_1 = 0.47 \text{ kcal/kg} \cdot \text{degree}$ ,<sup>\*</sup>  $c_2 = 0.24 \text{ kcal/kg} \cdot \text{degree}$ , and  $h_1 = 595 + 0.47t$ . According to relations (2-1-42) and (2-1-24) the heat-flux density  $j_q$  is

$$\vec{J}_q = -\frac{L_{qq}}{T} \text{grad } T - L_{q1} \text{grad} (\nu_1 - \nu_2)_T = -\lambda \text{grad } T - \frac{\lambda_{m1}}{c_{m1}} Q^* \text{grad } \rho_{10}, \quad (2-1-79)$$

where  $\lambda$  is the thermal conductivity of the mixture:

$$\lambda = \frac{L_{qq}}{T}, \quad (2-1-80)$$

and  $Q^*$  is the isothermal mass-transfer heat  $(Q^* = \frac{L_{q1}}{L_{11}} = \frac{k_T}{c_m})$ .

Equation (2-1-79) may now be substituted into (2-1-78) to obtain

$$c_p \rho \frac{dt}{d\tau} = \text{div} (\lambda \nabla t) + \text{div} (D\rho Q^* \text{grad } \rho_{10}) + (h_1 - h_2) I_1 - (c_1 - c_2) \vec{J}_1 \text{grad } t. \quad (2-1-81)$$

The left side of this differential heat-transfer equation is the sum of the time variation of the enthalpy ( $c_p \rho \partial t / \partial \tau$ ) plus the entropy transfer by the moving mixture ( $c_p \rho \vec{w} \text{grad } t$ ). The first term on the right side represents heat transfer by conduction, the second term represents heat transfer by diffusion heat conduction (the Dufour effect), and the third term represents

\* In the MKS system  $1 \text{ kcal/kg} \cdot \text{degree} = 4.1868 \cdot 10^3 \text{ joule/kg} \cdot \text{degree}$ .



a source or sink of heat due to phase or chemical transformation. The last term on the right side of (2-1-81) represents the enthalpy transfer due to diffusion. Usually the difference in specific heats ( $c_1 - c_2$ ) is small, so that the last term may be neglected. For a mixture of molecules with equal masses, governed by the same laws for interaction between any two molecules, the enthalpy transfer due to interdiffusion is taken into account by means of the overall thermal conductivity. In this case

$$\vec{J}_q = -\lambda \nabla t + \sum_k h_k \vec{J}_k = -\lambda' \nabla t, \quad (2-1-82)$$

where  $\lambda'$  is the thermal conductivity, equal to

$$\lambda' = \frac{1}{4} \left[ \left( 15 - \frac{6\rho D}{\eta} \right) \gamma - \left( 15 - 10 \frac{\rho D}{\eta} \right) \right] C_v \eta. \quad (2-1-83)$$

Here  $\gamma = C_p/C_v$ , and  $C_p$  and  $C_v$  are the molar heat capacities at constant pressure and at constant volume.

The dimensionless ratio  $\rho D/\eta$  is approximately unity ( $Pr_m \approx 1$ ), so that formula (2-1-83) simplifies to

$$\lambda' = \frac{1}{4} (9\gamma - 5) C_v \eta. \quad (2-1-84)$$

The quantity  $(9\gamma - 5)/4$  is known as Eucken's correction factor for the thermal conductivity of polyatomic molecules. The last term on the right side of equation (2-1-81), which may be written as  $D \rho \nabla \rho_{10} \nabla (h_1 - h_2)$ , becomes very important during gas dissociation at high temperatures, when the components of a gaseous mixture react with one another. The concentration gradient enters in as a result of the fact that the equilibrium composition of the gas varies with temperature. For example, in a gas which absorbs heat due to dissociation at increased temperatures, heat transfer occurs at the moment when the molecules dissociate in the high-temperature region and then diffuse toward the low-temperature region, where the concentration of dissociated molecules is lower. In the low-temperature region of the gas recombination takes place, as a result of which heat is released, only to be absorbed in the dissociation in the high-temperature region. Approximate calculations show that during the dissociation of diatomic molecules the thermal conductivity may be an order of magnitude higher than that of a non-reacting gas mixture.

In a dissociated gas at a high temperature the isobaric specific heat  $c_p = dh/dT$  becomes very high. In this case the approximate relation

$$\vec{J}_q = -\lambda \nabla t + \sum_k h_k \vec{J}_k \approx \rho \bar{D} \nabla h \quad (2-1-85)$$

is true, where  $\bar{D}$  is the average value of the diffusion coefficient.

Consequently, the last term on the right side of equation (2-1-81), which is  $D \rho \nabla \rho_{10} \nabla (h_1 - h_2)$  for a dissociated mixture, will be considerably larger than the first term,  $\text{div}(\lambda \nabla t)$ . For the zonal-calculation system differential equation (2-1-81) now becomes

$$\frac{dt}{d\tau} = a_p \nabla^2 t + \frac{DQ^*}{c_p} \nabla^2 \rho_{10} + \frac{(h_1 - h_2)}{c_p \rho} I_1 - \left( \frac{c_1 - c_2}{c_p} \right) D \nabla \rho_{10} \nabla t. \quad (2-1-86)$$

This equation, together with the differential mass-transfer equation

$$\frac{d\rho_{10}}{d\tau} = D \nabla^2 \rho_{10} + \frac{Dc_{m1}Q^*}{T} \nabla^2 t + \frac{1}{\rho} I_1, \quad (2-1-87)$$

constitutes the system of differential equations describing heat transfer and mass transfer in a binary gaseous mixture.

## 2-2. The Mass-Transfer and Heat-Transfer Equations for a Solution

The study of the diffusion of a solute or liquid under nonisothermal conditions is of more than theoretical interest, since it is practically important in the construction of [isotope] separation columns such as the one designed by Clusius and Dickel. In recent times the thermal-diffusion method of separation has also been used for colloidal solutions (for the separation of dyes, petroleum products, albumins, etc.).

The differential equations describing heat and mass transfer for a solute are similar to the differential equations of heat transfer and mass transfer for a binary gaseous mixture. The quantity  $\rho_{10}$  is here the relative concentration of the solute, and it equals the ratio of the volume concentration  $\rho_1$  to the density  $\rho$  of the solution ( $\rho_{10} = \rho_1/\rho$ ). The interdiffusion coefficient  $D$  equals the diffusion coefficient for the solute, and the quantity  $Dc_{m1}Q^*/T$  is the coefficient of thermal diffusion  $D_T$  (thus  $D_T = Dc_{m1}Q^*/T$ ). The ratio of the thermal-diffusion coefficient to the diffusion coefficient for the solute, known as the Soret coefficient, is denoted by  $\sigma$ :

$$\sigma = D_T/D. \quad (2-2-1)$$

Thus, the differential equations of heat and mass transfer for a moving solution in the absence of sources of solute are

$$\frac{dt}{d\tau} = a_p \nabla^2 t + \sigma D k_1 \nabla^2 \rho_{10}; \quad (2-2-2)$$

$$\frac{d\rho_{10}}{d\tau} = D \nabla^2 \rho_{10} + \sigma D \nabla^2 t, \quad (2-2-3)$$

where the coefficient  $k_1$  is defined as

$$k_1 = \frac{T}{(1 - \rho_{10})} \left( \frac{\partial \mu_1}{\partial \rho_{10}} \right)_{p, T}, \quad (2-2-4)$$

$\mu_1$  being the chemical potential for the solute. Here the heat transfer related to diffusion of the enthalpy of the solute has been neglected.

For a solution at rest ( $\vec{w} = 0$ ) the system of differential equations becomes

$$\frac{\partial t}{\partial \tau} = a_p \nabla^2 t + \sigma D k_1 \nabla^2 \rho_{10}; \quad (2-2-5)$$

$$\frac{\partial \rho_{10}}{\partial \tau} = D \nabla^2 \rho_{10} + \sigma D \nabla^2 t. \quad (2-2-6)$$

The Soret coefficient  $\sigma$  for liquid solutions is approximately the same order of magnitude ( $10^{-3}$  to  $10^{-5}$ ) as for gaseous mixtures. The diffusion coefficient  $D$  for the solute is approximately  $10^4$  times less than the interdiffusion coefficient for a binary gaseous mixture, and the same is true of the thermal-diffusion coefficient  $D_T$  as well. Thus it is relatively easy to determine the Soret effect for a solute by experiment. It is very difficult, however, to determine the Dufour coefficient ( $\sigma D k_1 = D_T k_1$ ) for a liquid solution

experimentally, since the thermal diffusivity of a liquid is only 100 times less than the thermal diffusivity of gas. Therefore, the Dufour coefficient  $D_T k_1$  is very small in comparison with  $a_p$ , so that the second term of the right side of equation (2-2-5) may be neglected.

Thus we are left with the following system of differential equations:

$$\frac{\partial t}{\partial \tau} = a_p \nabla^2 t; \quad (2-2-7)$$

$$\frac{\partial p_{i0}}{\partial \tau} = D(\nabla^2 p_{i0} + \sigma \nabla^2 t). \quad (2-2-8)$$

Solutions of this system are useful in the processing of experimental data on thermal diffusion in solutions.

### 2-3. The Equations For Mass and Heat Transfer in a Porous Body

Let us now derive the differential equations describing mass and heat transfer in a system consisting of a capillary-porous body\* and the matter bound to it. The matter bound to a porous body at positive temperatures ( $t > 0^\circ\text{C}$ ) may be a liquid, a vapor, or an inert gas, while at negative temperatures ( $t < 0^\circ\text{C}$ ) it may be ice, a supercooled liquid, a vapor, or a gas. Depending on the way in which the moisture is bound to the body, the freezing temperature of the liquid varies over a wide range. Therefore, in porous bodies with various kinds of moisture binding there is always, at negative temperatures, a certain amount of supercooled liquid.

The second feature typical of mass and heat transfer in a porous body is a partial filling with moisture of the pores and capillaries of the body, so that part of a capillary is filled with liquid or ice while the rest contains a mixture of vapor and gas. The amount of moisture in a given state varies during the mass and heat transfer, and thus in the derivation of the transfer equations it is necessary to take into account the moisture-concentration variation in the capillaries of the body.

Let us denote vaporous moisture by the subscript 1, liquid moisture by 2, moisture in the solid state by 3, inert gas by 0, and the capillary-body skeleton by the subscript c. The volume concentration of the bound matter (the inert gas, vapor, liquid, or ice) is the ratio of the mass  $M$  of the substance to the volume  $V$  of the body:

$$\omega = \frac{M}{V} = \frac{1}{V} \sum_{i=0}^3 M_i = \sum_{i=0}^3 \omega_i, \quad (2-3-1)$$

where  $M_i$  and  $\omega_i$  are the mass and volume concentrations of the bound matter in the state  $i$  (where  $i=0, 1, 2, 3$ ). The mass  $M$  of the bound matter in a porous body is the sum of the masses of the inert gas, vapor, liquid, and ice:

$$M = \sum_{i=0}^3 M_i. \quad (2-3-2)$$

The concentration  $\omega_i$  of any state  $i$  of the bound matter in a body can be

\* [Hereafter referred to simply as a porous body.]

expressed in terms of the density or concentration  $\rho_i$  of the bound matter itself and the volume porosity  $\Pi$  of the body (this being the pore volume per unit volume of the body):

$$\omega_i = \frac{M_i}{V} = \rho_i \Pi b_i, \quad (2-3-3)$$

where  $\rho_i$  is the density of the bound matter ( $\rho_0$  and  $\rho_1$  are the gas and vapor concentrations in the gas-vapor mixture,  $\rho_2$  is the liquid density, and  $\rho_3$  is the ice density)\* and  $b_i$  is the degree of filling of the capillaries. To a first approximation the quantity  $b_i$  equals the ratio of the volume of bound matter in the state  $i$  to the volume of the capillaries. In the theory of flow through porous media this parameter is called the saturation. However, such a definition of the coefficient  $b_i$  in our case would be somewhat arbitrary, since the moisture may be bound by adsorptive and osmotic forces not only to the walls of macroscopic capillaries but also to the internal and external surfaces of microscopic capillaries.

Equation (2-3-3) shows that the volume concentration of the bound matter is not determined uniquely by its density and porosity but also is a function of  $b_i$ , which varies during the mass and heat transfer. Thus we may conclude that differential transfer equations derived for the seepage of a homogeneous liquid or gas through a porous medium are not applicable to mass and heat transfer in a porous body in the presence of phase transitions, since for seepage processes  $b_i$  is considered to be unity or constant. In addition, the volume of the body varies with the amount of moisture absorbed (moist-body contraction), that is, the porosity of the body depends on the amount of bound matter.

Therefore, in most cases, instead of the volume concentration of bound matter, the relative concentration or the specific mass content  $u$  of the bound matter is introduced, the latter being defined as the ratio of the mass of bound matter to the mass  $M_0$  of the dry body (the mass of the body skeleton):

$$u = \frac{M}{M_0} = \frac{\omega}{\gamma}, \quad (2-3-4)$$

where  $\gamma$  is the density of the body.

The specific mass content is the sum of the specific mass contents of the bound matter in its various states\*\*:

$$u = \sum_{i=0}^3 u_i. \quad (2-3-5)$$

In relation to this it is important to note that under ordinary conditions, when the pressure of the moist air is nearly barometric, the mass of the air and vapor in the capillaries is negligibly small in comparison with the mass of liquid or ice. According to the calculations of Posnov, under normal conditions and for bodies with maximum porosity the mass of moist air in the capillaries of the body constitutes only  $10^{-5}\%$  of the mass of the liquid corresponding to the equilibrium moisture content of the body. Therefore, for a nonzero mass content  $u$  the quantity  $(u_0 + u_1)$  may be neglected and

$$u = u_2 + u_3. \quad (2-3-6)$$

\* For adsorption binding  $\rho_2$  is the density of the bound liquid.

\*\* If we neglect the mass of the adsorbed dry air at the walls of the capillaries, then the specific mass content of the bound matter equals the specific moisture content ( $u = u_1 + u_2 + u_3$ ), and there is no difference between the specific mass content and the specific moisture content.

Finally, in the positive-temperature range ( $t > 0^\circ\text{C}$ ) it is quite accurate to assume that the specific mass content of the body equals the specific mass content of the liquid ( $u = u_2$ ).

In general liquid in a porous body moves through the skeleton of the body by means of selective diffusion (molecular transfer). The motion is a combination of seepage under the action of gravitation or hydrostatic pressure (convective transfer) and capillary absorption.

The differential equation describing the mass transfer of the bound matter is similar to the differential transfer equation for one of the components of a moving gaseous mixture, except that instead of the concentration  $\rho_k$  in the expression for the convective component of the mass transfer it is necessary to substitute the volume concentration  $\omega_{ik}$  of the bound matter in the pores of the body.\* Thus we obtain

$$\frac{\partial \omega_i}{\partial \tau} = -\operatorname{div}(\omega_{ik}\vec{w} + \vec{j}_{im}) + I_i, \quad (2-3-7)$$

where  $\vec{w}$  is the linear velocity of macroscopic motion, calculated per unit surface of the body,  $\vec{j}_{im}$  is the density of the molecular flux of material, and  $I_i$  is the volume source (sink) strength for the  $i$ th substance, due to phase transitions.

The product  $\omega_{ik}\vec{w}$  represents the density of the convective flux  $\vec{j}_{ik}$  of the bound matter inside the porous body ( $\vec{j}_{ik} = \omega_{ik}\vec{w}$ ). Since this flux  $\vec{j}_{ik}$  is expressed in terms of the gradient of the corresponding macroscopic-transfer potential, equation (2-3-7) is a partial differential equation. This equation may now be written as

$$\frac{\partial \omega_i}{\partial \tau} = -\operatorname{div}(\vec{j}_{ik} + \vec{j}_{im}) + I_i. \quad (2-3-8)$$

In a porous body macroscopic and molecular transfer of the bound matter in various states take place simultaneously. If  $j_i$  is the flux density of the macroscopic-molecular transfer:

$$j_i = j_{ik} + j_{im}, \quad (2-3-9)$$

then we have

$$\frac{\partial(\gamma u_i)}{\partial \tau} = -\operatorname{div} \vec{j}_i + I_i. \quad (2-3-10)$$

If no chemical transformations associated with the formation of a noncondensing gas (dry air) are present, then the source term  $I_0$  drops out ( $I_0 = 0$ ). Also, if we neglect ice evaporation (sublimation) and the condensation of the vapor directly into ice (obliteration or desublimation), so that  $I_{31} = I_{13} = 0$ , then the ice source  $I_3$  related to the transition of liquid into ice may be expressed in terms of an ice ratio  $\epsilon_{32}$ . This ratio is defined as the ice mass  $M_3$  divided by the mass  $M$  of the entire bound matter:

$$\epsilon_{32} = \frac{M_3}{M} = \frac{u_3}{u_0 + u_1 + u_2 + u_3} = \frac{u_3}{u_2 + u_3}, \quad (2-3-11)$$

since the mass content ( $u_0 + u_1$ ) of moist air is negligible in comparison to the total mass content, that is,  $(u_0 + u_1) \ll (u_2 + u_3)$ . If the body does not contain ice ( $u_3 = 0$ ), then the ratio  $\epsilon_{32}$  is zero, whereas if all the liquid is transformed into ice ( $u_2 = 0$ ), then  $\epsilon_{32}$  is unity. In most cases  $\epsilon_{32}$  is less than unity ( $0 < \epsilon_{32} < 1$ ).

\* The volume concentration  $\omega_{ik}$  of the moving bound matter does not equal the total volume concentration  $\omega_k$ , defined as  $\omega_k = \gamma u_k$ . In the case of a pure seepage flow  $\omega_{ik} = \rho_i \Pi$ .

There is no macroscopic or molecular mass transfer of bound matter in the solid state ( $\vec{j}_3=0$ ); consequently, the differential equation of mass transfer for the solid phase is

$$\frac{\partial(\gamma u_s)}{\partial \tau} = I_3. \quad (2-3-12)$$

We may now use (2-3-11) to find an expression for the ice source  $I_3$ , existing due to transformation of liquid into ice ( $I_3 = -I_2$ ):

$$I_3 = \frac{\partial(\epsilon_{31}\gamma u)}{\partial \tau} = \frac{\partial}{\partial \tau} \left( \frac{\epsilon_{31}}{1-\epsilon_{31}} u_2 \gamma \right). \quad (2-3-13)$$

According to the studies of Neressova the ice ratio does not depend on the moisture content, but is a function only of the temperature. If we assume approximately that  $\epsilon_{32}$  is constant, then

$$I_{32} = -I_{23} = \epsilon_{32} \frac{\partial(\gamma u)}{\partial \tau} = \frac{\epsilon_{32}}{1-\epsilon_{32}} \cdot \frac{\partial(\gamma u_2)}{\partial \tau}. \quad (2-3-14)$$

The liquid source  $I_2$ , existing due to vapor condensation, can also be calculated approximately in terms of the divergence of the vapor-flux vector. Since the content of vapor and air ( $u_0+u_1$ ) is negligible in comparison with the liquid content ( $u_2$ ), it is possible to set  $\partial\gamma u_1/\partial\tau$  equal to zero in equation (2-3-10). We then obtain

$$I_2 = -I_1 = -\operatorname{div} \vec{J}_1. \quad (2-3-15)$$

On the other hand, this liquid source  $I_2$  can be expressed in terms of the local derivative  $\frac{\partial u_2}{\partial \tau}$  by means of the phase-transition coefficient  $\epsilon_2$  (where  $\epsilon_2 = \epsilon_{12} = \epsilon_{21}$ ). Just as in the case of mass transfer, for chemical transformations in open systems it is possible to write

$$du = d_e u + d_i u, \quad (2-3-16)$$

where  $d_e u$  is the variation in moisture content due to moisture transfer, and  $d_i u$  is the variation in moisture content due to phase transitions (vapor condensation or liquid evaporation). Let us now define the ratio

$$\beta_i = \frac{d_i u}{d_e u} = \frac{\epsilon_i}{1-\epsilon_i}, \quad (2-3-17)$$

where  $\epsilon_i$  is the coefficient (or criterion) for the liquid-to-vapor or vapor-to-liquid phase transition ( $\epsilon_i = \epsilon_{12} = \epsilon_{21}$ ). If  $\epsilon_i = 1$  (so that  $\beta_i = \infty$ ) then moisture is transferred in vapor form.

Since the moisture content  $u$  of a porous body is equal to the liquid content ( $u = u_2$ ), therefore for  $\epsilon_i = 1$  (and  $\beta_i = \infty$ ) the variation in moisture content is due to either liquid evaporation or vapor condensation (no liquid transfer). If  $\epsilon_i = 0$  (and  $\beta_i = 0$ ) the variation in moisture content is due only to liquid transfer (no phase transitions). Consequently,

$$\frac{\partial u_2 \gamma}{\partial \tau} = \frac{\partial_e u_2 \gamma}{\partial \tau} + \frac{\partial_i u_2 \gamma}{\partial \tau} = \frac{1}{1-\epsilon_2} \cdot \frac{\partial_e u_2 \gamma}{\partial \tau} = -\frac{1}{1-\epsilon_2} \operatorname{div} \vec{J}_2, \quad (2-3-18)$$

and so

$$\frac{\partial u_2 \gamma}{\partial \tau} = -\operatorname{div} \vec{J}_2 + \epsilon_2 \frac{\partial u_2 \gamma}{\partial \tau}. \quad (2-3-19)$$

A comparison of this equation with equation (2-3-10) for  $i=2$  shows that

$$I_2 = -I_1 = \epsilon_2 \gamma \frac{\partial u_2}{\partial \tau} = \epsilon_2 \gamma \frac{\partial u}{\partial \tau}. \quad (2-3-20)$$

For the one-dimensional case  $\text{div} \equiv \partial/\partial x$ ; when vectors  $\vec{j}_1$  and  $\vec{j}_2$  are parallel or antiparallel, coefficient  $\epsilon_2$  becomes

$$\epsilon_2 = \frac{\text{div} \vec{j}_1}{\text{div} \vec{j}_1 + \text{div} \vec{j}_2} = \frac{|\vec{j}_1|}{|\vec{j}_1| + |\vec{j}_2|}. \quad (2-3-21)$$

The motion of bound matter through a porous body is considered to be quite slow, so that the temperature of the liquid is virtually equal to the temperature of the capillary walls. The vapor in the capillaries of the body is in thermodynamic and molecular equilibrium with the liquid, with the result that its partial pressure in the hygroscopic range is determined from the sorption and desorption isotherms:

$$p_1 = f(u_2, T) \text{ for } 0 < u_2 < u_{m2}, \quad (2-3-22)$$

where  $u_{m2}$  is the maximum hygroscopic moisture content. In the moist-state range the vapor pressure equals the pressure  $p_s$  of saturated vapor, that is, it is a single-valued function of the temperature:

$$p_1 = p_s = f(T). \quad (2-3-22a)$$

The differential heat-transfer equation is obtained from the equation describing the transfer of internal energy. For constant pressure the local derivative of the volume concentration of the enthalpy of the system is equal to the divergence of the enthalpy flux:

$$\frac{\partial}{\partial \tau} \left( h_0 \gamma_0 + \sum_{i=0}^3 h_i \gamma u_i \right) = -\text{div} \left( \vec{j}_q + \sum_{i=0}^3 h_i \vec{j}_i \right), \quad (2-3-23)$$

where  $h_i$  is the specific enthalpy of the  $i$ th bound matter, and  $h_0$  and  $\gamma_0$  are the specific enthalpy and the density of the skeleton of the porous body. The density is

$$\gamma_0 = \frac{\gamma}{1 + \beta u_2}, \quad (2-3-24)$$

where  $\beta$  is the volume contraction. As  $\beta$  approaches zero  $\gamma_0$  approaches  $\gamma$ .

If the isobaric specific heat is

$$c_i = \left( \frac{dh_i}{dt} \right)_p, \quad (2-3-25)$$

then we have

$$\left( c_0 \gamma_0 + \sum_{i=0}^3 c_i \gamma u_i \right) \frac{\partial t}{\partial \tau} + \sum_{i=0}^3 h_i \frac{\partial (\gamma u_i)}{\partial \tau} = -\text{div} \vec{j}_q - \sum_{i=0}^3 \vec{j}_i c_i \nabla t - \sum_{i=0}^3 h_i \text{div} \vec{j}_i. \quad (2-3-26)$$

Now, let us multiply equation (2-3-10) by  $h_i$  and take the sum for  $i$  from 0 to 3 to obtain

$$\sum_{i=0}^3 h_i \frac{\partial (\gamma u_i)}{\partial \tau} = -\sum_{i=0}^3 h_i \text{div} \vec{j}_i + \sum_{i=0}^3 h_i I_i. \quad (2-3-27)$$

The differential heat-transfer equation is thus

$$c \gamma \frac{\partial t}{\partial \tau} = -\text{div} \vec{j}_q - \sum_{i=1}^3 h_i I_i - \sum_{i=0}^3 c_i \vec{j}_i \nabla t, \quad (2-3-28)$$

where  $c$  is the reduced specific heat of the body, \* defined as

$$c = c_0 \gamma_0 / \gamma + \sum_{i=0}^3 c_i u_i. \quad (2-3-29)$$

The heat sources  $\sum_{i=1}^3 h_i I_i$  exist due to phase transitions ( $I_0=0$ ). Moisture, except at the triple point, is in a two-phase state, so that the sum of the sources is

$$h_i I_{ij} + h_j I_{ji} = (h_i - h_j) I_{ij}, \quad \text{where } i, j = 1, 2, 3; I_{ii} = I_{jj} = 0.$$

From equation (2-3-28) it is possible to obtain as a particular case the Fourier-Kirchhoff heat-transfer equation for a moving liquid ( $i=2$ ). In this case

$$c \gamma_0 = c_2 u_2 \gamma_0 = c_2 \rho_2; \quad \vec{J}_2 = \rho_2 \vec{w};$$

so that

$$c_2 \rho_2 \frac{\partial t}{\partial \tau} + c_2 \rho_2 \vec{w} \nabla t = -\operatorname{div} \vec{J}_q. \quad (2-3-30)$$

In a porous body in the absence of liquid seepage the convective component of the heat transfer is small in comparison with the conductive component, provided that  $Re_e < 20$ . This value of the equivalent Reynolds number corresponds to a GrPr number of 1000 [GrPr is the Grashof number times the Prandtl number]. For GrPr less than 1000 the equivalent thermal conductivity equals the molecular thermal conductivity. In most cases of mass transfer in a porous body  $Re_e$  is considerably less than unity.

Consequently, the system of differential heat-transfer and mass-transfer equations becomes

$$c \gamma \frac{\partial t}{\partial \tau} = -\operatorname{div} (\vec{J}_q) - \sum_{i=0}^3 h_i I_i; \quad (2-3-31)$$

$$\frac{\partial (\gamma u_i)}{\partial \tau} = -\operatorname{div} \vec{J}_i + I_i; \quad (2-3-32)$$

$$\frac{\partial (\gamma \gamma)}{\partial \tau} = -\operatorname{div} \sum_{i=0}^3 \vec{J}_i, \quad (2-3-33)$$

since the sum of all the sources and sinks of the bound matter is zero:

$$\sum_{i=0}^3 I_i = 0. \quad (2-3-34)$$

For a final derivation of the system (2-3-31) to (2-3-33) it is necessary to express the fluxes  $\vec{J}_q$  and  $\vec{J}_i$  in terms of the gradients of the corresponding potentials. To do this it is necessary to take into account that in a porous body the mass content of noncondensing gas and vapor is negligible in comparison with the mass content of liquid and ice ( $u = u_2 + u_3$ , with  $u_0 = u_1 = 0$ ).

In addition, bound matter in the solid state does not move ( $\vec{J}_3=0$ ), and the source  $I_0$  is zero (no chemical transformations). In this case the mass-transfer equations may be written as

$$\operatorname{div} \vec{J}_0 = 0; \quad I_1 = -I_2 = \operatorname{div} \vec{J}_1; \quad (2-3-35)$$

\* In our previous publications the reduced specific heat was defined in terms of the relation

$$\bar{c} \gamma_0 = c_0 \gamma_0 + \sum_{i=0}^3 \gamma_0 c_i u_i. \quad \text{If the contraction coefficient is zero } (\beta=0), \text{ then } \bar{c} = c \text{ since } \gamma = \gamma_0.$$



$$\frac{\partial u_2 \gamma}{\partial \tau} = -\operatorname{div} \vec{J}_2 + I_2 = -\operatorname{div} \vec{J}_1 - \operatorname{div} \vec{J}_2; \quad (2-3-36)$$

$$\frac{\partial u_2 \gamma}{\partial \tau} = I_2 = \frac{\epsilon_{22}}{1 - \epsilon_{22}} \frac{\partial (\gamma u_2)}{\partial \tau} = \epsilon_{22} \frac{\partial (\gamma u)}{\partial \tau}. \quad (2-3-37)$$

Vapor moves through a porous body by means of diffusion.\* Here it is assumed that the vapor is in thermodynamic equilibrium with the bound liquid. In the hygroscopic range the partial vapor pressure is a function of the moisture content  $u$  and the temperature  $T$ , so that the relative vapor concentration in the capillaries of the body also depends on  $u$  and  $T$ :

$$p_{10} = p_{10} \frac{M_1^0}{M^0} = f(u, T). \quad (2-3-38)$$

In this case the density of the diffusion flux of vapor in a porous body is \*\*

$$\vec{J}_1 = -\epsilon D_p \nabla p_{10} = -a_{m1} \gamma [\nabla u + \delta_1 \nabla T], \quad (2-3-39)$$

where  $\epsilon$  is a dimensionless coefficient characterizing the resistance to vapor diffusion in a moist porous body,  $a_{m1}$  is the coefficient for the diffusion of vaporous moisture in a moist body, defined as

$$a_{m1} = \frac{\epsilon}{\gamma} D_p \left( \frac{\partial p_{10}}{\partial u} \right)_T, \quad (2-3-40)$$

and  $\delta_1$  is the thermal-gradient coefficient for vapor transfer:

$$\delta_1 = \left( \frac{\partial p_{10}}{\partial T} \right)_u / \left( \frac{\partial p_{10}}{\partial u} \right)_T. \quad (2-3-41)$$

In the derivation of (2-3-39) we also used the relation

$$\nabla p_{10} = \left( \frac{\partial p_{10}}{\partial u} \right)_T \nabla u + \left( \frac{\partial p_{10}}{\partial T} \right)_u \nabla T. \quad (2-3-42)$$

The flux  $\vec{J}_1$ , which is expressed in terms of gradients  $\nabla u$  and  $\nabla T$  in equation (2-3-39), corresponds to the hygroscopic range ( $u \leq u_{cm}$ ). In the moist-state range  $(\partial p_{10} / \partial u)_T$  is zero (the partial vapor pressure is independent of the moisture content), and thus the diffusion flux of vapor is

$$\vec{J}_1 = -a_{m1} \gamma \delta_1 \nabla T = -\epsilon D_p \left( \frac{\partial p_{10}}{\partial T} \right)_u \nabla T, \quad (2-3-43)$$

where

$$\frac{\partial p_{10}}{\partial T} = \frac{M_1^0}{M^0 p} \left( \frac{d p_s}{d T} \right), \quad (2-3-44)$$

$p_s$  being the pressure of saturated vapor at the temperature  $T$ .

Liquid transfer in a porous body takes place by means of selective diffusion and as a kind of capillary absorption under the action of a capillary potential, which is also a function of the moisture content and the temperature. Using transformations similar to the previous ones, we obtain

$$\vec{J}_2 = -a_{m2} \gamma [\nabla u + \delta_2 \nabla T], \quad (2-3-45)$$

where  $a_{m2}$  and  $\delta_2$  are, respectively, the coefficient for the diffusion of liquid

\* Macroscopic vapor transfer will be considered below.

\*\* In the expression for the diffusion flux of vapor we do not introduce a correction for the macroscopic transfer (the Stefan Flux), since the conditions for vapor diffusion in a porous body differ from those for vapor diffusion in a boundary layer when a liquid evaporates from a free surface. In addition, the macroscopic vapor transfer in a porous body is taken into account separately.

in the porous body and the thermal-gradient coefficient for nonisothermal liquid transfer.

It follows from (2-3-39) and (2-3-45) that the transfer of liquid and vapor in the hygroscopic range is described by the same law, that is, it is due to the two gradients  $\nabla u$  and  $\nabla T$ . Consequently, vapor and liquid transfer may be combined and expressed by a single mass-transfer law:

$$\vec{J}_m = \vec{J}_1 + \vec{J}_2 = -a_m \gamma (\nabla u + \delta \nabla T), \quad (2-3-46)$$

where  $a_m$  is the potential conductivity for the liquid and vapor mass transfer and  $\delta$  is once again the thermal-gradient coefficient. If all the forms of moisture transfer in a porous body, due to gradients  $\nabla u$  and  $\nabla T$ , are called arbitrarily moisture diffusion (molecular and capillary diffusion), then the coefficient  $a_m$  will represent a coefficient for moisture diffusion in a moist body.

If vectors  $\vec{J}_1$  and  $\vec{J}_2$  have the same directions, then coefficients  $a_m$  and  $\delta$  become

$$a_m = a_{m1} + a_{m2} \quad \text{and} \quad \delta = \frac{a_{m1}\delta_1 + a_{m2}\delta_2}{a_{m1} + a_{m2}}. \quad (2-3-47)$$

Since  $\delta$  may be determined experimentally as the ratio of the moisture-content gradient to the temperature gradient in the absence of mass transfer ( $\vec{J}_m = 0$ ), we have

$$\delta = -\frac{\nabla u}{\nabla T} = -\frac{\Delta u}{\Delta T} \quad \text{for} \quad \vec{J}_m = 0, \quad (2-3-48)$$

and so in the hygroscopic range  $\delta_1 = \delta_2$ . This fact makes it possible to represent vapor and liquid transfer in the hygroscopic range in terms of a single gradient of the mass-transfer potential  $\theta$ . If we neglect the Soret effect (the thermal diffusion of vapor and solute in a liquid), we may write

$$\vec{J}_m = -\lambda_m \nabla \theta, \quad (2-3-49)$$

where  $\lambda_m$ , a proportionality factor between the moisture-flux density  $\vec{J}_m$  and the potential gradient  $\nabla \theta$ , is called the mass conductivity, by analogy with the thermal conductivity  $\lambda$ , which is equal to  $\lambda = j_q / \nabla T$ . \* The potential gradient  $\nabla \theta$  can be expressed in terms of the gradients  $\nabla u$  and  $\nabla T$ :

$$\nabla \theta = \left( \frac{\partial \theta}{\partial u} \right)_T \nabla u + \left( \frac{\partial \theta}{\partial T} \right)_u \nabla T.$$

Equation (2-3-49) may thus be written as

$$\vec{J}_m = -\lambda_m \left( \frac{\partial \theta}{\partial u} \right)_T \nabla u - \lambda_m \left( \frac{\partial \theta}{\partial T} \right)_u \nabla T = -a_m \gamma \nabla u - a_m \gamma \delta \nabla T, \quad (2-3-50)$$

where  $a_m$  and  $\delta$  are

$$a_m = \frac{\lambda_m}{\gamma \left( \frac{\partial u}{\partial \theta} \right)_T} \quad \text{and} \quad \delta = \left( \frac{\partial u}{\partial \theta} \right)_T \left( \frac{\partial \theta}{\partial T} \right)_u. \quad (2-3-51)$$

By analogy with the specific heat, the ratio  $(\partial u / \partial \theta)_T$  may be called the isothermal specific mass (kg/kg · °M), denoted as

$$c_m = \left( \frac{\partial u}{\partial \theta} \right)_T. \quad (2-3-52)$$

The quantity  $a_m$  is thus obviously analogous to the thermal diffusivity  $a_p$ , since

$$a_m = \frac{\lambda_m}{c_m \gamma} \quad \text{and} \quad a_p = \frac{\lambda}{c_p \gamma}. \quad (2-3-53)$$

\* The mass-transfer potential  $\theta$  is measured on an experimental scale in mass-exchange degrees (°M); an absolutely dry body has a potential of 0°M, whereas conditions of maximum hygroscopic moisture content correspond to  $\theta = 100^\circ\text{M}$ .

This analogy led us, in our previous publications, to call  $a_m$  the "potential conductivity" for mass transfer, since the temperature conductivity [the thermal diffusivity] may also be called the potential conductivity for heat transfer.

The thermal-gradient coefficient is thus

$$\delta = c_m \left( \frac{\partial \theta}{\partial T} \right)_u, \quad (2-3-54)$$

the product of the specific mass times the temperature derivative of the mass-transfer potential.

According to the investigations of Nikitina, the thermodynamic mass-transfer potential corresponds to the free binding energy of the absorbed substance in a porous body; it is numerically equal to the work required for isothermal removal of a unit mass of material:

$$\theta_p = - \frac{RT}{M_1^0} \ln \varphi, \quad (2-3-55)$$

where  $\varphi$  is the relative vapor pressure in the porous body, defined as the partial vapor pressure  $p_1$  divided by the saturated-vapor pressure  $p_s$  at the given temperature ( $\varphi = p_1/p_s$ ). Consequently,  $\theta_p$  is a single-valued function of the air humidity. Since for a number of porous bodies the ratio  $u_p/u_{cm}$  is independent of the temperature, being a function of  $\varphi$ , therefore the mass-transfer potential  $\theta$  can be experimentally determined from the specific mass content [specific mass] of a standard body. In the moist-state range the capillary and diffusion-osmotic movement of the liquid can be described by means of the potential  $\theta$ . The macroscopic motion of the liquid and vapor due to the gradient of the total pressure is described by seepage laws. In this case the thermal-gradient coefficient defined in (2-3-48) will not obey equation (2-3-54), and also the thermal-gradient coefficients  $\delta_1$  and  $\delta_2$  for the vapor and liquid transfer will not be equal ( $\delta_1 \neq \delta_2$ ).

Let us now return to the system of differential mass-transfer equations. If the bound matter is in the liquid and vapor states only ( $u_3=0$ ), then, neglecting contraction of the body ( $\beta=0$ ), we obtain

$$\frac{\partial u}{\partial \tau} = - \operatorname{div} (a_m \nabla u + \delta a_m \nabla t), \quad (2-3-56)$$

$$c \gamma \frac{\partial t}{\partial \tau} = \operatorname{div} (\lambda \nabla t) + r_{12} \operatorname{div} (a_{m1} \nabla u + a_{m1} \gamma \delta \nabla t), \quad (2-3-57)$$

where  $r_{12} = h_1 - h_2$  is the specific vaporization heat, and  $\lambda$  is the thermal conductivity of the porous body, taking into account conductive heat transfer through the skeleton of the body and through the bound matter.

For a zonal calculation the system of mass-transfer and heat-transfer equations is

$$\frac{\partial u}{\partial \tau} = a_m \nabla^2 u + a_m \delta \nabla^2 t; \quad (2-3-58)$$

$$\frac{\partial t}{\partial \tau} = \left( a + \frac{a_{m1} \gamma r_{12}}{c} \right) \nabla^2 t + \frac{a_{m1}}{c} r_{12} \nabla^2 u. \quad (2-3-59)$$

A similar heat-transfer equation is obtained by using the relation involving the source term  $I_s$ , expressed in terms of the vaporization ratio  $s$  (where

$$I_s = s \gamma \frac{\partial u}{\partial \tau})$$

$$\frac{\partial t}{\partial \tau} = \left( a + \frac{a_m s r}{c} \right) \nabla^2 t + \frac{a_m s r}{c} \nabla^2 u, \quad (2-3-60)$$

the subscript 2 being omitted. A comparison of (2-3-60) and (2-3-59) shows that

$$\epsilon = a_{m1}/a_m. \quad (2-3-61)$$

This relation may be obtained directly from (2-3-18), provided we substitute the corresponding expressions for  $\vec{J}_1$  and  $\vec{J}_2$  and set  $\delta_1 = \delta_2$ , as is correct for the hygroscopic range.

The heat-transfer and mass-transfer equations are usually written as

$$\frac{\partial t}{\partial \tau} = a \nabla^2 t + \frac{\epsilon r}{c\gamma} \frac{\partial u}{\partial \tau}; \quad (2-3-62)$$

$$\frac{\partial u}{\partial \tau} = a_m \nabla^2 u + a_m \delta \nabla^2 t. \quad (2-3-63)$$

Equation (2-3-49) for the mass transfer of vapor and liquid, for the general case taking into account the simultaneous action of the two thermodynamic forces  $\nabla \theta$  and  $\nabla T$ , may be written as

$$\vec{J}_m = -\lambda_m \nabla \theta - \lambda_m \delta_\theta \nabla t = -a_m \gamma \nabla u - a_m \gamma \delta \nabla t, \quad (2-3-64)$$

where  $\delta_\theta$  is the thermal-gradient coefficient referred to the difference in mass-transfer potentials ( $^\circ M/^\circ C$ ), which is defined as

$$\delta_\theta = \left( \frac{\Delta \theta}{\Delta t} \right)_{j_m=0} \quad (2-3-65)$$

In this case the thermal-gradient coefficient  $\delta$  is\*

$$\delta = \left[ c_m \left( \frac{\partial \theta}{\partial T} \right)_u + \delta_\theta \right]. \quad (2-3-66)$$

The system of differential heat-transfer and mass-transfer equations will have the similar form:

$$\frac{\partial t}{\partial \tau} = a \nabla^2 t + \frac{\epsilon r}{c} c_m \frac{\partial \theta}{\partial \tau}; \quad (2-3-67)$$

$$\frac{\partial \theta}{\partial \tau} = a_m \nabla^2 \theta + a_m \delta_\theta \nabla^2 t. \quad (2-3-68)$$

In the moist-state range of the body ( $\theta > 100^\circ M$ ) the differential mass-transfer equation remains unchanged, while the heat-transfer equation can be simplified and reduced to the ordinary equation for heat conduction, since the coefficient  $a_{m1}$  is zero (see equation (2-3-59)). At negative temperatures the specific mass content (moisture content) is the sum of the moisture contents of the liquid and the ice ( $u = u_2 + u_3$ ).

The differential heat-transfer equation for a two-phase system of bound matter may be written as

$$\frac{\partial t}{\partial \tau} = a \nabla^2 t + \frac{r_{ij}}{c\gamma} I_{ji}, \quad (2-3-69)$$

where  $r_{ij}$  is the specific heat of the phase transition  $ij$ . The liquid source  $I_{21}$  for vapor condensation (the bound matter consists of supercooled liquid and vapor) is

$$I_{21} = -I_{12} = \epsilon \gamma \frac{\partial u_2}{\partial \tau}.$$

\* The quantity  $c_m \left( \frac{\partial \theta}{\partial T} \right)_u$  also represents the diffusion transfer of vapor in the moist-state range; it is equal to  $\epsilon D_p [\partial p_{10}/\partial T]_u / \lambda_m$ . However, this quantity can be isolated separately in (2-3-66), as is done in /1/.

For a liquid-ice system the source  $I_{32}$  is

$$I_{32} = \epsilon_{32} \gamma \frac{\partial u}{\partial \tau},$$

and the specific heat  $r_{23}$  of the phase transition equals the specific crystallization heat. Finally, for an ice-vapor system the material (ice) source is

$$I_{31} = -I_{13} = \gamma \frac{\partial u}{\partial \tau}.$$

Thus, the differential heat-transfer equation is

$$\frac{\partial t}{\partial \tau} = a \nabla^2 t + \frac{r \epsilon}{c} \frac{\partial u}{\partial \tau} = \left( a + \frac{a_m r \epsilon \delta}{c} \right) \nabla^2 t + \frac{a_m r \epsilon}{c} \nabla^2 u_2, \quad (2-3-70)$$

where  $\epsilon$  and  $r$  are, respectively, the coefficient [ratio] and specific heat of the phase transition. When no ice is present  $\epsilon = \epsilon_{12}$ ,  $r = r_{12} = r_{21}$ , and  $u = u_2$ . In the presence of ice ( $u_3 \neq 0$ ) and the phase transition from water to ice  $\epsilon = \epsilon_{32}$ ,  $r = r_{23} = r_{32}$ , and  $u = u_2 = u_3$ . For a two-phase ice-vapor system,  $r = r_{13} = r_{31}$  (the specific sublimation heat for ice) and  $\epsilon_{32} = 1$  (with  $u_2 = 0$ ). The differential mass-transfer equation is obtained by adding up equations (2-3-10) for the  $i$  components:

$$\frac{\partial u_i}{\partial \tau} = -\operatorname{div} \vec{J}_1 - \operatorname{div} \vec{J}_2, \quad (2-3-71)$$

from which it follows that

$$\frac{\partial u}{\partial \tau} = a_m \nabla^2 u_2 + a_m \delta \nabla^2 t. \quad (2-3-72)$$

Finally, it is also true that

$$\frac{\partial u}{\partial \tau} = \frac{\partial u_2}{\partial \tau} + \frac{\partial u_3}{\partial \tau} = \frac{1}{1 - \epsilon_{32}} \frac{\partial u_2}{\partial \tau}. \quad (2-3-73)$$

In the negative-temperature range, where all the liquid is transformed into ice ( $\epsilon_{32} = 1$ ), the mass transfer takes place by means of vapor, the condensation of which increases the ice content ( $\partial u_3 / \partial \tau > 0$ ), while conversely the sublimation of ice reduces its moisture content ( $\partial u_3 / \partial \tau < 0$ ). Differential equation (2-3-72) thus becomes

$$\frac{\partial u_2}{\partial \tau} = a_m \delta_1 \nabla^2 t. \quad (2-3-74)$$

This relation follows directly from equations (2-3-32), since  $I_1 = I_{31} = -I_{13}$  and  $\partial u_1 / \partial \tau = 0$ , because the variation in vapor content in the capillaries of the body without vapor condensation and ice vaporization does not affect the moisture content in the body.

When a moist body is heated intensively a gradient of the total pressure appears due to liquid evaporation at temperatures  $\geq 100^\circ\text{C}$ . A gradient of the total pressure inside the body may also exist at temperatures lower than  $100^\circ\text{C}$ , due to the effusive flow of air along microscopic capillaries within the body (thermomolecular pressure difference) or due to pressure differentials between media which are separated by a porous barrier (seepage through a porous medium).

The presence of a total-pressure gradient  $\nabla p$  inside a porous body sets up a macroscopic transfer of the vapor-gas mixture (vapor and air) which is a form of seepage through a porous medium. This vapor transfer is not taken into account by the mass-transfer law (2-3-64), and so it is necessary to introduce into this equation an additional term representing the convective-transfer flow of moist air.

According to Darcy's law the seepage flux of moist air through a porous body is

$$\vec{J}_s = -k \nabla p, \quad (2-3-75)$$

where  $k$  is the permeability coefficient for air ( $\text{kg/m} \cdot \text{h} \cdot \text{mm hg}$ ). The amount of vapor which is transferred by the seepage flow of air is

$$\vec{J}_{1s} = -k \rho_{10} \nabla p = -\lambda_{p1} \nabla p, \quad (2-3-76)$$

where  $\lambda_{p1}$  is the coefficient of macroscopic transfer of vapor

$$\lambda_{p1} = k \rho_{10} = k \frac{d}{1+d}, \quad (2-3-77)$$

$d$  being the moisture content of the air.

At temperatures  $t \geq 100^\circ\text{C}$  the moisture content  $d$  is very high ( $d \rightarrow \infty$ ), and so the transfer coefficient  $\lambda_{p1}$  becomes equal to the permeability coefficient for vapor ( $\lambda_p = k$ ), since the moist air is transformed into vapor.

The pressure gradient also causes a seepage transfer of liquid. This type of transfer includes liquid motion caused by the heating of entrapped air. The pressure in the bubbles of entrapped air is increased by the heating, and the air bubbles expand (relaxation of the total-pressure gradient), which causes a "forcing through" of the liquid in the direction of heat flow. This type of liquid transfer in a porous body is usually related to the thermal mass conduction (here, the thermal moisture conduction) and the thermal-gradient coefficient  $\delta$  enters into the calculations of it. However, it is best to calculate it in terms of a seepage transfer of liquid. Then, the seepage liquid flux  $\vec{J}_{2s}$  is

$$\vec{J}_{2s} = -\lambda_{p2} \nabla p, \quad (2-3-78)$$

where  $\lambda_{p2}$  is the transfer coefficient for the macroscopic process of liquid seepage.

The total flux of vapor and liquid caused by the total-pressure gradient is

$$\vec{J}_s = \vec{J}_{1s} + \vec{J}_{2s} = -\lambda_p \nabla p, \quad (2-3-79)$$

where  $\lambda_p = \lambda_{p1} + \lambda_{p2}$  is the coefficient of macroscopic moisture transfer. The total moisture flux caused by all the thermodynamic forces is

$$\vec{J}_m = -\lambda_m \nabla \theta - \lambda_m \delta_p \nabla t - \lambda_p \nabla p = -\lambda_m (\nabla \theta + \delta_p \nabla t + \delta_p \nabla p), \quad (2-3-80)$$

where  $\delta_p = \lambda_p / \lambda_m$ .

In this case the differential equation describing mass transfer is

$$c_m \gamma \frac{\partial \theta}{\partial \tau} = \text{div} [\lambda_m (\nabla \theta + \delta_p \nabla t + \delta_p \nabla p)]. \quad (2-3-81)$$

In order for the system of equations to be complete an additional equation is required, representing the variation in the seepage potential  $p$ . If we add up the differential mass-transfer equations for  $i=0, 1$  and take into account that  $I_0=0$  and  $I_1 = -\sigma \gamma \frac{\partial u}{\partial \tau}$ , then we obtain

$$\frac{\partial \gamma (u_0 + u_1)}{\partial \tau} = -\text{div} (\vec{J}_0 + \vec{J}_1) - \sigma \gamma \frac{\partial u}{\partial \tau}. \quad (2-3-82)$$

The molecular transfer of vapor and air (interdiffusion of vapor and air) is slight in comparison with the macroscopic seepage transfer. Thus, it may be assumed that, approximately,  $(\vec{J}_0 + \vec{J}_1)$  equals  $\vec{J}_i$ . The specific mass content  $(u_0 + u_1)$  of vapor-gas mixture in the capillaries of the body is determined

by the temperature  $T$  and the pressure  $p$ . We know that

$$(u_0 + u_1)\gamma = \omega_0 + \omega_1 = \rho \Pi b_i = \frac{\rho M^* \Pi}{RT} b_i, \quad (2-3-83)$$

where  $\rho$  is the density of the vapor-gas mixture ( $\rho = \rho_1 + \rho_2$ ). This equation may be differentiated to give

$$d(u_0 + u_1)\gamma = \frac{\Pi b_i M^*}{R} \left( dp - \frac{1}{T^2} dT \right) + \frac{\rho M^* \Pi}{RT} db_i. \quad (2-3-84)$$

Now, neglecting the last two terms and denoting  $\frac{\Pi b_i M^*}{\gamma R}$  as  $c_b$ , we obtain

$$\frac{\partial(u_0 + u_1)\gamma}{\partial \tau} = c_b \gamma \frac{\partial p}{\partial \tau}, \quad (2-3-85)$$

where the coefficient  $c_b$  characterizes the moist-air capacity of the porous body during the motion of a vapor-gas mixture. Consequently, differential equation (2-3-82) becomes

$$c_b \gamma \frac{\partial p}{\partial \tau} = \text{div}(k \nabla p) - \epsilon \gamma c_m \frac{\partial \theta}{\partial \tau}. \quad (2-3-86)$$

In the zonal system of calculation the differential heat-transfer and mass-transfer equations can be written as

$$\frac{\partial t}{\partial \tau} = a_q \nabla^2 t + \epsilon r \frac{c_m}{c_q} \frac{\partial \theta}{\partial \tau}; \quad (2-3-87)$$

$$\frac{\partial \theta}{\partial \tau} = a_m \nabla^2 \theta + a_m \delta_\theta \nabla^2 t + a_m \delta_p \nabla^2 p; \quad (2-3-88)$$

$$\frac{\partial p}{\partial \tau} = a_p \nabla^2 p - \epsilon \frac{c_m}{c_q} \frac{\partial \theta}{\partial \tau}, \quad (2-3-89)$$

where  $a_p$ , the coefficient of convective diffusion or the potential conductivity for the seepage flow of a vapor-gas mixture, is defined as  $a_p = k/c_b \gamma$ . For convenience the thermal parameters  $a$  and  $c$  are written with a subscript  $q$  (that is,  $a_q \equiv a$ ,  $c_q \equiv c$ ).

#### 2-4. The Equations for Laminar Flow of Immiscible Fluids through a Porous Medium

Studies of multiphase flow in a porous medium are of practical interest in the petroleum industry (the simultaneous flow of oil, water, and gas), the chemical industry (flow in countercurrent columns), and various other fields as well. However, the flow of mutually soluble fluids is a complex transfer phenomenon, and therefore in most cases mutually insoluble or immiscible fluids are considered. Darcy's seepage law is used as an initial assumption for each flowing phase. The flow is considered to be isothermal and laminar; adsorption and osmotic absorption by the skeleton of the porous medium may then be neglected, a fact characterized by the relation

$$\sum_i b_i = 1, \quad (2-4-1)$$

where  $b_i$  is the saturation factor.

The density  $\rho_i$  of a flowing phase is a function of the pressure  $p_i$ . Let us denote the derivative  $\partial \rho_i / \partial p_i$  as  $c_{\rho i}$ :

$$\frac{d \rho_i}{d p_i} = c_{\rho i}. \quad (2-4-2)$$

Darcy's law may be applied to each flowing phase

$$\vec{j}_i = k \frac{k_i}{\eta_i} (\nabla p_i - \vec{g} \rho_i), \quad (2-4-3)$$

where  $\vec{j}_i$  is the seepage-velocity vector,  $k$  is the overall permeability,  $k_i$  is the relative permeability for the  $i$ th phase, in fractions of the overall permeability,  $\eta_i$  is the viscosity, and  $\vec{g}$  is the gravitational acceleration (the gravitational-field-intensity vector).

For a two-phase flow ( $i=1, 2$ ) the differential transfer equations for immiscible liquids are

$$\Pi \frac{\partial (\rho_1 b_1)}{\partial \tau} = -\operatorname{div} (\rho_1 \vec{j}_1); \quad (2-4-4)$$

$$\Pi \frac{\partial (\rho_2 b_2)}{\partial \tau} = -\operatorname{div} (\rho_2 \vec{j}_2), \quad (2-4-5)$$

where  $\Pi$  is the porosity. These two equations are related by the expressions

$$b_1 + b_2 = 1 \text{ and } p_2 - p_1 = p_k(b_1), \quad (2-4-6)$$

where  $p_k$  is the capillary pressure.

If we assume that the quantity  $kk_i/\eta_i$  is constant and that  $c_{pi}$  is independent of the [position] coordinates, then the system of differential equations (2-4-4) and (2-4-5) may be written as

$$\frac{\partial \rho_1}{\partial \tau} = a_{p1} (\nabla^2 p_1 - c_{p1} \operatorname{div} \vec{g} \rho_1) - \frac{\rho_1}{b_1} \cdot \frac{\partial b_1}{\partial \tau}, \quad (2-4-7)$$

$$\frac{\partial \rho_2}{\partial \tau} = a_{p2} (\nabla^2 p_2 - c_{p2} \operatorname{div} \vec{g} \rho_2) + \frac{\rho_2}{1-b_1} \cdot \frac{\partial b_1}{\partial \tau}, \quad (2-4-8)$$

where  $a_{pi}$  is a parameter analogous to the potential conductivity for seepage transfer:

$$a_{pi} = \frac{kk_i}{\eta_i \Pi b_i c_{pi}}. \quad (2-4-9)$$

If gravitational effects are neglected, then this system of equations becomes

$$\frac{\partial \rho_1}{\partial \tau} = a_{p1} \nabla^2 p_1 - \frac{\rho_1}{b_1} \frac{\partial b_1}{\partial \tau}; \quad (2-4-10)$$

$$\frac{\partial \rho_2}{\partial \tau} = a_{p2} \nabla^2 p_2 + \frac{a_{p1} \rho_2 b_1}{\rho_1 (1-b_1)} \nabla^2 p_1 - \frac{b_1 \rho_2}{(1-b_1) \rho_1} \frac{\partial \rho_1}{\partial \tau}. \quad (2-4-11)$$

System of equations (2-4-10) and (2-4-11) is analogous to the system of differential heat-transfer and mass-transfer equations for a porous body.

## 2-5. The Equations for Neutron and Heat Transfer in an Absorbing Medium

As a first approximation to describing thermal-neutron transfer in a weakly absorbent [low-capture] medium let us consider it from the diffusion viewpoint. Neutron diffusion, however, has certain special features. For instance, the number of neutrons per unit volume (the neutron density  $n$ ) is relatively low. For a neutron flux of  $10^{12}$  neutrons/cm<sup>2</sup>·sec the neutron density is  $4.5 \cdot 10^6$  neutrons/cm<sup>3</sup>, which constitutes a high vacuum (let us recall that under normal conditions the number of molecules per cm<sup>3</sup> is  $2.7 \cdot 10^{19}$ ).



The rate of neutron flow in a medium is characterized by the neutron flux  $\Phi$  (neutrons/cm<sup>2</sup>·sec), a scalar quantity. For monoenergetic neutrons which have identical directions and velocities  $v$  (cm/sec), \* the neutron flux  $\Phi$  is

$$\Phi = nv. \quad (2-5-1)$$

The reason for the introduction of  $\Phi$  is the fact that the outcome of a nuclear reaction is independent of the direction from which the neutrons collide with the nuclei.

It should be noted that  $\Phi$  does not represent the actual number of neutrons which pass through one cm<sup>2</sup> in one second. However, integration of the equation for the distribution of thermal neutrons in an absorbing medium shows that the neutron flux in a given (selected) direction  $x$  is equal to  $(\Phi/4)_{+x}$ , with the condition that  $\Phi$  is constant. Because of symmetry the flux in the opposite direction  $(-x)$  will be  $-(\Phi/4)_{-x}$ . Thus, the total number of neutrons passing through unit surface per unit time, the neutron-flux density, is

$$\vec{j}_n = \left(\frac{\Phi}{4}\right)_{+x} - \left(-\frac{\Phi}{4}\right)_{-x} = \frac{1}{2} \Phi. \quad (2-5-2)$$

In contrast to  $\Phi$  the quantity  $\vec{j}_n$  is a vector. If the flux  $\Phi$  is variable, then the actual neutron flux (flux density) in the  $x$  direction is

$$\vec{j}_n = \frac{1}{4} \Phi - \frac{\lambda_d}{6} \left(\frac{\partial \Phi}{\partial x}\right) - \left[ \frac{1}{4} \Phi - \frac{\lambda_d}{6} \left(\frac{\partial \Phi}{\partial x}\right) \right] \approx \frac{\lambda_d}{3} \left(\frac{\partial \Phi}{\partial x}\right), \quad (2-5-3)$$

where  $\lambda_d$  is the scattering length (for graphite  $\lambda_d = 2.70$  cm). Equation (2-5-3) is valid for a large reactor, when the gradient  $\nabla \Phi$  is relatively small.

By analogy with diffusion theory it is possible to assume that the vector  $\vec{j}_n$  (the number of neutrons passing normally through an isoconcentration surface, per unit time per unit area) is directly proportional to the concentration gradient:

$$\vec{j}_n = -D \nabla n, \quad (2-5-4)$$

where  $D$  is the diffusion coefficient for the thermal neutrons (cm<sup>2</sup>/sec).

Usually the neutron flux  $\Phi$  is introduced in nuclear physics instead of the concentration ( $\Phi = nv$ , where  $v$  is the neutron velocity, assumed constant). In this case the diffusion equation becomes

$$\vec{j}_n = -D' \nabla \Phi, \quad (2-5-5)$$

where  $D' = D/v$  is a diffusion coefficient with the units of length (cm), since  $\vec{j}_n$  and  $\Phi$  have the same units.

For a weak absorber [low-capture medium] this diffusion coefficient is

$$D' = \frac{\lambda_d}{3(1 - \overline{\cos \theta})} = \frac{\lambda_{tr}}{3}, \quad (2-5-6)$$

where  $\overline{\cos \theta}$  is the average value of the cosine of the scattering angle, and  $\lambda_{tr}$  is the transport mean free path of the neutron. For example, for graphite  $\overline{\cos \theta} = 0.056$  and so  $\lambda_{tr} = 2.86$  cm, since  $\lambda_{tr} = \lambda_d / (1 - \overline{\cos \theta})$ . The quantity  $1/(1 - \overline{\cos \theta})$  is a correction factor to the preferred scattering direction, namely to the direction of neutron motion before collision with the nucleus. As the mass of the nucleus entering into the collision increases, this correction

\* The velocity  $v$  of a thermal neutron is 220 m/sec.

factor decreases ( $\overline{\cos\theta} \rightarrow 0$ ). For heavy isotopes the factor  $1/(1 - \overline{\cos\theta})$  goes to unity and the scattering length becomes equal to the transport length ( $\lambda_d = \lambda_{tr}$ ).

The differential equation describing neutron diffusion is

$$\frac{\partial n}{\partial \tau} = \text{div}(D\nabla\Phi) - \Sigma_a\Phi + I_n, \quad (2-5-7)$$

where  $I_n$  is the neutron-source strength (the rate of neutron supply per  $\text{cm}^3$  in 1 sec), and  $\Sigma_a\Phi$  is the rate of neutron absorption (neutron-sink strength), equal to the product of the macroscopic absorption section  $\Sigma_a$  times the neutron flux. For monoenergetic neutrons the differential diffusion equation becomes

$$\frac{\partial n}{\partial \tau} = D\nabla^2 n - \beta_a n + I_n, \quad (2-5-8)$$

where  $\beta_a$  is the coefficient of neutron absorption, equal to

$$\beta_a = \Sigma_a \sigma. \quad (2-5-9)$$

In an unbounded medium which does not contain disintegrated material the neutron source strength  $I_n$  is zero. If this is true, then the equation for neutron diffusion in the one-dimensional case will be similar to the differential equation describing heat conduction when a rod cools off via heat transfer from its lateral surface.

In order to solve the differential equation it is necessary to know the boundary conditions. At the interface between two absorbing media the neutron fluxes are equal, and the resultant neutron-flux densities in the

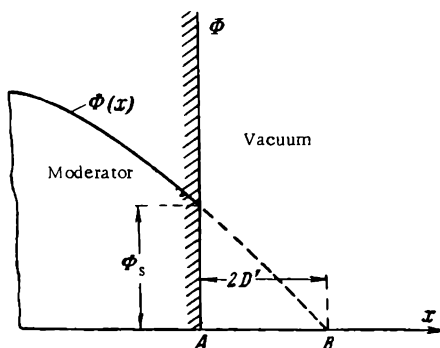


FIGURE 2-3. Boundary conditions of the third kind for neutron diffusion

direction normal to the interface are also equal (boundary conditions of the fourth kind). \* Near the boundary between a scattering medium and a vacuum the neutron flux varies in such a way that if a linear extrapolation is carried out the neutron flux goes to zero at a given distance from the boundary (Figure 2-3), which corresponds to boundary conditions of the third kind.

Let us consider this in more detail. If a reactor is placed in a vacuum, the neutron flux does not drop to zero at the external surface of the reactor, since some neutron leakage exists. At the external surface there is no

\* Concerning boundary conditions see below.

neutron diffusion into the reactor (zero neutron-flux density into the reactor). Thus, from (2-5-3) we obtain

$$\frac{1}{4}\Phi_s + \frac{D'}{2}\left(\frac{\partial\Phi}{\partial x}\right)_s = 0, \quad (2-5-10)$$

where the subscript  $s$  stands for the external surface of the reactor, and  $x$  is the direction normal to the reactor surface. This relation, which represents a boundary condition of the third kind, may be written as

$$D\left(\frac{\partial n}{\partial x}\right)_s + a_n n_s = 0, \quad (2-5-11)$$

where  $a_n = \frac{1}{2} \cdot v$  is a coefficient similar to the mass-exchange coefficient.

If the tangent to the distribution curve for the neutron flux  $\Phi(x)$  is extended until it intersects the  $x$  axis (Figure 2-3), then a distance of length  $AB = 2D' = D/a_n$  is cut off along the axis (the extrapolation distance). Near the interface between the absorbing medium and the vacuum the application of diffusion theory is somewhat arbitrary, but still the calculations based on the solution of the diffusion equation give results close to those of the exact transfer theory. For example, the exact theory describing neutron transfer gives an extrapolation distance of  $0.71 \lambda_{tr}$ , whereas in the above example  $AB = 2 \cdot \frac{1}{3} \lambda_{tr} = 0.666 \lambda_{tr}$ , which is close to the correct value.

Generally the neutron-source strength is a function of time and depends on the neutron flux  $\Phi$ . For an approximate calculation for a homogeneous reactor without a reflector, in the critical state, the so-called one-group theory gives a neutron-source strength of

$$I_n = k\Phi\Sigma_a = k\beta_a n, \quad (2-5-12)$$

where  $k$  is the neutron-multiplication constant for the reactor material, each neutron capture leading to the formation of  $k$  neutrons during the fission process. In other cases the expression for the neutron-source term will be more complicated. For example, if we take into account the probability of neutron leakage during braking, then the source strength  $I_n$  will be

$$I_n = k\beta_a n \exp(-B^2\tau), \quad (2-5-13)$$

where  $B^2$  is a geometrical parameter. If we take into account the probability of resonance capture and the effect of delayed neutrons, then the source term becomes still more complicated. Thus, the diffusion equation, in the absence of external neutron sources and with the above assumptions, may be written as

$$\frac{\partial n}{\partial \tau} = D\nabla^2 n - \beta_a n + k\beta_a n \exp(-B^2\tau). \quad (2-5-14)$$

The equation for heat transfer in an absorbing medium is the ordinary heat-conduction equation with a heat-source term  $I_q$  added:

$$c\gamma \frac{\partial t}{\partial \tau} = \lambda \nabla^2 t + I_q. \quad (2-5-15)$$

The heat-source strength  $I_q$  is mainly determined by the energy released during the fission of a nucleus, which may manifest itself in various ways (kinetic energy of the fission products and neutrons, energy of beta and gamma radiation, etc.).

The number of reactions, or the number of interactions between neutrons and nuclei per  $\text{cm}^3$ , inducing fission in one second are

$$v = nv\sigma N = nv\Sigma_a = \Sigma_a\Phi, \quad (2-5-16)$$

where  $N$  is the density of reacting atoms (atoms per  $\text{cm}^3$ ).

The total energy released during the fission of a single nucleus is approximately 188 Mev, of which 177 Mev are released immediately, while the release of 11 Mev is delayed (beta and gamma radiation of the fission products). The energy localized near the fission site is 173 Mev, of which 167 Mev go toward the kinetic energy of the fission products (89% of the total energy). This result shows that, to an accuracy of 10%, the heat-source strength  $I_q$  may be considered directly proportional to the neutron-source strength  $I_n$  ( $I_q = r_n I_n$ ). The proportionality factor  $r_n$ , the heat-source strength divided by the neutron-source strength, may be called the specific energy(heat) of the nuclear transformation.

The differential heat-transfer equation may now be written as

$$\frac{\partial t}{\partial \tau} = a\nabla^2 t + \frac{r_n}{c_V} I_n. \quad (2-5-17)$$

This differential equation is related to the neutron-diffusion equation, which may also be called the neutron-transfer equation,

$$\frac{\partial n'}{\partial \tau} = D\nabla^2 n' + I_n e^{\beta_a t}, \quad (2-5-18)$$

where  $n' = n \exp(-\beta_a \tau)$ . If as an approximation we express the source of thermal neutrons by means of relation (2-5-13), then the system of heat-transfer and neutron-transfer equations becomes

$$\frac{\partial t}{\partial \tau} = a\nabla^2 t + \frac{r_n k \beta_a}{c_V} n' \exp[-(\beta_a + B^2) \tau]; \quad (2-5-19)$$

$$\frac{\partial n'}{\partial \tau} = D\nabla^2 n' + k \beta_a n' \exp(-B^2 \tau). \quad (2-5-20)$$

## 2-6. The System of Differential Transfer Equations

The above systems of heat-transfer and mass-transfer equations can be represented by the following generalized system of differential transfer equations:

$$\frac{\partial \theta_1}{\partial \tau} + \vec{w} \nabla \theta_1 = R_{11} \nabla^2 \theta_1 + R_{12} \nabla^2 \theta_2 + R_{13} \nabla^2 \theta_3 + \Pi_1; \quad (2-6-1)$$

$$\frac{\partial \theta_2}{\partial \tau} + \vec{w} \nabla \theta_2 = R_{21} \nabla^2 \theta_1 + R_{22} \nabla^2 \theta_2 + R_{23} \nabla^2 \theta_3 + \Pi_2; \quad (2-6-2)$$

$$\frac{\partial \theta_3}{\partial \tau} + \vec{w} \nabla \theta_3 = R_{31} \nabla^2 \theta_1 + R_{32} \nabla^2 \theta_2 + R_{33} \nabla^2 \theta_3 + \Pi_3, \quad (2-6-3)$$

where  $\Pi_i$  ( $i=1,2,3$ ) are the source terms, which are interrelated by the expression  $\Pi_i = A_i \Pi_{i+1}$ , and  $R_{ij}$  are the thermophysical transfer coefficients, between which no reciprocal relations exist ( $R_{ij} \neq R_{ji}$ ). For specific cases these coefficients  $R_{ij}$  and the potentials  $\theta_j$  are:

a) for a moving binary gaseous mixture,

$$\theta_1 = t; \theta_2 = \rho_{10}; \theta_3 = 0; \Pi_1 = \frac{h_1 - h_2}{c_p \rho} I_1; \Pi_2 = \frac{1}{\rho_1} I_1; A_1 = \frac{h_1 - h_2}{c_p \rho} \rho_1; \quad (2-6-4)$$

$$R_{11}=a_p; R_{12}=\frac{DQ^*}{c_p}; R_{22}=D; R_{21}=\frac{DQ^*c_m}{T}; \quad (2-6-5)$$

b) for a molecular solution ( $\vec{w}=0$ ),

$$\vartheta_1=t; \vartheta_2=\rho_{10}; \vartheta_3=0; \Pi_1=\Pi_2=0; \quad (2-6-6)$$

$$R_{11}=a_p; R_{12}=\sigma D k_1; R_{21}=\sigma D; R_{22}=D; \quad (2-6-7)$$

c) for a porous body ( $\vec{w}=0$ ),

$$\vartheta_1=t; \vartheta_2=\theta; \vartheta_3=p; \Pi_1=\Pi_2=\Pi_3=0; \quad (2-6-8)$$

$$R_{11}=a_q + a_m \delta_\theta \epsilon r \frac{c_m}{c_q}; R_{12}=\epsilon r \frac{c_m}{c_q} a_m; R_{13}=a_m \epsilon r \delta_p \frac{c_m}{c_q}; \quad (2-6-9)$$

$$R_{21}=a_m \delta_\theta; R_{22}=a_m; R_{23}=a_m \delta_p; \quad (2-6-10)$$

$$R_{31}=-\frac{\epsilon c_m}{c_b} a_m \delta_\theta; R_{32}=-\frac{\epsilon c_m}{c_b} a_m; R_{33}=a_p - \frac{\epsilon c_m}{c_b} a_m; \quad (2-6-11)$$

d) for two-phase fluid flow through a porous medium ( $\vec{w}=0$ ),

$$\vartheta_1=\rho_1; \vartheta_2=\rho_2; \vartheta_3=0; \Pi_1=-\frac{\rho_1}{b_1}; \Pi_2=\frac{\rho_2}{1-b_1} \frac{\partial b_1}{\partial \tau};$$

$$A_1=-\frac{\rho_1(1-b_1)}{b_1 \rho_2}; \quad (2-6-12)$$

$$R_{11}=a_{p1}; R_{12}=R_{11}=0; R_{21}=a_{p2}; \quad (2-6-13)$$

e) for the transfer of thermal neutrons in a nuclear reactor,

$$\vartheta_1=t; \vartheta_2=n'; \Pi_1=\frac{r_n}{c\Gamma} I_n; \Pi_2=I_n e^{-\vartheta_{a^*}}; A_1=\frac{r_n}{c\Gamma} e^{\vartheta_{a^*}}; \quad (2-6-14)$$

$$R_{11}=a; R_{12}=R_{21}=0; R_{22}=D. \quad (2-6-15)$$

Consequently, the system of differential equations describing energy and mass transfer may be reduced, after certain restrictions and simplifications, to a system of equations expressed as

$$\frac{d\vartheta_i}{d\tau} = \frac{\partial \vartheta_i}{\partial \tau} + \vec{w} \nabla \vartheta_i = \sum_{j=1}^n R_{ij} \nabla^2 \vartheta_j + \Pi_i \quad (2-6-16)$$

$$(i, j = 1, 2, 3, \dots, n).$$

To solve such a system of equations it is necessary to know the initial state and the law of interaction between the surface of the body and its surroundings, as represented mathematically by the boundary conditions.

## 2-7. Boundary Conditions in Heat-Transfer Problems

First let us consider heat exchange between a solid body and a fluid flow. The differential heat-conduction equation establishes a relation between the temporal and spatial variations in the temperature of the body. To solve this equation it is necessary to know the temperature distribution inside the body at the initial moment (initial condition), the geometrical

shape of the body, and the law of interaction between the surroundings and the surface of the body (boundary condition). The combination of initial and boundary conditions constitutes the limiting conditions; the initial condition is a temporal limiting condition and the boundary condition is a spatial limiting condition.

At the surface of a body immersed in a flow of hot liquid or gas the [flow] velocity is zero ( $\vec{w}_s = 0$ ). \* The gas velocity  $w_c$  far from a surface is usually considered constant. The temperature distribution inside the body at the initial moment is given by the initial condition

$$t(x, y, z, 0) = f(x, y, z). \quad (2-7-1)$$

In many problems the temperature distribution at the initial moment is assumed to be uniform:

$$t(x, y, z, 0) = t_0 = \text{constant}. \quad (2-7-2)$$

The boundary condition may be specified in four ways:

a) The boundary condition of the first kind consists in specifying the temperature distribution over the body surface at any moment:

$$t_s(\tau) = f(\tau), \quad (2-7-3)$$

where  $t_s(\tau)$  is the temperature at the surface of the body. In the particular case when  $t_s(\tau) = t_c$  is constant, the temperature at the surface remains constant throughout the entire heat-exchange process. This may be attained by the artificial regulation of a constant temperature or by setting up special heat-exchange conditions between the surroundings and the surface of the body (see the boundary condition of the third kind).

b) The boundary condition of the second kind consists in specifying the thermal-flux density at each point on the body surface as a function of time:

$$q_s(\tau) = f(\tau). \quad (2-7-4)$$

The simplest case of the boundary condition of the second kind is when the thermal-flux density is constant:

$$q_s(\tau) = q_c = \text{constant}. \quad (2-7-5)$$

This type of heat exchange occurs when bodies are heated in high-temperature ovens, in which heat transfer takes place mainly according to the Stefan-Boltzmann law, the temperature of the body being considerably lower than the temperature of the radiating surfaces.

c) The boundary condition of the third kind consists in specifying the temperature of the surroundings and the law of heat exchange between the surface of the body and its surroundings. The boundary condition of the third kind is usually exemplified by the law for convective heat exchange between the surface of a body and its surroundings when the body is heated or cooled by a fluid flow about it. The law describing convective heat exchange is actually quite complicated, but to simplify the problem let us assume that Newton's law is valid. According to Newton's law the amount of heat transferred per unit time from a unit surface of the body to surroundings at a temperature  $t_c$  during cooling of the body ( $t_s > t_c$ ) is directly proportional to the difference between the temperature of the body surface and the

\* In the presence of a temperature drop along the body surface "thermal slip" occurs and the gas velocity at the body surface is not zero. However, the magnitude of the thermal-slip velocity is so small that it may be neglected.

temperature  $t_c(\tau)$  of the surroundings. The heat transferred is generally a function of time:

$$q_s(\tau) = \alpha [t_s(\tau) - t_c(\tau)], \quad (2-7-6)$$

where  $\alpha$  is a proportionality factor called the heat-exchange coefficient or heat-transfer coefficient.

For heating of the body it is possible to write a relation similar to (2-7-6), but in which the quantities  $t_s(\tau)$  and  $t_c(\tau)$  are interchanged. The heat-exchange coefficient is numerically equal to the amount of heat released (or absorbed) by a unit area of the body surface per unit time, for a difference of  $1^\circ$  between the temperatures of surface and surroundings. According to the law of energy conservation this amount of heat must equal the amount of heat supplied to (or moved inward from) the surface of the body by heat conduction, per unit time per unit surface area:

$$q_s(\tau) = \alpha [t_s(\tau) - t_c(\tau)] = -\lambda \left( \frac{\partial t}{\partial n} \right)_s, \quad (2-7-7)$$

where  $n$  is the normal to the surface of the body.

The boundary condition is usually written as

$$\lambda \left( \frac{\partial t}{\partial n} \right)_s + \alpha [t_s(\tau) - t_c(\tau)] = 0. \quad (2-7-8)$$

The heat-exchange coefficient depends on the velocity and temperature of the surrounding fluid, on the state of the surface (it varies along the surface of the body), and on other factors. However, as an approximation we may assume that the heat-exchange coefficient is constant, temperature-independent, and equal over the whole surface of the body.

From the boundary condition of the third kind it is possible to obtain, as a particular case, the boundary condition of the first kind. As the ratio  $\alpha/\lambda$  approaches infinity (the heat-exchange coefficient  $\alpha \rightarrow \infty$  or the thermal conductivity  $\lambda \rightarrow 0$ ) we have

$$t_s(\tau) - t_c(\tau) = \lim_{\frac{\alpha}{\lambda} \rightarrow \infty} \left[ \frac{1}{\alpha/\lambda} \left( \frac{\partial t}{\partial n} \right)_s \right] = 0, \quad (2-7-9)$$

or  $t_s(\tau) = t_c(\tau)$ , that is, the temperature of the body surface equals the temperature of the surroundings throughout the entire heat-exchange process.

This situation is observed not only if  $\frac{\alpha}{\lambda} \rightarrow \infty$  but also if  $\frac{\alpha}{\lambda} l \rightarrow \infty$ , where  $l$  is a characteristic dimension of the body. In the particular case when the temperature of the surroundings is constant equation (2-7-9) gives the boundary condition of the first kind in its simplest form:

$$t_s = t_c = \text{constant}. \quad (2-7-10)$$

d) The boundary condition of the fourth kind is exemplified by a heat exchange between the body and its surroundings which obeys the heat-conduction law or else by heat exchange between solid bodies in contact, when the temperatures of the touching surfaces are the same (ideal thermal contact). When fluid flows around a solid body the heat transfer from the fluid to the surface of the body which occurs in the immediate vicinity of the surface (laminar boundary layer or laminar sublayer) obeys the heat-conduction law (molecular heat transfer), that is, a heat exchange takes place which

corresponds to the boundary condition of the fourth kind:

$$t_s(\tau) = [t_c(\tau)]_s. \quad (2-7-11)$$

In addition to equality of the temperatures the thermal fluxes are also equal:

$$-\lambda_c \left( \frac{\partial t_c}{\partial n} \right)_s = -\lambda \left( \frac{\partial t}{\partial n} \right)_s, \quad (2-7-12)$$

where the index  $c$  stands for the (fluid) surroundings. Boundary condition (2-7-8), representing heat exchange between the surface of the solid body and the surrounding fluid, may now be written as

$$\lambda_c \left( \frac{\partial t_c}{\partial n} \right)_s + \alpha [t_s(\tau) - t_c(\tau)] = 0. \quad (2-7-13)$$

In this equation all the parameters and coefficients refer to the surroundings, since according to (2-7-11) the temperature of the surface of the body equals the temperature of the surrounding fluid at the surface.

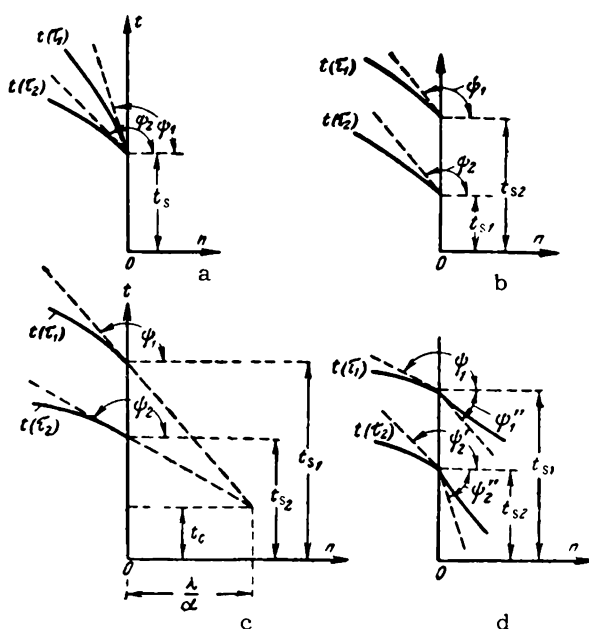


FIGURE 2-4. The various types of boundary conditions

- a)  $t_s = t_c$  constant,  $\tan \phi$  variable; b)  $t_s$  variable,  $\tan \phi = q/\lambda$  constant; c)  $t_s$  variable,  $\tan \phi$  variable,  $\lambda/\alpha$  constant; d)  $t_s$  variable,  $\tan \phi'/\tan \phi''$  constant

A graphical interpretation of the four types of boundary conditions, for the simplest case (constant  $t_c$ ), is given in Figure 2-4. The scalar magnitude of the thermal-flux vector is proportional to the absolute value of the temperature gradient, which is numerically equal to the tangent of the slope angle of the curve for the temperature distribution along the normal to the isothermal surface:

$$\left( \frac{\partial t}{\partial n} \right)_s = \tan \psi_n. \quad (2-7-14)$$



Figure 2-4 shows an area element  $\Delta A$  of the surface of the solid body and also the temperature distributions near the surface  $t(\tau_1)$  and  $t(\tau_2)$  at two different times during the cooling process. The normal  $n$  is directed away from the surface (external normal) of the body, so that the derivatives  $\partial t/\partial n$  are negative.

The boundary condition of the first kind specifies the function  $t_s(\tau)$ ; in the simplest case  $t_s$  is constant. The slope of the temperature curve at the surface of the body is measured, and then the amount of heat given off by the surface of the body is determined (Figure 2-4, a).

Problems involving boundary conditions of the second kind are just the opposite. The slope of the temperature curve at the surface of the body is given (Figure 2-4, b) and the temperature of the surface is determined from it.

In problems involving boundary conditions of the third kind the temperature of the surface and the slope of the temperature curve are variable, but all lines drawn tangent to the temperature curve must pass through some point  $C$  on the external normal (Figure 2-4, c). Then, it follows from the boundary condition that

$$\tan(90^\circ - \psi_s) = -\left(\frac{\partial t}{\partial n}\right)_s = \frac{t_s(\tau) - t_c}{\lambda/a}. \quad (2-7-15)$$

The tangent of the angle complementary to the slope angle of the temperature curve at the surface of the body is equal to the opposite side  $t_s(\tau) - t_c$  divided by the adjacent side  $\lambda/a$  of the corresponding right triangle (Figure 2-4, c). The adjacent side  $\lambda/a$  is constant, whereas the opposite side  $[t_s(\tau) - t_c]$  varies continuously during the heat-exchange process and is directly proportional to  $\tan \psi_s$ . Consequently, the position of reference point  $C$  remains unchanged.

In problems involving boundary conditions of the fourth kind (Figure 2-4, d) the ratio of the slopes of the temperature curves in the body and in the medium, at the interface between them, assuming perfect thermal contact (the tangents at the surface pass through the same point), is given:

$$\frac{\tan \psi'_s}{\tan \psi''_s} = \frac{\lambda}{\lambda_c} = \text{constant}, \quad (2-7-16)$$

where the prime refers to the body and the double prime refers to the medium.

The differential heat-conduction equation together with the initial and boundary conditions completely determine the problem. In other words, if the geometrical shape of the body and the initial and boundary conditions are known, then it is possible to solve the equation completely, that is, to specify the temperature-distribution function inside the body at any moment. The temperature  $t_c$  of the surroundings must also be given. If the temperature of the moving fluid varies as a result of heat transfer from the solid body, then it becomes necessary to solve not only the heat-conduction equation for the solid body but also, simultaneously, the equation describing heat transfer in the moving medium and the Navier-Stokes and continuity equations. The solution of the latter equations is necessary when the fields of temperature and velocity in the moving medium are introduced into the problem.

Thus, the existence of specified limiting conditions makes it possible in the simplest cases to obtain an analytical solution describing heat conduction,

by determining the form of the function

$$t=f(x, y, z, \tau). \quad (2-7-17)$$

The function  $f(x, y, z, \tau)$  must satisfy the initial and boundary conditions as well as the differential equation. When this function is substituted for  $t$  in the differential heat-conduction equation, the equation is transformed into an identity. According to the uniqueness theorem, if some function  $f(x, y, z, \tau)$  satisfies a differential equation and the initial and boundary conditions, then it represents a unique solution of the problem.

In conclusion let us consider some methods for calculating the heat losses of a body which becomes cooled during heat exchange.

a) First calculation method. A surface element  $dA$  gives off during a time  $d\tau$  an amount of heat

$$-\lambda \left( \frac{\partial t}{\partial n} \right) dA d\tau. \quad (2-7-18)$$

To find the amount of heat  $\Delta Q$  released by the body during a time interval  $\Delta\tau = \tau_2 - \tau_1$  it is necessary to integrate the quantity (2-7-18) over the entire surface  $A$  and over the time interval  $\Delta\tau$ :

$$\Delta Q = - \int_{\tau_1}^{\tau_2} \int_A \lambda \left( \frac{\partial t}{\partial n} \right) dA d\tau. \quad (2-7-19)$$

If the temperature and temperature gradient do not vary along the surface, then equation (2-7-19) simplifies to

$$\Delta Q = Q_2 - Q_1 = -\lambda A \int_{\tau_1}^{\tau_2} \left( \frac{\partial t}{\partial n} \right) d\tau. \quad (2-7-20)$$

b) Second calculation method. During a time  $d\tau$  a surface element  $dA$  gives off to its surroundings an amount of heat

$$a(t_s - t_o) dA d\tau. \quad (2-7-21)$$

To find the total amount of heat released, it is necessary to integrate over the whole surface and over the time interval:

$$\Delta Q = \int_{\tau_1}^{\tau_2} \int_A a(t_s - t_o) dA d\tau. \quad (2-7-22)$$

If the temperature is constant over the surface, then the formula for the heat transfer simplifies to

$$\Delta Q = \bar{a} A \int_{\tau_1}^{\tau_2} (t_s - t_o) d\tau, \quad (2-7-23)$$

where  $\bar{a}$  is the average heat-exchange coefficient over the surface. If the heat-exchange coefficient  $a$  does not vary over the surface of the body, then the local coefficient equals the average value ( $a = \bar{a}$ ).

c) Third calculation method. During a time  $\Delta\tau$  a volume element  $dv = dx dy dz$  becomes cooled from a temperature  $t_1$  to  $t_2$ , and releases a heat

$$c\gamma = (t_1 - t_2) dv. * \quad (2-7-24)$$

\* [This should read  $c\gamma (t_1 - t_2) dv$ .]

Integration over the entire volume  $V$  gives the total amount of heat  $\Delta Q$  given off by the body during  $\Delta\tau$ :

$$\Delta Q = c\gamma \int_{(V)} (t_1 - t_2) dv = c\gamma V \cdot \frac{1}{V} \int_{(V)} (t_1 - t_2) dv. \quad (2-7-25)$$

If the average (integral) temperature over the entire volume of the body is defined as

$$\bar{t} = \frac{1}{V} \int_{(V)} t dv, \quad (2-7-26)$$

then we may write

$$\Delta Q = c\gamma V (\bar{t}_1 - \bar{t}_2). \quad (2-7-27)$$

The heat transfer from the body during a time  $\tau$  after beginning ( $\tau=0$ ) of the cooling is

$$Q - Q_0 = c\gamma V (\bar{t} - \bar{t}_0), \quad (2-7-28)$$

where  $\bar{t}_0$  is the average (integral) initial temperature. Consequently, the calculation of the heat transfer reduces to a determination of the average temperature over the volume.

In the case of symmetrical heat exchange, when the temperature field is one-dimensional, the average (integral) temperature may be determined from the following formulas:

a) for a plate with dimensions  $2R \times 2H \times 2L$  (origin of coordinate system at plate center),

$$\begin{aligned} \bar{t}(\tau) &= \frac{1}{V} \int_{(V)} t(x, y, z, \tau) dv = \frac{1}{2R \cdot 2H \cdot 2L} \int_{-R}^{+R} \int_{-H}^{+H} \int_{-L}^{+L} t(x, y, z, \tau) dx dy dz = \\ &= \frac{1}{2R} \int_{-R}^{+R} t(x, \tau) dx = \frac{1}{R} \int_0^R t(x, \tau) dx; \end{aligned} \quad (2-7-29)$$

b) for a sphere of radius  $R$  (with a volume element  $dv = r^2 \sin \theta d\theta d\psi dr$ ),

$$\bar{t}(\tau) = \frac{3}{4\pi R^3} \int_0^R \int_0^{2\pi} \int_0^\pi t(r, \theta, \psi, \tau) r^2 \sin \theta d\theta d\psi dr = \frac{3}{R^3} \int_0^R r^2 t(r, \tau) dr; \quad (2-7-30)$$

c) for a cylinder of radius  $R$  and length  $L$  (with a volume element  $dv = r dr d\theta dz$ ),

$$\bar{t}(\tau) = \frac{1}{\pi R^2 L} \int_0^R \int_0^{2\pi} \int_0^L t(r, \theta, z, \tau) r dr d\theta dz = \frac{2}{R^2 L} \int_0^R \int_0^L r t(r, z, \tau) dr dz. \quad (2-7-31)$$

For an infinite cylinder  $L \gg 2R$  this simplifies to

$$\bar{t}(\tau) = \frac{2}{R^2} \int_0^R r t(r, \tau) dr. \quad (2-7-32)$$

## 2-8. Boundary Conditions in Problems of Mass and Heat Transfer

The transfer of mass is determined by a difference in mass-transfer potential. The potential corresponding to diffusion transfer in a gaseous mixture is the ratio of the chemical potential  $\mu$  to the absolute temperature. If we neglect thermal-diffusion and diffusive-heat-conduction effects, then mass transfer is determined by the gradient  $\nabla \mu_s$  of the specific mass content, while molecular heat transfer is determined by the temperature gradient.

Just as in the case of the boundary conditions for heat exchange, the mass-exchange interaction of the surface of a body with its surroundings may be represented by four kinds of boundary conditions. Boundary conditions of the first kind correspond to the case when the diffusive-mass-transfer potential at the surface of the body equals the mass-transfer potential in the surroundings:

$$\mu_s = \mu_c. \quad (2-8-1)$$

Boundary conditions of the second kind are represented by the mass exchange during the drying of a moist body, in which the drying intensity is constant for a while and then decreases. In this case the boundary conditions specify the mass-flux density as a function of time:

$$\vec{j}_m = f(\tau). \quad (2-8-2)$$

In particular cases  $\vec{j}_m$  may be constant.

Mass-exchange boundary conditions of the third kind are similar to heat-exchange boundary conditions. For example, at the boundary between a scattering medium and a vacuum the neutron flux is

$$D \left( \frac{\partial n}{\partial x} \right)_s + \alpha_s n_s = 0, \quad (2-8-3)$$

where  $\alpha_s$  is a constant which may be called the coefficient of neutron exchange.

Boundary conditions of the fourth kind characterize molecular exchange between two media. In the case of neutron exchange the conditions are

$$\Phi_1 = \Phi_2 \quad \text{and} \quad D_1 (\nabla \Phi_1)_s = D_2 (\nabla \Phi_2)_s, \quad (2-8-4)$$

where subscripts 1 and 2 denote the contacting media, and  $s$  denotes the boundary between them.

Let us consider the boundary conditions for mass and heat exchange in some particular cases.

1. When liquid evaporates from a free surface which is immersed in a laminar flow of moist air, the interaction of the liquid surface with the moist air obeys boundary conditions of the third kind. The moist air may be considered as a binary vapor-gas mixture ( $\rho_1 + \rho_2 = \rho$ ) with a constant total pressure ( $p = \text{const}$ ). The amount of evaporating liquid is determined from a relation similar to Newton's law for convective heat exchange:

$$\vec{j}_m = \alpha_\mu (\tilde{\mu}_s - \tilde{\mu}_c) = \alpha_m (\rho_{1s} - \rho_c), \quad (2-8-5)$$

where  $\tilde{\mu} = \mu_1 - \mu_2$  is the reduced chemical potential,  $\alpha_\mu$  is the mass-exchange coefficient calculated for the chemical-potential difference, and  $\alpha_m$  is the

mass-exchange coefficient (m/hr), a quantity similar to the heat-exchange coefficient, referred to the difference in the volume enthalpy concentrations  $a_h$  (m/hr). This amount of vapor, which leaves the surface of the liquid, can be calculated as follows. The vapor produced during vaporization diffuses into the moist air, the diffusion-flux density being equal to

$$\vec{j}_{\text{diff}} = -\rho D_{12} \nabla \rho_{10}. \quad (2-8-6)$$

The presence of the gradient  $\nabla \rho_{10}$  causes in turn an equal but opposite gradient of the mass content of the air:

$$\nabla \rho_{10} = -\nabla \rho_{20}, \text{ where } \rho_{10} + \rho_{20} = 1, \quad (2-8-7)$$

and consequently also causes a diffusion flux of air:

$$\vec{j}_{\text{diff}} = -\vec{j}_{\text{diff}} = -\rho D_{21} \nabla \rho_{20}. \quad (2-8-8)$$

If the vapor diffuses freely into the surrounding air, then for the latter the liquid surface represents an impenetrable barrier. As a result, the amount of dry air at the surface continually increases. Since the moist-air pressure is constant, therefore in order to fulfill this condition it is necessary for the entire vapor-gas mixture to take part in a macroscopic (convective) transfer. Via this convective motion vapor and air are simultaneously released from the liquid surface into the surroundings. If the velocity of this convective motion is denoted as  $\vec{w}_n$ , then the total vector flux, which is the sum of the molecular and convective vapor fluxes, may be written as

$$\vec{j}_1 = D_{21} \rho \nabla \rho_{10} + \rho_1 \vec{w}_n. \quad (2-8-9)$$

The total air flux at the water surface is zero:

$$\vec{j}_2 = D_{21} \rho \nabla \rho_{20} + \rho_2 \vec{w}_n = 0. \quad (2-8-10)$$

Now, keeping in mind relation (2-8-7) and the equality of the interdiffusion coefficients ( $D_{12} = D_{21}$ ), we obtain from (2-8-9) and (2-8-10) an expression for the convective-transfer velocity:

$$\vec{w}_n = -\frac{D_{12}}{1 - \rho_{10}} \nabla \rho_{10}. \quad (2-8-11)$$

Thus, the total vapor flux is

$$\vec{j}_1 = -D_{12} \rho \frac{1}{1 - \rho_{10}} \nabla \rho_{10}. \quad (2-8-12)$$

Here, the convective vapor transfer is taken into account by the factor

$\left(\frac{1}{1 - \rho_{10}}\right)$ . If the molecular vapor transfer is taken as unity, then the fraction corresponding to the convective vapor transfer is  $\rho_{10}/(1 - \rho_{10})$ . The amount of vapor given by (2-8-12) equals the amount of vapor given by (2-8-5). Thus, the two equations may be equated to obtain the boundary condition in the form

$$D_{12} \rho \left(\frac{1}{1 - \rho_{10}}\right)_s [\nabla \rho_{10}]_s + \alpha_m \rho [(\rho_{10})_s - (\rho_{10})_c] = 0. \quad (2-8-13)$$

Let us now compare this relation with the boundary condition for heat exchange, which may be written as

$$\alpha_p (\nabla h)_s + \alpha_h \rho (h_s - h_c) = 0, \quad (2-8-14)$$

where  $h$  is the specific enthalpy.

All the parameters and coefficients refer to the surroundings. A comparison of (2-8-13) and (2-8-14) shows that the boundary condition for mass exchange contains a correction for the convective mass transfer (the Stefan flux), the factor  $1/(1 - \rho_{10})$ . In the case of pure heat exchange, which is not complicated by mass exchange, when a plane-parallel gas stream flows over the surface of the body, the component  $[\vec{w}_n]_s$  of the velocity vector at the surface of the body is small, and can usually be neglected. Therefore, boundary condition (2-8-14) does not contain a similar correction for the convective heat transfer.

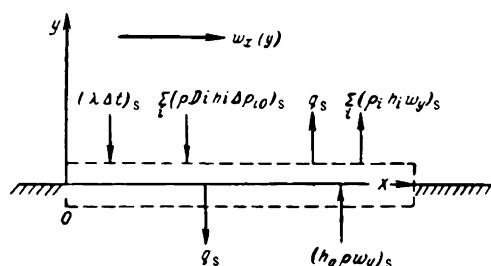


FIGURE 2-5. Boundary conditions for mass-exchange cooling of a porous wall

2. The boundary conditions for a system in which there is mass-exchange cooling are obtained from the equation for energy balance at the boundary surface of the body. Mass-exchange cooling is understood to mean the cooling of a heated boundary layer by conveying masses of cold air to it. In porous cooling the coolant gas is admitted to the boundary layer through a porous wall. In evaporation cooling the coolant gas is generated at the surface of the solid body by sublimation or by chemical reactions in which the wall material itself takes part. In liquid-film cooling gas is obtained by evaporation of liquid from the surface of a film which covers the wall.

An analysis of the solutions and the related calculations describing mass-exchange cooling has shown that heat exchange with a wall may be reduced considerably by supplying a coolant mass. Particularly suitable for this purpose are gases with a low molecular weight, especially hydrogen.

Let us consider that a fluid (gaseous mixture) flows past a plane porous wall. In order to cool the boundary layer a coolant gas is fed through the porous wall. In this case, according to Eckert [2/], the boundary conditions for a multicomponent gaseous mixture are (Figure 2-5):

$$q_s = \left( \lambda \frac{\partial t}{\partial y} \right)_s + \left[ \sum_i h_i \rho_i D_i \frac{\partial \rho_{i0}}{\partial y} \right]_s + h_{s0} (\rho w_y)_s - \sum_i (\rho_i h_i w_{yi})_s - q_r, \quad (2-8-15)$$

where  $q_s$  is the density of the heat flux going off into the wall;  $q_r$  is the density of the radiation heat flux from the wall to the surroundings,  $h_0$  is the specific enthalpy of the coolant gas, and the subscript  $s$  once again denotes the surface of the wall. The first term on the right side of balance equation (2-8-15) represents the amount of heat supplied to the wall surface by conduction, while the second term determines the amount of enthalpy

transferred by interdiffusion. The third term represents the amount of enthalpy supplied by the coolant gas and the fourth term represents macroscopic transfer of the mixture enthalpy  $h_p$  (where  $h_p = \sum_i p_i h_i$ ), taking place normal to the surface and with a velocity  $w_y$ .

For a binary gaseous mixture ( $i=1, 2$ ) this enthalpy-balance equation becomes

$$q_s = \left( \lambda \frac{\partial t}{\partial y} \right)_s + \left[ \rho D_{12} (h_1 - h_2) \frac{\partial p_{10}}{\partial y} \right]_s + h_{0s} (\rho w_y)_s - [(p_1 h_1 + p_2 h_2) w_y]_s - q_r. \quad (2-8-16)$$

We may now use (2-8-9) and (2-8-10) to determine the total macroscopic-molecular flux of a two-component gas (air and coolant gas):

$$j_m = j_1 + j_2 = (\rho_1 w_y)_s - \left( \rho D_{12} \frac{\partial p_{10}}{\partial y} \right)_s = [(\rho_1 + \rho_2) w_y]_s = (\rho w_y)_s. \quad (2-8-17)$$

Here it is assumed that the gas (air) washing the porous wall does not diffuse into the wall in the way that air diffuses into the liquid when liquid evaporates from a free surface. Boundary condition (2-8-16) then becomes

$$q_s = \left( \lambda \frac{\partial t}{\partial y} \right)_s - (h_1 - h_s) (\rho w_y)_s - q_r. \quad (2-8-18)$$

For evaporation cooling the quantity  $(\rho w_y)_s$  equals the velocity of recession of the evaporation surface into the wall, and  $h_{1s}$  is the enthalpy of the solid body or liquid.

3. For the heating of a moist porous body, when the internal macroscopic-molecular moisture transfer is due to gradients of the potentials  $\theta$ ,  $t$ , and  $p$ , the boundary conditions for heat and mass exchange are

$$-\lambda_q (\nabla t)_s + [j_q(\tau)]_s - [r j_{m2}(\tau)]_s = 0; \quad (2-8-19)$$

$$\lambda_m [(\nabla \theta)_s + \delta_\theta (\nabla t)_s + \delta_p (\nabla p)_s] + [j_m(\tau)]_s = 0; \quad (2-8-20)$$

$$p_s = p = \text{constant}, \quad (2-8-21)$$

where  $\lambda_q$  and  $\lambda_m$  are the thermal and mass conductivities of the porous body,  $j_q(\tau)$  is the amount of heat supplied to the surface of the solid body,  $j_m(\tau)$  is the density of the moisture flux leaving the surface of the body (the vaporization rate), and  $j_{m2}(\tau)$  is the liquid-flux density.

Boundary condition (2-8-19) constitutes a heat-balance equation. The heat  $\vec{j}_q(\tau)$  supplied to the surface of the body is used for liquid evaporation  $r \vec{j}_{m2}(\tau)$  and for heating the body  $[-\lambda_q \nabla t]_s$ . This equation is similar to boundary condition (2-8-18) for the case of evaporation cooling, since the heat flux  $q_s$  at the surface of the body is approximately equal to  $\left[ -\lambda_q \left( \frac{\partial t}{\partial y} \right) \right]_s$ , while the amount of supplied heat  $j_q$  equals  $[a(t_c - t_s)]$ . If the heat exchange between the surface of the body and its surroundings takes place according to the convection law, then

$$[j_q(\tau)]_s = a(t_c - t_s). \quad (2-8-22)$$

Similarly, the vaporization rate may be written as

$$[j_m(\tau)]_s = a_m[p_s - p_c] = a_\theta(\theta_s - \theta_c), \quad (2-8-23)$$

where  $\alpha_g$  is the mass-exchange coefficient referred to a difference in the potentials  $\theta$ .

Boundary condition (2-8-20) constitutes a mass-balance equation, while boundary condition (2-8-21) represents the constancy of the total pressure of the moist air.

## 2-9. Methods of Solving Differential Transfer Equations

The system of differential transfer equations together with the initial and boundary conditions represents in analytical form the basic features of a given process, that is, it constitutes a mathematical model. Solution of the model makes it possible to obtain a complete picture of the distribution of transfer potentials in the body or system of bodies, to trace the time variation of the potential fields, and on this basis to analyze in detail the kinetics and dynamics of the process. No empirical methods of study or approximate methods of semiempirical nature can replace these analytical methods. The great successes achieved in thermophysics during recent years are directly related to the widespread use of the analytical theory, the importance of which increases constantly. Therefore, the working out of reliable and effective methods for solving boundary-value problems in transfer theory represents one of the vital and important problems in thermophysics.

Of extreme value are methods for solving the systems of differential heat-transfer and mass-transfer equations in their finite form (the various integration methods). A complete solution makes it possible to investigate quite simply the effects of individual parameters on the course of the process and to find relations between the most important characteristics. In cases when it is impossible to solve a problem in this way, methods of numerical solution or methods of experimental analogs must be used. The importance of numerical methods for the solution of various boundary-value problems has especially increased in recent years in connection with the intensive development and implementation of electronic computers. The choice of the solution method depends on the specific problem, on the requirements imposed on the calculation data, and on the estimated time necessary to obtain a solution of given accuracy.

Since this monograph is devoted to working out an analytical theory of heat and mass transfer, let us now consider the basic analytical methods for solving differential transfer equations.

Until recently the equations describing heat conduction and diffusion were usually solved using the method of separation of variables, the method of instantaneous sources, or methods based on applications of Green's function, Dirac's function, and others. These classical methods first seek a general solution and then attempt to apply it to the conditions of a specific problem. A detailed discussion of classical methods for solving the transfer equations is given in the basic work by Tikhonov and Samarskii /7/. The solutions obtainable by classical methods are, however, not always convenient for practical application. Thus, it is sometimes necessary to obtain approximate relations, in which the regime parameters of the process must be



separated from the physical characteristics of the body or system of bodies which interacts with its surroundings. These relations, which are so important in practice, are difficult to obtain from the classical solutions. Even greater complications arise when systems of differential heat-transfer and mass-transfer equations are solved by classical methods. Due to the exigencies of engineering research engineers and physicists have found necessary, during the last decades, a widespread application of operational methods of solution. The basic rules and theorems of the operational calculus were derived by Vashchenko-Zakharchenko /8/, and they have found very wide application in electrical engineering, thanks to the work of Heaviside. This method was so effective that it made it possible to solve many problems which were considered insoluble before its appearance and also to obtain solutions of some solved problems in forms considerably more adaptable to numerical calculations.

Later, operational methods were applied in heat engineering to solve various problems of nonstationary [unsteady] heat conduction /9 to 11/ and in chemical engineering to solve unsteady-state-diffusion problems /12/. In recent years these methods have been used more and more in hydrodynamics, neutron-transport theory, and other fields.

A rigorous mathematical basis for the operational methods was derived by Efros and Danilevskii /13/, Ditkin /14, 15/, Doetsch /16, 17/, van der Pol /18/, and others. At present these may be considered as independent methods for solving the equations of mathematical physics, equivalent in rigorousness to the classical methods. In particular, the operational method of Vashchenko-Zakharchenko and Heaviside is equivalent to the Laplace integral-transformation method.

The Laplace transformation method consists in studying not the function itself (the original) but its modification (the transform). This modification or transform is obtained by multiplying by some exponential function and integrating it between given limits. Thus, the Laplace transformation is an integral transformation, defined as

$$L[f(\tau)] = \{f(s)\}_L = \int_0^{\infty} f(\tau) e^{-s\tau} d\tau, \quad (2-9-1)$$

where  $L[f(\tau)] = \{f(s)\}_L$  is the transform of the original function  $f(\tau)$ , and  $s = \xi + i\eta$  is some complex quantity. In order for the transform to exist the integral on the right side of (2-9-1) must converge. This imposes certain restrictions on the function  $f(\tau)$ ; for instance, it must be a piecewise-continuous function. The basic properties of Laplace transformations may be found in the previously cited references.

If a problem has been solved by means of transforms, the original function may also be determined from the transform (inverse transformation). This is generally carried out using the inversion formula

$$f(\tau) = L^{-1}[\{f(s)\}_L] = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{s\tau} \{f(s)\}_L ds. \quad (2-9-2)$$

The integration in equation (2-9-2) is carried out along a straight line  $\sigma = \text{constant}$  in the complex  $s = \xi + i\eta$  plane, a line parallel to the imaginary axis and located in the half-plane  $\text{Re } s \geq \sigma_1 > \sigma_0$ . Methods for performing this

integration are described in detail in special handbooks on the theory of functions of a complex variable. In most practical cases inverse transformations can be carried out without using line integral (2-9-2).

The original function may be found quite rapidly from its transform if the latter is a transform which has been tabulated [15, 17, 19]. In this case it is necessary to know a relation which makes it possible to find the original function when its transform has the form  $\Phi(s)/\Psi(s)$ , where  $\Psi(s)$  is an  $n$ th-degree polynomial with respect to  $s$ . This problem may be solved using expansion theorems. In particular, if  $\Psi(s)$  has simple roots  $s_n$ , then the appropriate expansion theorem is

$$f(\tau) = L^{-1} \left[ \frac{\Phi(s)}{\Psi(s)} \right] = \sum_{n=1}^{\infty} \frac{\Phi(s_n)}{\Psi'(s_n)} e^{s_n \tau};$$

if a root  $s_m$  of the polynomial  $\Psi(s)$  has a multiplicity  $k$ , then the original of the corresponding term is given by the formula

$$f_m(\tau) = L^{-1} \left[ \frac{C_m}{(s - s_m)^k} \right] = \frac{1}{(k-1)!} \lim_{s \rightarrow s_m} \left\{ \frac{d^{k-1}}{ds^{k-1}} \left[ \frac{\Phi(s) \cdot (s - s_m)^k}{\Psi(s)} e^{s\tau} \right] \right\}.$$

The application of Laplace integral transformations to the solution of differential heat-conduction and mass-conduction equations, as well as to the solution of systems of differential heat-transfer and mass-transfer equations for the presence of phase and chemical transformations, has a number of advantages over the classical methods for integrating differential equations and also over certain other methods of integral transformation.

First, the way in which Laplace integral transformations are applied remains the same for problems of quite different nature and for different body shapes. Thus, the solution method is more straightforward, and does not require special skill and a new approach for the solution of each new type of problem. Second, Laplace integral transformations make it possible to solve with equal proficiency problems involving boundary conditions of the first, second, third, and fourth kinds, without the introduction of any new assumptions or transformations. Third, the existence of a large number of simple theorems enables us to obtain results which are the most suitable under the specific conditions; in particular, solutions in a form convenient for calculation with small or large time intervals may be obtained.

Fourth, the method makes it possible to solve with particular ease problems having simple initial conditions. Laplace transformations are applied most effectively with respect to the time coordinate, and also with respect to the spatial coordinate for a body which is unlimited or semi-limited in extent. Fifth, the effectiveness of solving various problems using Laplace transformations is increased considerably by the existence of detailed transform tables.

Of course, Laplace integral transformations also have certain disadvantages. In particular, difficulties arise during the solution of problems in which the initial conditions are specified as functions of the spatial coordinates or when it is necessary to solve certain multidimensional problems. In relation to this, a number of methods of integral transformation with respect to spatial coordinates, as dictated by the geometrical shape of the body, have been suggested. Outside of the USSR such transformations

have been proposed by Doetsch /20/, Sneddon /21/, Tranter /22/, and others, and have been used by these authors to solve various problems in mathematical physics. A number of works in this direction have also been published in the Soviet Union /23 to 27 and others/.

If the transformation is made with respect to the spatial coordinate  $x$ , then the integral transformation of a function  $f(x)$  may be represented as

$$\{f(p)\}_{F,H} = \int_0^{\infty} K(p, x) f(x) dx. \quad (2-9-3)$$

If the kernel  $K(p, x)$  of the transformation is expressed in terms of  $\sin px$  or  $\cos px$ , then this integral transformation is called a Fourier sine transformation or a Fourier cosine transformation. If the kernel of the transformation is a Bessel function  $K(p, x) = x J_0(px)$ , then it is called a Hankel transformation. In the particular case when the integration limits are from  $-\infty$  to  $+\infty$  and when  $K(p, x) = e^{ikx}$ , we obtain a complex Fourier integral transformation. Complex Fourier transformations are convenient for problems involving infinite bodies; Fourier sine transformations should be used when the value of the function is specified at the surface of the body (boundary conditions of the first kind), whereas Fourier cosine transformations are useful in solving differential transfer equations with boundary conditions of the second kind. Hankel transformations are used when the body has axial symmetry. The practical application of the above integral transformations does not involve any particular difficulty once good transform tables /28, 15/ have been compiled.

In cases when the use of a Fourier transformation is justified but the values of the required transforms are not available, the originals of the transforms may be found using the following quite simple inversion formulas:

for a complex Fourier transformation,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_p(t) e^{-itx} dt,$$

for a Fourier sine transformation,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(p) \sin px \, dp,$$

for a Fourier cosine transformation,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(p) \cos px \, dp,$$

for a Hankel transformation,

$$f(r) = \int_0^{\infty} r \{f_s(p)\}_H J_0(pr) \, dp.$$

It is characteristic of all these transformations that the upper integration limit is infinity. Whereas in the Laplace transformation (2-9-1), which is in most cases carried out with respect to the time coordinate, the infinite

integration limit is due to the very nature of an unsteady process, in the Fourier and Hankel transformations (2-9-3), carried out with respect to the spatial coordinates, the presence of an infinite limit restricts the range of application of these methods. In other words, integral transformation (2-9-3) can be used successfully only in systems having semi-infinite extent. In addition, it should be noted that when a Fourier transformation is applied, especially a sine or cosine transformation, convergence of the integrals is extremely important, since the convergence conditions here are more rigid than the convergence conditions for the corresponding integrals in a Laplace transformation.

The limited application of the Fourier and Hankel integral transformations and in part of the Laplace transformations, on one hand, together with the acute necessity of solving problems with a finite range of variation of the variables, on the other hand, have led to the development of methods involving finite integral transformations. Even in cases when these methods pertain to the range of problems which are soluble by classical methods (using a Fourier or Fourier-Bessel series) they should be preferred. The simplicity of the solution technique, together with its "standardness", ensures that methods using finite integral transformations are much more advantageous than classical methods.

The type of finite integral transformation denoted as

$$\{f(p)\}_{F,H} = \int_a^b K(p, x) f(x) dx$$

was first formulated by Koshlyakov /29/. Later, a comprehensive theory describing such integral transformations was worked out by Grinberg /30/, who generalized the methods used to include the case of a discontinuous variation in the properties of the medium in the direction of the coordinate with respect to which the transformation is carried out. A detailed development of integral transformations with finite limits has been carried out by Sneddon /21/, Tranter /22/, Doetsch /20/, Scott /31/, Conte /32/, and others.

If integration takes place from 0 to  $l$ , then the kernels of the finite Fourier sine and cosine transformations and the finite Hankel transformation are respectively:

$$K(p, x) = \sin \frac{n\pi x}{l};$$

$$K(p, x) = \cos \mu x$$

(for type-II boundary conditions  $\mu = \frac{n\pi}{l}$ , while for type-III boundary conditions  $\mu$  represents the roots of the equation  $\mu \tan l\mu = \frac{\alpha}{\lambda}$ ); and

$$K(p, x) = r J_\nu(\mu x),$$

where  $\mu$  is the root of the equation  $J_\nu(l\mu) = 0$ ; for type-III boundary conditions the roots  $\mu$  are determined from the equation

$$\frac{J_\nu(\mu l)}{J'_\nu(\mu l)} = -\frac{\mu}{\alpha/\lambda}. \quad (2-9-4)$$

The inversion formula is usually found by means of an expansion of the function in the series of orthogonal functions of the corresponding Sturm-Liouville problem. Therefore solutions obtained by these methods have

the same fundamental disadvantages as solutions obtained by classical methods. For instance, the inversion formulas are:

for sine transformations,

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{l};$$

for cosine transformations,

$$f(x) = \frac{1}{l} f_e(0) + \frac{2}{l} \sum_{n=1}^{\infty} f_e(n) \cos \frac{n\pi x}{l}$$

for type-II boundary conditions, and

$$f(x) = 2 \sum_{\mu} \frac{\mu^2 + (\alpha/\lambda)^2}{l[\mu^2 + (\alpha/\lambda)^2] + \alpha/\lambda} f_e(\mu) \cos \mu x.$$

for type-III boundary conditions, where the summation is over all the positive roots of the equation  $\mu \tan l\mu = \alpha/\lambda$ ;

and for Hankel transformations,

$$f(x) = \frac{2}{l^2} \sum_{\mu} f_H(\mu) \frac{J_{\nu}(\mu x)}{[J'_{\nu}(l\mu)]^2},$$

where the summation is over all the positive roots of  $J_{\nu}(l\mu) = 0$ , or

$$f(x) = \frac{2}{l^2} \sum_{\mu} f_H(\mu) \frac{\mu^2}{(\alpha/\lambda)^2 + \left[\mu^2 - \frac{\nu^2}{l^2}\right]} \frac{J_{\nu}(\mu x)}{[J'_{\nu}(l\mu)]^2},$$

where the summation is over all the positive roots of (2-9-4).

To overcome the previously mentioned difficulties, various methods of approximate integral transformations have been worked out, in which the direct and inverse transformations are accomplished using approximate formulas. At the same time, it has been suggested that a method of so-called finite Laplace integral transformations, or rather finite Green integral transformations, be developed. Let us consider this in more detail.

The inversion formula for the Laplace integral transformation is in general the Riemann-Mellin integral (2-9-2). This formula makes it possible to obtain solutions in some desired form, such as a closed form. The method consists in choosing the kernel  $K(p, x)$  of the integral transformation in accordance with the differential equation and boundary conditions, thereby taking into account the geometrical shape of the body and the law describing its interaction with its surroundings. In other words, the transformation kernel is a Green function for the given problem. The transform of the function  $f(x)$  is obtained by means of the integral transformation

$$\{f(p)\}_0 = \int_0^l K(p, x) f(x) dx,$$

while the inverse transformation is performed according to formula (2-9-2), but with  $\{f(p)\}_0$  substituted in place of  $\{f(s)\}_L$ .

This means of integral transformation has a physical justification. Actually, any integral transformation carried out with respect to spatial

coordinates represents, from the physical point of view, some sort of averaging of the physical quantity being studied. It is quite natural, moreover, that this averaging should be carried out not only in accordance with the nature of the process and the shape of the body (with the form of the differential equation) but also in accordance with the boundary conditions.

In this case the solution for the transform of the function will be of independent interest, since such a transformation corresponds, from the physical point of view, to a transition from an analysis of the actual values of the given functions (the differential equation and the uniqueness conditions) to average values, in perfect accordance with the actual conditions of the physical problem. Thus, integral-transformation methods have another, quite considerable, advantage over classical methods, since they make it possible to derive a number of regularities in the development of physical processes on the basis of analysis of the solution for the averaged values of a given physical quantity (analysis of the solution for the transform). This fact brings the results of analytical methods closer to the results of the methods used in similarity theory, a subject which will be discussed in the next chapter.

Integral transformations possess special advantages for the solution of systems of partial differential equations. The means of solution of a system of equations is in principle the same as that of an individual equation and is carried out in several successive operations. For example, for one-dimensional heat-transfer and mass-transfer problems, in which the parameters are functions of the spatial coordinates and of time, the following operations must be carried out.

1. By analyzing the system of equations and the boundary conditions a suitable integral transformation or group of integral transformations is selected.

2. The differential equations and boundary conditions are multiplied by the chosen transformation kernel and the resulting expressions are integrated, within appropriate limits, with respect to the variable to be eliminated. As a result, instead of the system of partial differential equations in terms of the original functions, we obtain a system of ordinary differential equations in transforms, which takes into account the initial (when using the Laplace transformation) or boundary (when using the Fourier transformations) conditions.

3. The system of ordinary differential equations is solved with respect to the transformed functions. (If a solution of the obtained system is difficult, then a suitable integral transformation with respect to a second independent variable must be applied to it. As a result of this transformation we obtain a system of algebraic equations, whose solution is more elementary. After expressions for the doubly transformed functions are determined, inverse transformations are applied to them. The solution obtained will constitute a solution of the required system of ordinary differential equations.)

4. More precise expressions for the arbitrary constants involved in solving the system of equations are obtained, the boundary conditions for the given problem being used to obtain them.

5. Known relations between the transform and the original function or else inverse-transformation formulas are used to find the original functions. This represents a final solution of the problem.

The procedure for solving systems of differential heat-transfer and mass-transfer equations with various boundary conditions will be illustrated in detail in later chapters of this text.

The methods of mathematical physics, especially integral-transformation methods, make possible effective solutions of only a relatively narrow range of problems in transfer theory. When systems of differential equations with quite general boundary conditions are considered, exact methods of solution involve serious difficulties, and these become insurmountable in the case of nonlinear problems. In such situations it is necessary

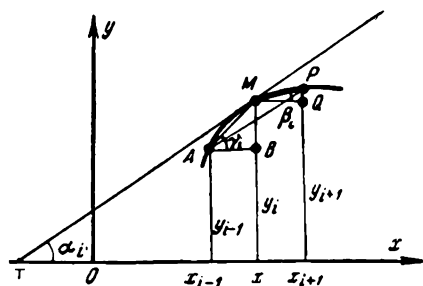


FIGURE 2-6. Determination of the derivative of a function  $f(x)$ .

to turn to a numerical method of solution. It is important to note that the use of numerical methods often makes it possible to get along without a simplified treatment of the mathematical model of the process. At present the most valuable method of obtaining approximate solutions for equations of heat and mass transfer is the method of finite differences or, as it is also called, the net-point method.

In the method of finite differences derivatives are replaced by their approximate values, these being expressed in terms of the differences between values of the function at individual discrete points, the nodes of the net. As a result of these transformations the differential equation is replaced by an equivalent relation involving finite differences, whose solution may be obtained by carrying out certain simple algebraic operations. The final result is given by an expression which states the value of the "future" potential at a given point (node) as a function of time, of its "present" potential, and of the "present" potential of the adjacent nodal points. A repetition of these same operations to calculate the potential fields is very convenient for the application of modern calculating techniques, as a result of which the efficiency of the operation may be increased many times.

An approximate replacement of the first and second derivatives by difference ratios may be carried out in the following simple way. Consider the function  $y=f(x)$  whose curve is shown in Figure 2-6. If  $\alpha_i$  is the angle between the positive  $x$  direction and the tangent to the curve at a point  $M(x_i, y_i)$ , then the derivative of the function at  $x=x_i$  is given by the formula

$$y'_i = \tan \alpha_i.$$

Let us specify two adjacent points on the curve  $A(x_{i-1}, y_{i-1})$  and  $P(x_{i+1}, y_{i+1})$  in such a way that the differences  $x_i - x_{i-1} = x_{i+1} - x_i = h$  are quite small, and let us replace, as an approximation,  $\alpha_i$  by  $\beta_i$  or  $\gamma_i$  (or, alternatively, let us consider one of the secants  $MP$  or  $AM$  instead of the tangent  $MT$ ). Then we obtain

$$y'_i \approx \tan \beta_i = \frac{QP}{MQ} = \frac{y_{i+1} - y_i}{h}, \quad (2-9-5)$$

or

$$y'_i \approx \tan \gamma_i = \frac{BM}{AB} = \frac{y_i - y_{i-1}}{h}. \quad (2-9-6)$$

Now, if we replace, again as an approximation, the slope of tangent  $MT$  by the slope of secant  $AP$ , then we have

$$y'_i \cong \frac{y_{i+1} - y_{i-1}}{2h}. \quad (2-9-7)$$

The right sides of (2-9-5), (2-9-6), and (2-9-7) are called respectively the forward difference ratio, the backward difference ratio, and the symmetric difference ratio.

An approximate value for the second derivative  $y''_i$  of the function  $y=f(x)$  for  $x=x_i$  may be obtained in an elementary way if we replace the portion  $AP$  of the curve by the broken line  $AMP$ , which has two slopes at point  $M$ :

$$y''_i \approx \frac{1}{h} \left( \frac{y_{i+1} - y_i}{h} - \frac{y_i - y_{i-1}}{h} \right) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}. \quad (2-9-8)$$

It should be evident that formulas (2-9-5) to (2-9-8) for replacing the derivatives by difference ratios are not the only ones possible; sometimes it is convenient to use other substitutions. However, in numerical integrations of the heat-conduction and mass-conduction equations it is the above which are most often used.

Let us consider, as an example, the one-dimensional equation describing heat conduction in a thin insulated rod of length  $L$ :

$$\frac{\partial t(x, \tau)}{\partial \tau} = a_q \frac{\partial^2 t(x, \tau)}{\partial x^2}, \quad (0 \leq x \leq L). \quad (2-9-9)$$

Since  $t(x, \tau)$  is a function of two variables, we may use the rectangular net shown in Figure 2-7. Along the  $x$  axis a segment of length  $L$  is laid off and divided into  $n$  equal parts. The resulting intervals each have a length of  $h=L/n$ , and the division points (nodes) along the  $x$  axis are at  $x=0$ ,  $x=h$ , ...,  $x=L$ .

On the ordinate axis is plotted the time  $\tau$ , which is divided into equal intervals  $l$ . When lines parallel to the coordinate axes are drawn through

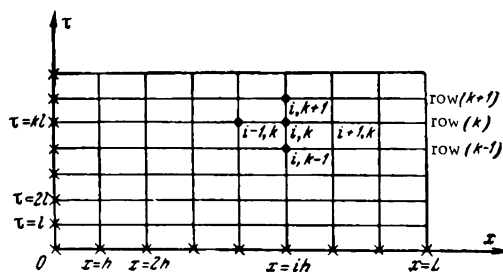


FIGURE 2-7. Rectangular net

the nodes situated along the axes (in Figure 2-7 these nodes are marked by small crosses), a rectangular net is formed. The values of function  $t$  at the nodes lying on the axes and on the line parallel to the ordinate axis and situated at a distance  $L$  from it are found from the initial and boundary conditions.

An approximate numerical integration of equation (2-9-9) using the net-point method consists in specifying an approximate value of function  $t$  at



each node of the net. Let us call  $t_{i,h}$  the actual value of the temperature at some point  $x=ih$  along the rod at the moment  $\tau=kl$ , that is, at the node marked  $i, k$  in Figure 2.7. We may now replace the partial derivatives  $\frac{\partial t}{\partial \tau}$  and  $\frac{\partial^2 t}{\partial x^2}$  at the point  $(ih, kl)$  by the corresponding difference ratios given in formulas (2-9-5) and (2-9-8):

$$\frac{\partial t_{i,h}}{\partial \tau} = \frac{t_{i,h+1} - t_{i,h}}{l} + \varepsilon_1;$$

$$\frac{\partial^2 t_{i,h}}{\partial x^2} = \frac{t_{i-1,h} - 2t_{i,h} + t_{i+1,h}}{h^2} + \varepsilon_2,$$

where  $\varepsilon_1$  and  $\varepsilon_2$  are residual terms which approach zero as  $l$  and  $h$  approach zero. Thus, for the node  $(ih, kl)$  differential equation (2-9-9) may be replaced by the following relation:

$$\frac{t_{i,h+1} - t_{i,h}}{l} + \varepsilon_1 = a_q \left( \frac{t_{i-1,h} - 2t_{i,h} + t_{i+1,h}}{h^2} + \varepsilon_2 \right)$$

or

$$t_{i,h+1} = \left( 1 - \frac{2la_q}{h^2} \right) t_{i,h} + \frac{la_q}{h^2} (t_{i-1,h} + t_{i+1,h}) + lR, \quad (2-9-10)$$

where  $R = a\varepsilon_2 - \varepsilon_1$ .

Next we drop the residual term  $lR$  in (2-9-10) to obtain the difference equation

$$\vartheta_{i,h+1} = \left( 1 - \frac{2la_q}{h^2} \right) \vartheta_{i,h} + \frac{la_q}{h^2} (\vartheta_{i-1,h} + \vartheta_{i+1,h}), \quad (2-9-11)$$

in which  $\vartheta_{i,h}$  represents the approximate value of  $t_{i,h}$  at the same node  $(ih, kl)$ . Formula (2-9-11) makes possible a calculation of  $\vartheta$  at the nodes of some horizontal row  $(k+1)$  from the values of  $\vartheta$  corresponding to the row  $(k)$  just preceding it [see Figure]. Therefore, with the aid of formula (2-9-11) we can find the values of  $\vartheta$  at the nodes of the first horizontal row (for  $\tau=l$ ) from the values of the temperature at the nodes along the  $x$  axis (for  $\tau=0$ ), the latter values being known from the boundary conditions. Once we obtain the values of  $\vartheta$  in the first row the same formula gives the values at the nodes in the second horizontal row (that is, for  $\tau=2l$ ). This process of expanding the table can be continued as far as desired, since the values of the temperature at the nodes of the lines  $x=0$  and  $x=L$  are known from the boundary conditions.

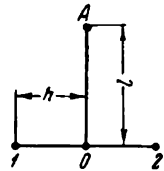


FIGURE 2-8. Net for the derivation of formula (2-9-12)

Formula (2-9-11) can be derived by applying the Fourier and Newton laws to a composition of the heat balances for the elements into which the body is divided. This method for deriving the working formulas is widely used at present by many investigators outside of the Soviet Union. Following Panov /33/, let us rewrite formula (2-9-11) in the more convenient form (Figure 2-8)

$$\vartheta_A = \left( 1 - \frac{2la_q}{h^2} \right) \vartheta_0 + \frac{la_q}{h^2} (\vartheta_1 + \vartheta_2). \quad (2-9-12)$$

Next, by choosing various ratios between the intervals  $l$  and  $h$  it is possible to obtain from (2-9-12) a series of particular relations. Thus, for example,

$$\text{for } l = h^2/3a_q, \quad \vartheta_A = \frac{\vartheta_1 + \vartheta_2 + \vartheta_3}{3};$$

$$\text{for } l = h^2/6a_q, \quad \vartheta_A = \frac{\vartheta_1 + 4\vartheta_0 + \vartheta_2}{6}; \quad (2-9-13)$$

$$\text{for } l = h^2/12a_q, \quad \vartheta_A = \frac{\vartheta_1 + 10\vartheta_0 + \vartheta_2}{12}; \quad (2-9-14)$$

and in general

$$\text{for } l = \frac{h^2}{pa_q}, \quad \vartheta_A = \frac{\vartheta_1 + (p-2)\vartheta_0 + \vartheta_2}{p}. \quad (2-9-15)$$

Equation (2-9-12) becomes particularly simple for  $p=2$ :

$$\vartheta_A = \frac{\vartheta_1 + \vartheta_2}{2}. \quad (2-9-16)$$

Formula (2-9-16), known as Schmidt's formula, has an important practical advantage over formula (2-9-13) and especially over (2-9-14), since for a given  $h$  the value of  $l$  is the highest; consequently, when (2-9-16) is used, the amount of computational work is reduced by a factor of three in comparison with (2-9-13) and by a factor of six in comparison with (2-9-14). It should be noted that, due to its simplicity, formula (2-9-16) is widely used for graphical solutions of unsteady-state transfer problems [34, 35]. Studies have shown that for  $p=1$  a divergent calculation scheme is obtained. In general it should be noted that in the solution of unsteady-state partial differential equations of parabolic type the ratio between  $h$  and  $l$ , and also the rounding-off error in the numerical solution, are of prime importance, since they determine the convergence and stability of the solutions obtained. Rigorous theoretical considerations indicate that:

- 1) it is possible to use formula (2-9-15) only for  $p \geq 2$ ;
- 2) the largest interval of  $l$  corresponds to formula (2-9-11), in which  $p=2$ ;
- 3) the most accurate formula is (2-9-13), in which  $p=6$ ; formula (2-9-16) was first obtained in 1938 by Panov.

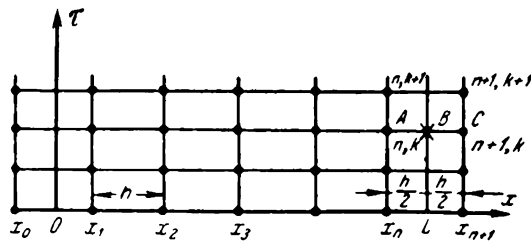


FIGURE 2-9. Calculation diagram for heat conduction with boundary conditions of the third kind

The net given in Figure 2-7 for the numerical integration of equation (2-9-9) is convenient when a problem has boundary conditions of the first kind; in this case the boundary lines  $x=0$  and  $x=L$  are a part of the net itself. If the equation is solved with boundary conditions of the third kind (that is, the value of the potential of the medium is given), then calculation experience and theoretical studies have shown that to raise the accuracy of the determination of the potential at the boundaries additional nodal points,

lying outside the region being studied, must be introduced. For example, if equation (2-9-9) is to be solved with the boundary conditions

$$\left(\frac{\partial t}{\partial x}\right)_{x=0} = \frac{\alpha}{\lambda} [t(0, \tau) - t_0] \text{ and } \left(\frac{\partial t}{\partial x}\right)_{x=L} = -\frac{\alpha}{\lambda} [t(L, \tau) - t_0],^* \quad (2-9-17)$$

then the net (Figure 2-9) should be constructed so that the right boundary line lies half way between the two lines  $x=x_n$  and  $x=x_{n+1}$ , while the left boundary lies half way between the lines  $x=x_0$  and  $x=x_1$  (at which  $x=h/2$ ). In this case the values  $\vartheta_{n+1, h}$  and  $\vartheta_{0, h}$ , which are values of the function located at points outside the region being studied, can be introduced. The derivative  $\left(\frac{\partial t}{\partial x}\right)_{x=L}$  in the second condition of (2-9-17), which corresponds to point  $B(L, kl)$ , may be replaced by a symmetric difference ratio:

$$\left(\frac{\partial t}{\partial x}\right)_{x=L} = \frac{t_{n+1, h} - t_{n, h}}{h/2 + h/2} + g',$$

while the temperature at the surface itself, that is at  $t(L, kl)$ , may be considered to be the arithmetic mean of the temperatures at points  $A$  and  $C$ :

$$t(L, kl) = \frac{t_{n, h} + t_{n+1, h}}{2}.$$

Thus, condition (2-9-17) becomes

$$\frac{t_{n+1, h} - t_{n, h}}{h} + g' = -\frac{\alpha}{\lambda} \left[ \frac{t_{n, h} + t_{n+1, h}}{2} - t_0 \right],$$

or, in terms of the approximate values  $\vartheta$ , we obtain after certain transformations

$$\vartheta_{n+1, h} = \frac{\left(1 - \frac{h}{2} \frac{\alpha}{\lambda}\right) \vartheta_{n, h} + h \frac{\alpha}{\lambda} t_0}{1 + \frac{h}{2} \frac{\alpha}{\lambda}}. \quad (2-9-18)$$

This formula is also used to find approximate values of the function at the nodes of the auxiliary line  $x = L + \frac{h}{2}$ .

The temperature along the boundary line itself  $x=L$  is determined by the formula

$$\vartheta(L, kl) = \frac{\vartheta_{n, h} + \vartheta_{n+1, h}}{2},$$

which may be transformed to give

$$\vartheta(L, kl) = \frac{\vartheta_{n, h} + \frac{h}{2} \frac{\alpha}{\lambda} t_0}{1 + \frac{h}{2} \frac{\alpha}{\lambda}} = \frac{2S\vartheta_{n, h} + ht_0}{2S + h},$$

where

$$S = \lambda/2.$$

The temperature of the nodes of the auxiliary line  $x = -\frac{h}{2}$  is found using the formula

$$\vartheta_{0, h} = \frac{\left(1 - \frac{h}{2} \frac{\alpha}{\lambda}\right) \vartheta_{1, h} + h \frac{\alpha}{\lambda} t_0}{1 + \frac{h}{2} \frac{\alpha}{\lambda}}.$$

\* [The first condition should apparently be  $\left(\frac{\partial t}{\partial x}\right)_{x=0} = -\frac{\alpha}{\lambda} [t(0, \tau) - t_0].$ ]

Along the left surface we obtain

$$\vartheta(0, kl) = \frac{\vartheta_{1,k} + \frac{h}{2} \frac{\alpha}{\lambda} t_0}{1 + \frac{h}{2} \frac{\alpha}{\lambda}} = \frac{2S\vartheta_{1,k} + ht_0}{2S + h}.$$

The technique for solving differential equations including source terms does not differ from the one described above. The method of finite differences makes it possible to solve successfully one-, two-, and three-dimensional problems. Cases in which a square net is plotted for the range of variation of  $x$  and  $y$  were studied comprehensively by Mikeladze /36/. Triangular and polar nets have been considered by Yushkov /37, 38/ and others /39/. It should be noted that polar nets are especially convenient for solving problems with axial symmetry. A determination of the temperature field in three-dimensional space is given in /40/ for the case of constant thermophysical characteristics and in /41 to 43/ for the case of variable characteristics. All these problems have been discussed in great detail in references /35/ and /44/.

As was shown by Yushkov, the method of finite differences enables an effective solution of the system of differential heat-transfer and mass-transfer equations, either for constant coefficients /35, 45, 46/ or for variable coefficients /47/.

Although numerical methods have great potentiality, until quite recently their extended application to the solution of transfer equations involved a large amount of calculational work. However, the rapid development and availability of computers has greatly increased the importance of these methods in studies of transfer phenomena. Modern electronic discrete-type computers are designed to carry out laborious computational operations, while certain continuous-type [analog] computers can also set up physical models.

The use of models to investigate transfer phenomena is based on the formal identity, from the point of view of an analytical representation, of a series of processes. This identity is a consequence of a far-reaching correspondence in the behavior of compared analog systems. "The unity of nature," wrote Lenin, "is observed in the 'remarkable similarity' of the differential equations referring to different classes of phenomena."\* This fact enables us to investigate processes taking place in a real prototype by means of other processes taking place in a model. A solution obtained using a model analog naturally will not be analytical in nature, but it can be carried out successfully by experimental means and then expressed in terms of the parameters of the initial problem. The most common models used at present are those constructed on the basis of hydraulic, electrical, mechanical, and acoustical analogies of the processes.

A hydrodynamic analogy based on the identity (in a formal mathematical sense) between the stream function and velocity potential of an ideal liquid taking part in irrotational flow and the heat-flux function and temperature in a system without heat sources has been used by Moore and others to solve two-dimensional steady-heat-conduction problems /39/. Later the range of application of this model has been extended to systems with distributed sources /48/. In 1928 Emanuel, and somewhat later Budrin, designed and constructed models based on the analogy of the mathematical

\* Lenin, V. I. *Sobranie sochinenii* (Collected Works), Vol. 14, p. 276. — Fourth Edition.

relations describing the temperature distribution in a solid body and the pressure-head distribution in water moving through a capillary tube /49/. The model units, known as hydraulic integrators, made it possible to solve the problem of unsteady heat conduction and mass conduction. Luk'yanov later worked out a number of integrators for the solution of two- and three-dimensional problems in heat and mass conduction /50/, and Budrin /51/ developed hydrostatic integrators for the solution of nonlinear transfer equations of parabolic type.

Koil /52/ developed a method of air-aerodynamic analogy which is similar to the method of hydrodynamic analogy. The operation principle of his device is similar to that of Budrin's hydrostatic integrators. The amount of heat and temperature in a heat-conducting system here correspond to the amount of air and the pressure. Other types of hydrodynamic analogy, for example types based on the correspondence between heat transfer and fluid transfer in a porous body (between the laws of Fourier and Darcy), have also been proposed /53/. The value of hydrodynamic models is reduced due to the bulkiness of the equipment and the complexity of its operation, and due to certain difficulties involved in solving problems with variable thermophysical characteristics.

The most widespread analogy used to study phenomena of heat transfer in matter is the electrical analogy. The equipment necessary for such studies is portable and cheap and it can be constructed quite simply and with high accuracy. In addition, electrical instruments are excellent, and methods for measuring electrical quantities have been highly perfected. The electrical analogy is based on the formal resemblance between differential equations describing heat conduction and mass conduction, on one hand, and the electrical-conduction equation, on the other. Moreover, the analogy between the above processes is deeper than just identity of the respective transfer equations. This is evident from a comparison of the electrical, thermal, and mass quantities and laws listed in Table 2-1. An analysis of the transfer equations and of Table 2-1 shows that it is possible to reproduce in full unsteady potential fields of heat transfer and mass transfer by means of electrical analogies, for different kinds of boundary conditions as well as for different types of source distributions.

Electrical analogies may be implemented using any of various experimental devices. The method using modeling equipment in which the geometry of the prototype is reproduced while the model is prepared of a material with continuous conductivity is called a geometrical analog, or field-modeling using the method of continuous media. If the model is an equivalent electrical circuit with lumped constants, then the setup is called a modeling circuit.

A convenient model for finding the potential field in a conductor is the electrolytic bath. The quite wide application of electrolytic baths has ensured homogeneity of the liquid modeling medium, the possibility of creating large-scale models, and relatively easy access to internal points in the liquid when volume fields are modeled.

Electrical analogs using liquid models make use of the ionic conductivity of electrolytes. An electrolyte (a weak solution of salts, acids, and alkalis; any of various vitriol solutions; etc.) with a constant resistivity is used as the conductor. The model may be three- or two-dimensional, but its shape must be identical to the shape of the prototype body. The boundary of the

bath must have a potential proportional to the temperature at the boundary of the prototype, and this is accomplished using a metal conductor through which electrical current is conveyed to the electrolyte. Using such a model, Langmuir studied the heat transfer through the walls of a shell in the form of a parallelepiped /54/. When transfer potentials in a nonuniform field are to be modeled, an electrolyte with variable concentration or a bath with variable depth may be used /55/.

TABLE 2-1

Process characteristic	Type of transfer process		
	Electrical	Mass	Thermal
Potential	$\varphi, \text{ v}$	$\theta, ^\circ\text{M}$	$t, ^\circ\text{C}$
Motive force	$\nabla\varphi, \text{ v/m}$	$\nabla\theta, ^\circ\text{M/m}$	$\nabla t, ^\circ\text{C/m}$
Charge	Electrical charge	Number of moles of diffusing component	Amount of entropy
Conductivity	Electrical conductivity	Mass conductivity	Thermal conductivity
Resistance	Electrical resistance, $R_e = \frac{1}{\lambda_e} \frac{L}{A}, \text{ ohms}$	Resistance to material transfer $R_m = \frac{1}{\lambda_m} \frac{L}{A} \frac{\text{ohm} \cdot \text{hr}}{\text{kg (or mole)}}$	Thermal resistance $R_q = \frac{1}{\lambda} \frac{\text{degree} \cdot \text{hr}}{\text{kcal}}$
Flux density	$j_e = \frac{\Delta\varphi}{R_e}, \text{ amperes}$	$j_m = \frac{\Delta\theta}{R_m} \frac{\text{kg (or mole)}}{\text{hr}}$	$j_q = \frac{\Delta t}{R_q} \text{ kcal/hr}$
Specific capacity	Electrical capacity $C_e, \text{ farads}$	Mass capacity $C_m, \text{ kg}$ (or mole)/ $^\circ\text{M}$	Heat capacity $C_q,$ kcal/degree

Modeling equipment operating on the geometrical-analog principle may also make use of models made from solid electrically conductive materials and coatings. In modeling plane-parallel fields some use has been made of models consisting of thin sheets of electrically conductive material. The sheet may be a piece of metal foil, metallized paper, or normal paper coated with a layer of electrically conductive graphite of given resistance (for example, Teledel'tos paper). The sheet is cut into a shape identical to the prototype, and electrodes are cemented on or painted on using a high-conductivity paint. By suitable choice of the paint, the required boundary potentials may be obtained. The sources are attached by means of foil electrodes, cemented on using a conductive adhesive, in accordance with a drawing on the opposite side of the sheet. Areas with different thermal or mass conductivities are reproduced by perforating the sheet with square holes or by gluing on individual sections of paper several layers thick.

In modeling three-dimensional problems solid continuous media are used. However, dispersed media of various conductivities may also be used, for example, graphite powder mixed with quartz sand or colloidal masses (gelatin) /55/.

The possibility of using electrical networks for modeling was first demonstrated in 1926 by the Russian mathematician Gershgorin. One of the advantages of a network is that the coordinates of the network points are electrical, not geometrical. This makes it possible to solve, using rectangular circuits, problems in any coordinate system. The flexibility of a circuit is very convenient during preparation of a model and ensures its reliability during operation.

To obtain the model, the given body is divided into a number of volume elements, just as in the method of finite differences. The value of the potential is then obtained for a finite number of selected points, that is, the continuous potential field in the body is replaced by its "equivalent" lumped values. The electrical network (the modeling circuit) consists of electrical capacities connected in parallel, these being lumped at the nodal points of the circuit. Sources of current and matter are reproduced by including supply sources at one or more nodal points in the circuit in the case of concentrated sources or at all the nodal points in the case of uniformly distributed sources.

For the modeling of steady and unsteady heat-transfer and mass-transfer potential fields, various types of universal-electrical-network models, known as electrical integrators, have been developed and constructed. For example, to study plane steady potential fields with boundary conditions of the first, second, and third kinds, the EI-11 and EI-12 electrical integrators designed by Gutenmakher may be used, while unsteady fields with coefficients which are functions of the coordinates may be studied using EI-22, EI-21, and EI-31 electrical integrators. These models have enabled the solution of many important problems in different branches of science and engineering /56/.

Effective modeling equipment has also been constructed outside of the USSR, for instance those of Beuken in Holland, Paschkis, McCann, and Liebmann in the USA, Lawson and Jackson in England, and Fischer and Müller in Germany.

Of particular interest is the model designed by Liebmann /57/ to solve unsteady-state problems using a network of resistors. In this model the solution could be interrupted, so that in this case the calculation is performed using an explicit finite-difference spatial and temporal approximation. The latter model makes it possible, as does Budrin's hydrostatic integrator, to determine a temperature field while at the same time taking into account that the anisotropy and the variation in the thermophysical characteristics of the material may be temperature-dependent.

The disadvantage of Liebmann's model is its relatively limited flexibility, since for each new problem it is necessary to change the values of all the resistors. This disadvantage is eliminated in the new static (capacitive and ohmic) integrators designed at the Thermophysics Research Laboratory of the Kazakh State University im. S. M. Kirov /58, 59/. A detailed summary of applications of various electrical models to the solution of thermophysical problems is given in /59/.

Other types of analogies are also used in the study and calculation of heat-transfer and mass-transfer processes. In particular, the membrane analogy is quite widely used outside of the Soviet Union to determine the volume-stress state of a body and to solve problems in steady-state heat conduction with heat sources /39/.

Methods of numerical solution and analogy are used in cases when it is impossible to solve a problem in a closed form [analytically] or when the solutions obtained are so complicated that they cannot be used for practical calculations. The choice between the methods of numerical solution and analogy depends on the specific problem, on the requirements imposed on the calculation data, and on the estimated time needed to solve the formulated problem with a specified accuracy.



## Chapter III

### THE FUNDAMENTALS OF SIMILARITY THEORY

#### 3-1. The Problem Posed

In the study of different physical phenomena two methods of investigation are used which make it possible to obtain quantitative laws describing the phenomena at hand. The first method consists of a generalization of empirically observed regularities, while the second method is based on a rigorous theoretical consideration of the problem.

The main advantage of the first method is the reliability of the partial results obtained, within the limits of the experimental accuracy. The accuracy of an experiment can always be determined and fitted appropriately to the practical requirements. Moreover, during the experiment the greatest attention may be focused on the direct relation between the quantities which are of most practical interest.

The main disadvantage of this method of investigation is the limited value of its results. The results of a specific experiment cannot be applied to any other phenomenon which differs to a certain degree from the one studied. Actually, since the results of any specific experiment reflect all the individual features of the corresponding phenomenon, it is impossible to decide directly from the experiment which features are the most significant and how their variations affect the development of the phenomenon. Therefore, conclusions based on an analysis of the results of an experimental study of a given phenomenon cannot be extended to other phenomena. Unjustified attempts to extend experimental results beyond the limits of the investigation very often lead to errors. Thus, rigorously speaking, if the experimental method is used, each specific phenomenon must serve as an independent object of study. This represents an intrinsic disadvantage of the method and leads to considerable practical difficulty, a difficulty which is particularly evident in the creation of new processes and equipment.

If it were only possible to supplement experimental investigation by a scientifically established means of generalizing the data of a single experiment, then all the basic difficulties would at once disappear and we would obtain a very powerful means of solving various practical problems. Actually, however, all the generalizations made on this basis are highly approximate ones.

The basis for a generalization of experimental results should really be sought in mathematical physics, that is, in the branches of theoretical physics which study phenomena in their most general form by setting up and analyzing existing differential equations. Mathematical physics

represents the second method for establishing quantitative laws, a method which is widely used in modern science.

In deriving the differential equations of mathematical physics the most general laws of nature are usually drawn upon, and these laws are given a form corresponding to the specific features of the given phenomenon. When a phenomenon is considered from the point of view of the fundamental laws of physics, then naturally many of its specific features must be ignored. The general laws of physics themselves are the result of an extremely wide generalization of experimental data. The application of these general laws to the study of phenomena makes it possible, of course, to obtain very general relations between the important parameters of the phenomenon.

A cogent example is the derivation of the differential heat-conduction equation. While deriving this equation we ignored all the specific conditions of the phenomenon and considered only an isolated differential volume of the body. To derive the equation only a single experimental fact was necessary, namely that redistribution of heat in the medium is possible only for a nonzero temperature gradient. This fact was described using Fourier's hypothesis (law), and then it became possible to apply the energy-conservation law to study the temperature field in the body.

The differential heat-conduction equation describes the mechanism by which heat is redistributed in a material medium (it is essentially a mathematical model of this mechanism). This differential equation is thus the most general relation between the quantities which are essential to the phenomenon, and it characterizes those properties which are inherent to all the phenomena in this class (in our case, the class of heat-conduction phenomena). Consequently, every phenomenon (independently of its individual characteristics) whose basis is this mechanism of heat redistribution is described by this general equation.

This is precisely the reason why no data on the specific values of individual quantities characteristic of any single phenomenon enter into the differential equation. The variables entering into the equation may assume very different values, each of which corresponds to some particular phenomenon.

It is thus perfectly obvious that any differential equation (or system of equations) represents a mathematical model for an entire class of phenomena. Consequently, a class is here understood to mean the whole set of phenomena characterized by the same process mechanism. Correspondingly, the integration of any differential equation can lead to an infinite number of different solutions which satisfy it. In order to obtain from the multitude of possible solutions the one particular solution corresponding to the given specific phenomenon, it is necessary to possess data additional to those contained in the initial differential equation. To obtain such data, all the features of the given phenomenon which distinguish it from the entire class of similar phenomena must be known. These additional conditions, which together with the differential equation or its solution uniquely determine a particular phenomenon, are called uniqueness conditions.

It is evident that uniqueness conditions must contain all the distinctive features of the given specific phenomenon. Naturally, these distinctive features do not depend on the mechanism of the process (which is common to the entire class of phenomena) and are specified in relation to the conditions of a particular problem.

A specific phenomenon has the following characteristics which distinguish it from the entire class of phenomena to which it belongs:

1. Every given system has definite dimensions and shape. Therefore, the uniqueness conditions must involve the geometrical properties of the system.

2. The bodies comprising a given system possess definite physical properties. Therefore, to determine a given phenomenon it is necessary to know all the physical constants of the bodies which are important in this phenomenon.

3. Every process exists and develops in time. In order to determine the state of a system at a certain moment, it is necessary to know its state at some previous moment which is taken as initial. The uniqueness conditions must therefore include time-dependent conditions which characterize the state of the system at the starting (initial) moment. A complete picture must be had of the distribution of variables throughout the entire volume of the system at the initial moment.

4. A system being studied always interacts, to some extent, with its surroundings. Very often this interaction is the reason for the appearance in the system of the process which is of interest. It is obvious that in order to describe completely a specific phenomenon it is also necessary to know the conditions of interaction between the system and its surroundings, in other words, to know the conditions at the boundaries of the system.

The four above conditions, together with the differential equation (or system of equations), determine uniquely a given specific phenomenon. Consequently, it may be concluded that mathematical physics gives, in the form of a differential equation, the most general relations between the quantities characterizing a phenomenon. These relations are so general that they cannot be used directly to study a specific phenomenon. In order to use the relations contained in the differential equation, it is thus necessary to solve this equation and then to fit the solution to the uniqueness conditions. Such a solution will contain the amount of information required in practice. The part played by the experimental method under such conditions is merely to verify the individual results given by theoretical physics.

However, in most cases it is impossible to find a solution which satisfies the differential equation and the uniqueness conditions. This is due to the complexity of the differential equations, and consequently, in the final analysis, to the complexity of the phenomena themselves. The Fourier-Kirchhoff differential equation describing heat transfer, which has to be solved simultaneously with the Navier-Stokes equation, is an example. It is impossible to solve this system of equations in its general form, and, up to the present, exact solutions have been obtained only for certain very simple particular cases.

In the study of a complex physical phenomenon simplifications of the initial differential equations are sometimes introduced. However, this leads in general to a distortion of the model of the actual mechanism of the phenomenon, which is difficult to estimate and which lowers appreciably the practical value of the solution obtained.

It is now clear that, in contrast to the experimental method, which enables the study of the properties of only one single specific phenomenon, mathematical physics possesses means of investigation which make possible

an examination of the most general properties of a phenomenon, properties which are characteristic of the whole class of phenomena. The disadvantage of the experimental method consists in the impossibility of extending the results obtained in a given experiment to phenomena which are different from the one studied. The disadvantage of the methods of mathematical physics consists in the impossibility of passing from the class of phenomena described by a differential equation to a single specific phenomenon, which is described in addition by the corresponding uniqueness conditions. Neither of the methods separately can, in general, be used effectively to solve the problems encountered in practice.

If, without losing the advantages of the two methods, we could unify them into a single method, then it would be possible to obtain a highly universal means of studying different natural phenomena. This unification of the two methods is accomplished by similarity theory. Consequently, the theory of similarity solves the problem posed earlier, since it represents a study of methods leading to a scientific generalization of the data of a single experiment.

### 3-2. Transformation Factors and Similarity Criteria

The foregoing analysis has led to two important concepts, that of a class of phenomena and that of a single phenomenon. As stated above, the transition from a class of phenomena to a single phenomenon is accomplished by adding uniqueness conditions to the differential equation. This separates a single specific phenomenon (for example, the phenomenon of heat propagation in the wall of a building) from an infinite number of similar phenomena (for example, heat-conduction phenomena in general).

It is evident that it is impossible to extend the results of a single experiment to all the phenomena in the class as a whole, since within every class there are phenomena which differ greatly from one another. Thus, for example, the heat conduction in the wall of a building is something quite different from the heat conduction in the metal blank [billet] rolled out in a blooming mill (although the two phenomena belong to the same class). Consequently, in the theory of similarity the special concept of a group of phenomena is introduced. A group is more limited than a class but is more general than a single phenomenon. A group unites all the phenomena to which it is possible to apply the results of a single experiment. Let us clarify how a group may be differentiated from the entire class of phenomena.

Obviously, a phenomenon in a given class acquires individual features only when uniqueness conditions are defined for it. Consequently, every limitation of a class of phenomena (every narrowing of its limits down toward the limits of a group) can be achieved by some appropriate assignment of uniqueness conditions. A direct assignment of uniqueness conditions in the form of numerical values of the individual quantities involved results in a narrowing of the class to a single phenomenon. Clearly, such a restriction as this is excessive for our purposes.

Let us consider the principle behind the construction of a phenomena group, using geometrical figures as an example. Figure 3-1 shows several rectangles. The concept of a "rectangle" obviously defines a whole class of

plane figures united by some common property (all four angles must be right angles). In order to isolate a single figure from this whole class of figures, it is necessary to specify the numerical values  $l_1$  and  $l_2$  of the sides. These numerical values represent, in the present case, uniqueness conditions. A group of figures is obtained if the sides of the basic figure are multiplied by some factor  $k_l$ . By assigning different values to this factor, we obtain a whole series of figures (Figure 3-1, a).

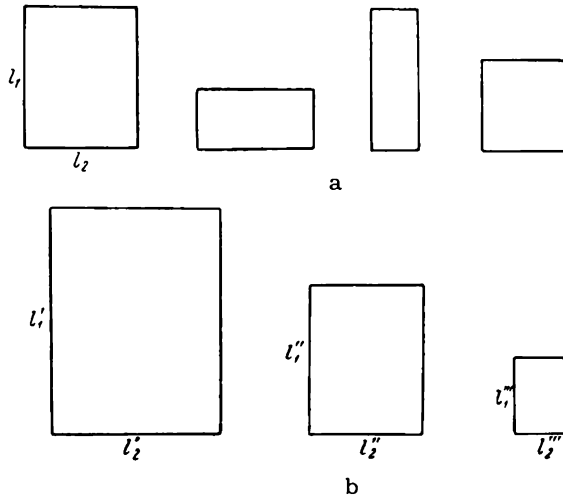


FIGURE 3-1. Class and group of plane rectangular figures

a) class of figures (sides of arbitrary lengths); b) group of similar figures

The figures in Figure (3-1, b) are similar to one another, since their sides are proportional:

$$\frac{l'_1}{l'_2} = \frac{l''_1}{l''_2} = k_l. \quad (3-2-1)$$

Thus, by multiplying the sides of the basic figure by some factor  $k_l$ , which may have any value but which must be the same for both sides, we obtain a group of similar figures. The quantity  $k$  is called a transformation factor. For a group of figures constructed in this manner, each figure differs from another one in the same group only by its scale, and distortions of the geometrical shape do not occur. Each point on a given figure thus corresponds to a similar point on the other figure.

Obviously, all the above discussion refers not just to two-dimensional figures but to three-dimensional systems as well. In geometry this type of figure transformation is called similar. This term is also used in similarity theory, but here it has a wider scope, since it also refers to the transformation of physical quantities.\*

We have considered a similar transformation of the geometrical properties of a system and have obtained a group of systems whose geometrical

\* If the transformation factors for different variable dimensions of a figure (or body) are not equal to one another, then a transformation will obviously distort the shape and the resulting figures (bodies) will no longer be similar to the basic figure (basic body). This type of transformation is called affine transformation.

characteristics are similar. An analogous method must be used for a similar transformation of all the other characteristics entering into the uniqueness conditions. First let us consider a transformation of the time coordinate, which (as do the spatial coordinates) constitutes one of the physical variables involved in every problem.

The time may be transformed similarly if it is multiplied by a transformation factor  $k_t$ . Then, the physical variables corresponding to a time  $\tau'$  for one phenomenon must equal the same variables in the second phenomenon at some moment  $\tau''$ . The ratio

$$\frac{\tau''}{\tau'} = k_t \quad (3-2-2)$$

is not in general unity, so that similar moments of time for two different phenomena in a given group are nonsimultaneous. The time intervals which separate these moments from a common beginning of the process have some constant ratio satisfying condition (3-2-2).

If different values are given to  $k_t$ , the time scale is changed. Time similarity of phenomena is called homochronicity (uniformity in time). It is obvious that in the special case  $k_t = 1$  similar moments of time occur simultaneously, that is, the courses of the processes are synchronous.

To construct a group of phenomena whose physical parameters are similar, it is necessary to multiply the value of each parameter by the corresponding transformation factor. If the physical parameters have different values at different points in the system, then the whole parameter field must be similar. Thus, for example, if the thermal diffusivity of one system has the values  $a'_1, a'_2, a'_3$  at different points, then at similar points in another system of the given group it must have values  $a''_1, a''_2, a''_3$  which satisfy the requirement

$$\frac{a''_1}{a'_1} = \frac{a''_2}{a'_2} = \frac{a''_3}{a'_3} = \dots = \frac{a''_n}{a'_n} = k_a. \quad (3-2-3)$$

It is thus clear that similarity of initial conditions implies similarity of the fields of all the variables at the initial moment of the process.

In the same way, similarity of the boundary conditions implies a similar transformation of all the variables describing conditions at the boundaries of the system. Moreover, under unsteady-state conditions, comparisons must be made at homochronous (similar) moments.

The above discussion refers to scalar quantities. When a vector is subjected to a similar transformation, its direction must remain unchanged. This is a direct implication of the geometrical concept of similarity.

When the uniqueness conditions for different phenomena belonging to the same class are subjected to similar transformations, different phenomena groups in the given class are obtained.

Let us now establish what common properties are possessed by individual phenomena within a given group. As noted previously, differences between the properties of phenomena in a given class are determined entirely by the uniqueness conditions. However, in setting up a group we chose the uniqueness conditions so that they differed only in their scales. Consequently, a given phenomenon differs from other phenomena of the same group only by the scale of its characteristic quantities [parameters]. Thus it is obvious that all the phenomena of a given group simply represent a single phenomenon projected onto different scales. All the phenomena included in a single group will be called mutually similar phenomena.

The basic theorem of the theory of similarity states that:

*Two phenomena are similar provided they are described by the same system of differential equations and they have similar uniqueness conditions.*

The first part of the theorem (that the phenomena must be described by the same system of differential equations) indicates that the two phenomena belong to the same class. The second part (that the phenomena must have similar uniqueness conditions) indicates that the two phenomena are in the same group. This theorem was established by Kirpichev and Gukhman, two investigators who have contributed a great deal to the development of similarity theory, and is called in their honor the Kirpichev-Gukhman theorem. The theorem was first published in 1931 /1, 2/.

A group of mutually similar phenomena, that is, a set of phenomena which differ only in scale, constitutes a range within whose limits the extension of the results of a single experiment is permissible and advisable.

It is obvious that one of the requirements of the basic theorem, identity of the equations describing the phenomena, will be satisfied by substituting in the equation either the values referring to the basic phenomenon or other values obtained by multiplying the first values by arbitrary numbers (the transformation factors  $k$ ). However, in order to satisfy simultaneously both requirements of the theorem, it is necessary to restrict somewhat the choice of the transformation factors, namely, each combination of these factors expressed as a dimensionless number must equal unity.

As an example let us consider the heat transfer between a plane plate and a plane-parallel liquid flow around it. The heat-transfer equation for steady-state conditions and with a boundary condition of the third kind is\*

$$w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y} = a \left( \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} \right); \quad (3-2-4)$$

$$\lambda \frac{\partial t}{\partial y} + a \Delta t = 0, \quad (3-2-5)$$

where  $\Delta t$  is the temperature drive ( $\Delta t = t_s - t_c$ ). The basic system is described by the parameter values  $w'_x, w'_y, x', t', a', y', \lambda', a', \Delta t'$ , while the second system is described by the values  $w''_x = k_w w'_x$ ;  $w''_y = k_w w'_y$ ;  $x'' = k_l x'$ ;  $y'' = k_l y'$ ;  $t'' = k_t t'$ ;  $a'' = k_a a'$ ;  $\lambda'' = k_\lambda \lambda'$ ;  $a'' = k_a a'$ ;  $\Delta t = k_t \Delta t'$ , each equal to the product of the first-system value times the corresponding transformation factor. If phenomena (') and (') are to be similar, then equation (3-2-4) and boundary condition (3-2-5) must remain valid for a transition from phenomenon (') to phenomenon (').

Consequently,

$$w'_x \frac{\partial t'}{\partial x'} + w'_y \frac{\partial t'}{\partial y'} = a' \left( \frac{\partial^2 t'}{\partial x'^2} + \frac{\partial^2 t'}{\partial y'^2} \right); \quad \lambda' \frac{\partial t'}{\partial y'} + a' \Delta t' = 0; \quad (3-2-6)$$

$$w''_x \frac{\partial t''}{\partial x''} + w''_y \frac{\partial t''}{\partial y''} = a'' \left( \frac{\partial^2 t''}{\partial x''^2} + \frac{\partial^2 t''}{\partial y''^2} \right); \quad \lambda'' \frac{\partial t''}{\partial y''} + a'' \Delta t = 0.$$

Now, the (') quantities may be expressed in terms of the (') quantities multiplied by the corresponding transformation factors to obtain

$$\frac{k_w k_t}{k_l} \left( w'_x \frac{\partial t'}{\partial x'} + w'_y \frac{\partial t'}{\partial y'} \right) = \frac{k_a k_t}{k_l^2} a' \left( \frac{\partial^2 t'}{\partial x'^2} + \frac{\partial^2 t'}{\partial y'^2} \right); \quad (3-2-7)$$

$$\frac{k_\lambda k_t}{k_l} \lambda' \frac{\partial t'}{\partial y'} + k_a k_t a' \Delta t' = 0. \quad (3-2-8)$$

\* For simplicity liquid flow at constant pressure ( $\partial p / \partial x = 0$ ) is considered.

A comparison of formulas (3-2-8) and (3-2-6) shows that they may be true simultaneously. However, all the equations are satisfied only when

$$\frac{k_w k_t}{k_i} = \frac{k_a k_t}{k_i^2} \quad \text{or} \quad \frac{k_w k_i}{k_a} = 1; \quad (3-2-9)$$

$$\frac{k_\lambda k_t}{k_i} = k_a k_t \quad \text{or} \quad \frac{k_a k_i}{k_\lambda} = 1. \quad (3-2-10)$$

This is an important result, since the selection of the  $k$  factors is restricted by conditions (3-2-9) and (3-2-10). Consequently, the identity of the given components for all analogous points of different phenomena guarantees invariance of the equation for a similar transformation of all the variables. This is the only requirement which must be satisfied in order for a similar transformation of the uniqueness conditions to result in similarity of the phenomena.

If we substitute in the values of the factors and separate variables, then we obtain the dimensionless parameters

$$\frac{w'l'}{a'} = \frac{w''l''}{a''} = \dots = \text{idem}; \quad (3-2-11)$$

$$\frac{a'l'}{\lambda'} = \frac{a''l''}{\lambda''} = \dots = \text{idem}. \quad (3-2-12)$$

Dimensionless numbers such as (3-2-11) and (3-2-12) are called similarity criteria.\* It is standard practice to denote them by the first two letters of the surnames of scientists who have contributed greatly to the development of the corresponding fields of knowledge. Criteria which have not been so designated will be denoted by the letter  $K$ . Criterion (3-2-11) is called the Péclet number and denoted by  $Pe$ :

$$Pe = \frac{wl}{a}. \quad (3-2-13)$$

Criterion (3-2-12) is called the Nusselt number  $Nu$ :

$$Nu = \frac{al}{\lambda}. \quad (3-2-14)$$

For similar phenomena these similarity criteria must have the same values at analogous points of similar systems.

The quantity  $l$  entering into the similarity criteria may refer to any dimension. It is only important that the criteria for all the systems be calculated using the same rule. Usually some characteristic dimension of the system (plate thickness, pipe diameter, etc.) is chosen. The characteristic dimension introduced into the criterion is generally called the determining dimension of the system.

In most cases the method discussed above gives the similarity criteria necessary for the study of a given problem. However, this approach to the analysis of physical phenomena is somewhat formal in nature. Recently, thanks to the studies of Gukhman /3/, similarity theory has been put on a more solid physical basis. Because of this, the theory acquires even greater value as a theory of generalized variables. Let us consider this in more detail.

\* [See Appendix II for a list of the similarity criteria (dimensionless numbers) used in this text.]



### 3-3. The Similarity of Heat-Exchange and Mass-Exchange Phenomena

The system of differential heat-transfer and mass-transfer equations and the uniqueness conditions involve a large number of variables, since the corresponding phenomena are very complex. However, not all the variables are essential to the development of processes of heat and mass transfer.

For each problem there exists a set of characteristic variables (specific for the given problem), in terms of which the problem must be considered. A transition to these parameters makes it possible to reduce all the quantities essential for the process to a small (minimum possible) number of independent and dependent variables. Similarity theory has as its purpose the development of general methods for determining these variables. It provides general methods for the direct transformation of differential operators into simple algebraic expressions.

According to the theory formulated by Gukhman, this transformation method is essentially a replacement of the real process by a simple fictitious scheme in which all the differential operators retain constant values in space and time for a given state. In this case the expressions for the derivatives are replaced by ratios of finite quantities. Consequently, if  $\frac{dy}{dx}$  is constant, then  $\frac{dy}{dx} = \frac{Y}{X}$ , while if  $\frac{d^2y}{dx^2}$  is constant, we have  $\frac{d^2y}{dx^2} \sim \frac{Y}{X^2}$ , where the symbol  $\sim$  indicates proportionality. In general,

$$\frac{d^m y}{dx^m} \sim \frac{Y}{X^m}. \quad (3-3-1)$$

Gukhman explains this idea by means of the following example. For steady-state heat exchange between a plate and its surroundings, Newton's law states that

$$\alpha(t_0 - t_s) = -\lambda \left( \frac{\partial t}{\partial x} \right)_s, \quad (3-3-2)$$

where  $\alpha$  and  $\lambda$  are respectively the coefficients of heat exchange and heat conduction. As previously, the subscript "s" refers to the surface of the body, while "c" refers to the surroundings. If  $\frac{\partial t}{\partial x}$  is constant, then

$$\alpha \Delta t = \lambda \frac{\Delta t}{l}, \quad (3-3-3)$$

where  $l$  is the plate thickness and  $\Delta t$  is the temperature drive ( $\Delta t = t_0 - t_s$ ), from which we obtain

$$\frac{\Delta t}{\Delta t} = \frac{\alpha l}{\lambda} = \text{Bi}, \quad (3-3-4)$$

$\Delta t$  representing the temperature drop over the distance  $l$ .

For a curvilinear temperature distribution, in which case  $\frac{\partial t}{\partial x}$  is variable, we may assume that

$$\left( \frac{\partial t}{\partial x} \right)_s = -\epsilon \frac{\Delta t}{l}, \quad (3-3-5)$$

where  $\epsilon$  is the distortion factor or degree of distortion, a quantity which depends only on the configuration of the temperature-distribution curve:

$$\frac{\Delta t}{\Delta t} = \frac{1}{\epsilon} \text{Bi} = f(\text{Bi}), \quad (3-3-6)$$

since for similar distributions  $\epsilon$  has equal values and is a function only of the Biot number  $Bi$ .

From Fourier's heat-conduction equation

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2} \quad (3-3-7)$$

we obtain, by similar means, the ratio

$$\frac{\Delta_i t}{\Delta_i \tau} = \frac{a \tau}{l^2} = Fo, \quad (3-3-8)$$

where  $\Delta_i t$  is the temperature variation during the time  $\tau$ . Here, the principle of the transition was to assume that in the fictitious scheme the derivatives are equal to the ratio of the finite quantities. From a rigorous analytical point of view, this transformation corresponds only to the case of a constant derivative with zero initial conditions.

Consequently, if the temperature distributions in two bodies are mutually similar, then the quantities  $a\tau/l^2$  for the bodies must have the same value. Therefore, if the initial temperature distribution is given, the subsequent distributions depend on three quantities: the time  $\tau$ , the thermal diffusivity  $a$ , and the characteristic dimension  $l$  of the body. It should be noted that not the individual values of  $a$ ,  $\tau$ , and  $l$  are important but only their combination in the ratio  $a\tau/l^2$ . This combination is a generalized variable, or similarity criterion, and is called the Fourier number. The physical interpretation of the Fourier number is that it represents a generalized time, and thus it is one of the homochronicity criteria.

A simultaneous analysis of differential equation (3-3-7) and boundary condition (3-3-2) leads to the relation

$$\frac{\Delta_i t}{\Delta_i \tau} = \varphi \left( Bi, Fo, \frac{x}{l} \right), \quad (3-3-9)$$

where the ratio  $x/l$  is a parametric criterion. Ratio (3-3-8) is a ratio of temperature differences which have been transformed with respect to time and space. Thus, the Fourier number is a ratio of transformed temperature differences.

As a result, it was suggested that similarity criteria be derived with respect to the transforms rather than to the original functions. It is known that the Heaviside operator  $p$ , introduced with respect to the time variable  $\tau$ , gives the following relations:

$$\frac{dt}{d\tau} \doteq pt - t_0; \quad \frac{d^m t}{d\tau^m} \doteq p^m t - p^{m-1} t(0) \dots t^{m-1}(0); \quad (3-3-10)$$

$$\frac{1}{p} \doteq \tau; \quad \frac{1}{p^2} \doteq \frac{1}{2!} \tau^2; \quad \frac{1}{p^m} \doteq \frac{\tau^m}{m!},$$

where  $\doteq$  signifies operational correspondence. Consequently, if the derivatives are replaced by the ratio of the given physical quantity divided by the  $m$ th power of the variable, then this actually represents a transition from the original function to its Carson-Heaviside transform. This corresponds essentially to the analytical transformation itself. Operational methods are mathematical methods which transform the symbols of one operation to the symbols of another operation. The operational method consists in transforming the operator, not the quantities:

$$\frac{d^m t}{d\tau^m} \doteq p^m t \doteq \frac{t}{\tau^m} m!, \quad (3-3-11)$$

if  $t(0)=t'(0)=\dots t^{m-1}(0)=0$ .

The Heaviside-Carson method is a method of integral transformation, and it is defined by the relations

$$F(p) = p \int_0^{\infty} f(\tau) e^{-p\tau} d\tau; \quad (3-3-12)$$

$$f(\tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{E(p)}{p} e^{p\tau} dp, \quad (3-3-13)$$

where  $F(p)$  is the transform of the function  $f(\tau)$ . The Heaviside method is an interesting one in that it makes it possible to break down operations and to introduce fractional differentiators and integrators. For example,

$p \doteq \frac{d}{d\tau}$ ,  $p^{\frac{1}{2}} \doteq \frac{1}{\sqrt{\pi\tau}}$ . The integration operation is replaced by the operator  $i$ ;

under certain conditions an interrelation exists between these operators:

$p^{-1} = i \doteq \tau$ ;  $p^{-1} = i^2 \doteq \frac{\tau^2}{2}$ , etc.

Therefore, to obtain similarity criteria it is possible to use not only a system of differential equations but also mixed systems of integral-differential equations. This fact constitutes a definite advantage of operator methods.

Usually, the Heaviside-Carson or Laplace transformation methods are applied to unsteady-state processes, so that the transformation is carried out with respect to the time coordinate (integration being over limits from zero to  $\infty$ ). These processes are only beginning to be studied at present, and steady-state heat-transfer and mass-transfer processes have been of greatest interest. The Heaviside-Carson transformation may also be applied to the spatial coordinates, provided they vary from 0 to  $\infty$  (an infinite body).

When bodies of finite dimensions are involved, the finite integral transformations of Fourier, Hankel, and others are applied. However, only in certain particular cases is it possible to write the relation

$$\frac{d^2 t}{dx^2} \doteq p^2 t. \quad (3-3-14)$$

Moreover, a Fourier cosine or sine transformation does not eliminate the derivative  $\frac{\partial t}{\partial x}$  or any other odd derivative, since the integral  $\int f(x) \frac{\sin px}{\cos px} dx$  cannot be expressed in terms of  $F(p)$ . A finite Hankel integral transformation enables the elimination of the set of terms

$$F(t) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial t}{\partial r} \right) - \frac{n^2 t}{r^2}. \quad (3-3-15)$$

We thus obtain the relation

$$F(t) \doteq -p^2 t + \varphi(p, t_w), \quad (3-3-16)$$

where  $\varphi(p, t_w)$  is a function depending on the boundary conditions. Here, the subscript  $w$  denotes the surface of the body.

Consequently, a transition from differential equations to algebraic equations consist not only in replacing the partial derivative  $\frac{\partial^m t}{\partial x^m}$  by the expression  $p^m t$  but also in introducing additional relations which take into account the boundary conditions (the interaction between the body and its surroundings).

Physically, this means a transition from the actual values of the given quantities (differential equations and uniqueness conditions) to average values, calculated in accordance with the specific conditions of the physical problem using the methods of operator transformation.

Heaviside's operator method has a certain advantage over methods of integral transformations, from the standpoint of its use in the theory of operator similarity. During the evolution of the operational calculus the original point of view of Heaviside was for a time largely supplanted by the works of Carson, Bromwich, Doetsch, and van der Pol, who based their investigations on the Laplace transformation and the Mellin integral. However, a complete return to the initial point of view of Heaviside was made in 1946 by the Polish mathematician Mikusinski /5/. In the operational calculus of Heaviside and Mikusinski the operator  $p = \frac{d}{d\tau}$  is utilized to reduce a differential equation to an algebraic equation.

More recently, Ditkin /6/ introduced the Bessel operator  $B = \frac{d}{d\tau} \tau \frac{d}{d\tau}$ , closely related to the Bessel equation, which makes it possible to solve a number of differential and integral equations.

Let us now return to Gukhman's method of transition from a real process to a very simple model scheme, by means of the relation  $\frac{d^m t}{dx^m} \sim \frac{t}{x^m}$ . It should be noted that this method makes it possible to obtain similarity criteria for a whole class of phenomena, since differential equations describe a class of similar phenomena. If  $\frac{d^m t}{dx^m}$  is replaced by  $p^m t$  and the boundary conditions are taken into account, then it is possible to obtain not only the set of similarity criteria for a given law of interaction between the body and its surroundings but also to establish the interrelation between the similarity criteria and thus to find the basic similarity criteria.

Therefore, solutions for the transform, which may be obtained in an overwhelming majority of cases, constitute basic initial relations for determining the relation between the generalized variables. As is well-known, the difficulty of an analytical investigation consists in the transition from the transform to the original function [inverse transform] and not in obtaining the solution for the transform. Moreover, from the solution for the transform it is possible to obtain a number of approximate calculational relations, by using transform tables and by simplifying (by approximation) the solution for the transform.

Thus, when operator methods are applied to the differential equations and the uniqueness conditions are taken into account, it is possible to obtain relations between the average values of the basic similarity criteria for heat and mass exchange. The simultaneous application of similarity-theory methods and operator methods has enabled a further development of the theory of heat and mass exchange, on the basis of the methods of operator similarity.

### 3-4. Similarity Criteria for Heat and Mass Transfer

The similarity criteria for heat and mass transfer are obtained from the differential transfer equations and the uniqueness conditions. For a binary vapor-gas mixture, neglecting thermal diffusion and diffusive heat conduction, the equations describing heat and mass transfer are

$$\frac{\partial \vec{w}}{\partial \tau} + \vec{w} \nabla \vec{w} = \vec{g} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{w}; \quad (3-4-1)$$

$$\frac{\partial t}{\partial \tau} + \vec{w} \nabla t = a_p \nabla^2 t; \quad (3-4-2)$$

$$\frac{\partial \rho_{10}}{\partial \tau} + \vec{w} \nabla \rho_{10} = D_{12} \nabla^2 \rho_{10}, \quad (3-4-3)$$

with the corresponding uniqueness conditions (zero velocity at the surface of the body and constant velocity far from the surface).

In the simplest case the boundary conditions for heat and mass transfer are

$$\lambda (\nabla t)_s + \alpha (t_s - t_c) = \lambda' (\nabla t')_s + F \alpha (t'_s - t_c) = 0; \quad (3-4-4)$$

$$D_{12} \rho \left( \frac{1}{1 - \rho_{10}} \right)_s (\nabla \rho_{10})_s + \alpha_m \rho [(\rho_{10})_s - (\rho_{10})_c] = 0, \quad (3-4-5)$$

where the subscript  $s$  refers to the surface of the body and subscript  $c$  denotes values in the surroundings, far from the surface of the body. The relation  $t_s = t'_s$  is thus valid.

Let us now pass from the real transfer process to a model process in which the differential parameters are constant in space and time. In such a case  $\frac{\partial w}{\partial \tau} \sim \frac{w}{\tau}$ ;  $\nabla w \sim \frac{w}{l}$ ;  $\nabla^2 w \sim \frac{w}{l^2}$ ;  $\nabla^2 t \sim \frac{t}{l^2}$ ; etc., where  $w$  is some reference velocity, for example the velocity of the mainstream far from the wall, and  $t$  is a reference temperature (the flow temperature). Using this method, we obtain from equation (3-4-1) the following criteria:

$$\frac{w \tau}{l} = \text{Ho}; \quad \frac{g l}{w^2} = \text{Fr};^* \quad \frac{\rho}{\rho w^2} = \text{Eu}; \quad \frac{w l}{\nu} = \text{Re}, \quad (3-4-6)$$

where Ho is the homochronicity criterion [Strouhal number] (all the similarity criteria involving time are called homochronicity criteria, since they determine the temporal transformation factor in terms of multipliers made up of the other physical quantities); Fr is the Froude number; Eu is the Euler number; Re is the Reynolds number.

Consequently, for two or more hydromechanically similar systems the quantities Ho, Fr, Eu, and Re have the same values at all similar points. In practical applications of similarity criteria it is advisable to give some of these criteria a more convenient form, namely one which enables a direct determination of the quantities involved in the criteria.

For example, in studies of the fluid motion caused by a difference in the densities of individual particles it is impossible to measure the velocity. In this case, instead of the Froude number it is more convenient to use the Galileo number Ga or the Archimedes number Ar. The latter is obtained

\* [The Froude number is usually defined as the reciprocal of this quantity.]

by dividing the Galileo number by  $\frac{\rho - \rho_0}{\rho}$ , where  $\rho$  and  $\rho_0$  are the fluid densities at two points in the system:

$$Ga = Fr Re^2 = \frac{g l^3}{\nu^2}; \quad (3-4-7)$$

$$Ar = Ga \frac{\rho - \rho_0}{\rho} = \frac{g l^3 (\rho - \rho_0)}{\nu^2 \rho}. \quad (3-4-8)$$

If the density difference is determined by the temperature difference  $\Delta t$ , then  $\frac{\rho - \rho_0}{\rho} = \beta \Delta t$ , where  $\beta$  is the bulk expansion coefficient for the mixture. When this value is substituted into formula (3-4-8), we obtain the Grashof number Gr:

$$Gr = \beta \frac{g l^3}{\nu^2} \Delta t. \quad (3-4-9)$$

It should be noted that Fr, Ga, Ar, and Gr are actually identical and that these are four different forms of the same criterion.

The Euler number is usually expressed in a different form, in which the pressure drop (hydrodynamic resistance) in a liquid or gas pipeline enters in instead of the pressure. The relation between the similarity criteria for forced isothermal motion is thus generally represented as

$$Eu = f(Re). \quad (3-4-10)$$

From equation (3-4-2) and boundary condition (3-4-4) we obtain the Fourier Number Fo, the Peclet number Pe, the Lewis number Le, the Nusselt number Nu, the Biot number Bi, and the parametric criterion  $K_0$ :

$$Fo = \frac{a_p \tau}{l^2}; \quad Pe = \frac{wl}{a_p}; \quad Le = \frac{D}{a_p}; \quad Nu = \frac{al}{\lambda};$$

$$Bi = \frac{al}{\lambda}; \quad K_0 = (c_1 - c_2)/c_p. \quad (3-4-11)$$

Consequently, for thermal similarity between systems the quantities Fo, Pe, Nu, and Le have the same values at all similar points of the systems.

The Péclet number may be transformed and represented as the product of the Reynolds number times the Prandtl number Pr (where  $Pr = \nu/a$ ):

$$Pe = \frac{wl}{a} = \frac{wl}{\nu} \frac{\nu}{a} = Re Pr. \quad (3-4-12)$$

In the experimental study of a heat-exchange process which is not complicated by mass exchange ( $Le = 0$ ), the important quantity is the heat-exchange coefficient  $a$ . Therefore, the relation between the similarity criteria must be

$$Nu = f(Fo, Pe). \quad (3-4-13)$$

A necessary prerequisite for thermal similarity is mechanical similarity. Thus, the Reynolds and Grashof numbers must be introduced into the argument of the criterial relation. The relation between the similarity criteria for processes of pure heat exchange consequently may be written as

$$Nu = f(Fo, Re, Gr, Pr). \quad (3-4-14)$$

For a vapor-gas mixture, in which the lift force is determined not only by the temperature difference but also by the moisture-content difference (the density of moist air is lower than the density of dry air), the Grashof number must be replaced by the Archimedes number.

The parametric criterion  $K_c = (c_1 - c_2)/c_p$  for moist air may be replaced by the two criteria  $K_M = M_1/M_2$  and  $K_d = d_1$ , where  $M_1$  and  $M_2$  are the molecular weights of the binary-mixture components (dry air and vapor) and  $d$  is the moisture content, since the isobaric specific heat of a mixture is a function of the molecular weights of the mixture components and of the moisture content:

$$c_p = c_{p10} + c_2 p_{10} \quad \text{and} \quad p_{10} = \frac{d}{1-d}. \quad (3-4-15)$$

From mass-transfer equation (3-4-9) and boundary condition (3-4-5) we obtain the following criteria:

$$Fo_m = \frac{D_{12}\tau}{l^2}; \quad Pe_m = \frac{wl}{D_{12}}; \quad Nu_m = \frac{\alpha_m l}{\lambda_m}; \quad K_1 = \frac{1}{1-p_{10}}, \quad (3-4-16)$$

where  $Fo_m$ ,  $Pe_m$ , and  $Nu_m$  are the mass-exchange (diffusion) Fourier, Péclet and Nusselt numbers (compare with the heat-exchange criteria defined in (3-4-11)). Obviously,

$$Pe_m = \frac{wl}{D_{12}} = \frac{wl}{\nu} \frac{\nu}{D_{12}} = Re Pr_m, \quad (3-4-17)$$

where  $Pr_m = \frac{\nu}{D_{12}}$  is the mass-exchange (diffusion) Prandtl number. By means of the Lewis number, the mass-exchange Fourier number  $Fo_m$  may be expressed in terms of the heat-exchange Fourier number:

$$Fo_m = \frac{D_{12}\tau}{l^2} = \frac{\alpha\tau}{l^2} \frac{D_{12}}{\alpha} = Fo Le. \quad (3-4-18)$$

Criterion  $K_1$  may be replaced by the moisture content  $d$  of the air near the surface of the body, since  $d$  is a single-valued function of  $p_{10}$  (see formula (3-4-15)).

The processing of experimental data on mass exchange is usually carried out by means of some relation between the similarity criteria:

$$Nu_m = f(Fo_m, K_1, Re, Ar, Pr_m). \quad (3-4-19)$$

When applied to individual problems criterial relations (3-4-14) and (3-4-19) may often be simplified. For example, for steady-state processes the criteria  $Fo$  and  $Fo_m$  drop out, while for forced turbulent motion it is usually possible to neglect the effect of free motion, with the result that  $Gr$  and  $Ar$  drop out.

When liquid evaporates from a free surface in the boundary layer of a vapor-gas mixture, very small water droplets will form and fall down (macroscopic dispersion of the liquid by a turbulent air flow). This effect, a transfer into the boundary layer of the liquid phase, is particularly evident during the drying of porous bodies, when the presence of excess pressure inside the body facilitates the macroscopic dispersion of the liquid. In this case a heat-sink term, representing evaporation of the liquid phase in the boundary layer, appears in the differential equation describing heat transfer in this layer (see formula (2-1-78)):

$$\frac{(h_1 - h_2) I_1}{c_p \rho} = \frac{r_{12} I_1}{c_p \rho}. \quad (3-4-20)$$

A comparison of this expression for the heat sink with the first term on the right side of differential heat-transfer equation (3-4-2) leads us to define the Gukhman number

$$Gu = \frac{r_{12} I_1}{\lambda T_0} = \frac{\lambda(T_c - T_{as})}{\lambda T_0} = \frac{T_c - T_{as}}{T_0}, \quad (3-4-21)$$

since the heat, transferred by conduction, required to evaporate the very small liquid particles is

$$r_{12}I_1 = \lambda(T_c - T_{as}), \quad (3-4-22)$$

where  $T_{as}$  is the adiabatic-saturation temperature of air. The Gukhman number is a variable quantity, since  $T_c$  varies both across and along the boundary layer. However, for steady-state conditions with a given temperature drive the average Gukhman number may be introduced, equal to  $(T_c - T_{as})/T_c$ , where  $T_c$  is the absolute temperature of the flow. The Gukhman number also should be introduced into the criterial relation for the mass-exchange criterion  $Nu_m$ , since the source strength  $I_1$  enters into the differential mass-transfer equation. The criterial relations for  $Nu$  and  $Nu_m$  for the steady state will thus be

$$Nu = f(Re, Ar, Pr, Gu); \quad (3-4-23)$$

$$Nu_m = f(Re, Ar, Pr_m, Gu). \quad (3-4-24)$$

In a rigorous study of heat and mass exchange it is necessary to take into account diffusive heat conduction, by introducing an extra term into heat-transfer equation (3-4-2). Thermal diffusion is taken into account in the differential mass-transfer equation by means of an additional term on the right side,

$$\frac{DQ_{cm1}}{T} \nabla^2 t.$$

These additional terms give us two new criteria, the Soret number and the Dufour number, defined as\*

$$So = c_m Q^* = k_T \quad \text{and} \quad Du = Q^*/c_p T_c, \quad (3-4-25)$$

where  $k_T$  is the thermal-diffusion constant.

The differential heat-transfer and mass-transfer equations for a porous body

$$\frac{\partial t}{\partial \tau} = a_q \nabla^2 t + \epsilon \frac{c_m}{c_q} \frac{\partial \theta}{\partial \tau} + \frac{c_p k}{c_q \gamma} \nabla p \nabla t; \quad (3-4-26)$$

$$\frac{\partial \theta}{\partial \tau} = a_m (\nabla^2 \theta + \delta_\theta \nabla^2 t + \delta_p \nabla^2 p); \quad (3-4-27)$$

$$\frac{\partial p}{\partial \tau} = a_p \nabla^2 p - \epsilon \frac{c_m}{c_b} \frac{\partial \theta}{\partial \tau}, \quad (3-4-28)$$

and the boundary conditions

$$-\lambda_q (\nabla t)_s + j_q(\tau) - (1 - \epsilon) r j_m(\tau) = 0; \quad (3-4-29)$$

$$\lambda_{m1} (\nabla \theta)_s + \delta_\theta (\nabla t)_s + \delta_p (\nabla p)_s + j_m(\tau) = 0; \quad (3-4-30)$$

$$p_s = p = \text{constant}, \quad (3-4-31)$$

where

$$j_q = \alpha_q (t_c - t_s) \quad \text{and} \quad j_m = \alpha_\theta (\theta_s - \theta_c), \quad (3-4-32)$$

heat-exchange parameters being denoted by the subscript  $q$  and mass-exchange parameters being denoted by  $m$ , lead to several new similarity criteria.

1. The criteria for homochronicity of the transfer-potential fields, called the Fourier numbers for the corresponding type of heat and mass transfer,

\* In /7/  $So$  and  $Du$  are defined differently; however, their physical meanings (see below) are the same in both cases.



are defined as

$$Fo_q = a_q \tau / R^2, \quad Fo_m = a_m \tau / R^2, \quad Fo_p = a_p \tau / R^2, \quad (3-4-33)$$

where  $R$  is a characteristic dimension of the body. These criteria are interrelated by means of the quantities  $Lu$  and  $Lu_p$ , to be defined next.

2. The interrelation criteria for mass and heat transfer are [the Lukomskii numbers]

$$Lu = \frac{a_m}{a_q} \quad \text{and} \quad Lu_p = \frac{a_p}{a_q}; \quad (3-4-34)$$

consequently,

$$Fo_m = Fo Lu \quad \text{and} \quad Fo_p = Fo Lu_p. \quad (3-4-35)$$

3. The Kossovich number is defined as

$$Ko = \frac{r c_m \Delta \theta}{c_q \Delta t} = \frac{r \Delta u}{\Delta t}. \quad (3-4-36)$$

4. The Posnov numbers for moisture transfer by mass conduction ( $Pn_\theta$ ) and by seepage ( $Pn_p$ ) are defined as

$$Pn_\theta = \frac{\delta_\theta \Delta t}{\Delta \theta} \quad \text{and} \quad Pn_p = \frac{\delta_p \Delta t}{\Delta \theta}. \quad (3-4-37)$$

The Posnov number is usually defined as the relative drop in moisture content of the body caused by a temperature difference:

$$Pn = \frac{\delta \Delta t}{\Delta u}, \quad (3-4-38)$$

where  $\delta$  is the thermal-gradient coefficient

$$\delta = c_m \left[ \delta_\theta + \left( \frac{\partial \theta}{\partial T} \right)_u \right]. \quad (3-4-39)$$

If this relation is substituted into the definition of the criterion, then we obtain

$$Pn = Pn_\theta + \left( \frac{\partial \theta^*}{\partial T^*} \right)_u, \quad (3-4-40)$$

where  $\left( \frac{\partial \theta^*}{\partial T^*} \right)_u$  is the relative mass-transfer temperature coefficient

$$\left( \frac{\partial \theta^*}{\partial T^*} \right)_u = \left[ \frac{\partial (\theta / \Delta \theta)}{\partial (T / \Delta T)} \right]_u. \quad (3-4-41)$$

If the mass-transfer temperature coefficient is zero, then  $Pn$  and  $Pn_\theta$  are equal. In analytical calculations the product  $\epsilon Pn Ko$  is often introduced as a generalized variable, and some authors call it the Fedorov number

$$Fe = \epsilon Ko Pn = \frac{\epsilon \delta r}{c_q}. \quad (3-4-42)$$

This criterion does not depend on the choice of the heat-transfer and mass-transfer potentials and is determined only by the thermophysical parameters  $\epsilon$ ,  $\delta$ ,  $r$ , and  $c_q$ .

5. Finally, from differential equation (3-4-28) the generalized variable

$$K = Ra = \frac{c_b \Delta p}{c_m \Delta \theta} = \frac{Bu}{Ko} \quad (3-4-43)$$

is obtained, where the Bulygin number is defined as

$$Bu = \frac{r c_b \Delta p}{c_q \Delta T}. \quad (3-4-44)$$

From boundary conditions (3-4-29) to (3-4-32) we obtain two additional Biot numbers, the heat-exchange criterion

$$\text{Bi}_q = \frac{aR}{\lambda_q} = \frac{a_h R}{a_q} \quad (3-4-45)$$

and the mass-exchange criterion

$$\text{Bi}_m = \frac{a_g R}{\lambda_m} = \frac{a_u R}{a_m}. \quad (3-4-46)$$

The Biot numbers vary from zero to infinity ( $0 < \text{Bi} < \infty$ ). If the heat fluxes  $j_q(\tau)$  and  $j_m(\tau)$  cannot be determined from (3-4-32), then instead of  $\text{Bi}_q$  and  $\text{Bi}_m$  two Kirpichev numbers are used, the mass-exchange criterion

$$\text{Ki}_m = \frac{j_m(\tau) R}{\lambda_m \Delta \theta} = \frac{j_m(\tau) R}{a_m \gamma \Delta u} \quad (3-4-47)$$

and the heat-exchange criterion

$$\text{Ki}_q = \frac{j_q(\tau) R}{\lambda \Delta T} = \frac{j_q(\tau) R}{a \gamma \Delta h}. \quad (3-4-48)$$

If the fluxes  $j_q(\tau)$  and  $j_m(\tau)$  are determined by the laws describing convective heat and mass exchange, then the following interrelations exist between the above criteria:

$$\text{Ki}_q = \text{Bi}_q \frac{T_c - T_s}{\Delta T} \quad \text{and} \quad \text{Ki}_m = \text{Bi}_m \frac{\theta_s - \theta_c}{\Delta \theta}. \quad (3-4-49)$$

The quantities  $\Delta T$ ,  $\Delta u$ ,  $\Delta \theta$ , and  $\Delta p$  which enter into the similarity criteria for heat and mass transfer are selected in accordance with the conditions of the specific problem.

### 3-5. The Physical Meaning of the Similarity Criteria for Heat and Mass Transfer

Let us now consider the physical significance of the basic similarity criteria for heat and mass transfer. The Reynolds number may be written as

$$\text{Re} = \frac{wl}{\nu} = \frac{\rho w^2}{\eta w/l}. \quad (3-5-1)$$

Consequently, this criterion represents the ratio of the inertia forces to the friction forces.

The physical meaning of the Nusselt number may be represented in various ways. For example, this criterion may be regarded as the ratio of the density of the actual heat flux (determined in terms of the heat-exchange coefficient  $a$ ) to the density of the heat flux which would exist under conditions of pure heat conduction in a layer of thickness  $l$ :

$$\text{Nu} = \frac{al}{\lambda} = \frac{a}{\lambda/l}. \quad (3-5-2)$$

If we consider the heat exchange between two walls between which there is a gap of thickness  $l$  filled with a medium of thermal conductivity  $\lambda$ , then the Nusselt number is the ratio of the apparent thermal conductivity  $\lambda_a = al$  to the actual thermal conductivity  $\lambda$ :

$$\text{Nu} = \frac{al}{\lambda} = \frac{\lambda_a}{\lambda}. \quad (3-5-3)$$

Consequently, the Nusselt number indicates the increase in heat-exchange intensity due to convection, in comparison with pure heat conduction in a medium at rest. For  $Nu=1$  the heat-exchange process is identical to pure thermal conduction through a layer of thickness  $l$ .

It was noted earlier that the Nusselt number differs from the Biot number ( $Bi=al/\lambda'$ ) only in that it contains the gas thermal conductivity  $\lambda$  in its denominator rather than the solid-body thermal conductivity  $\lambda'$ . The Biot number is a decisive criterion, while the Nusselt number is not. This is the main difference between the two. Moreover, it should be noted that the Biot number is not related to any supplementary conditions; it is obtained directly from the boundary condition. In deriving the Nusselt number from the boundary condition, on the other hand, it is assumed that heat is transferred through the gas layer adjacent to the surface by conduction.

The Biot number may be represented as the ratio of some important dimension  $l$  to the distance  $\lambda/a$  from a reference point to the surface of the body (see Figure 2-4):

$$Bi = \frac{al}{\lambda'} = \frac{l}{(\lambda'/a)}. \quad (3-5-4)$$

In addition, the Biot number may be regarded as the ratio of the temperature drop  $\Delta t$  in the wall to the temperature drive  $\Delta t$ . Under steady-state conditions  $Bi$  is exactly equal to this ratio (see formula (3-3-4)). Finally, the Biot number represents the ratio of the thermal resistance  $\frac{\lambda}{l}$  of the wall to the thermal resistance to heat transfer  $\frac{1}{a}$  at the surface.

The previous statements about the Nusselt number can be extended to the mass-exchange Nusselt number, which is defined as

$$Nu_m = \frac{a_m l}{\lambda_{m1}} = \frac{a_m l}{D_{12}}, * \quad (3-5-5)$$

where  $\lambda_{m1}$  is the mass conductivity of a binary mixture ( $\lambda_{m1}=L_n$ ), and  $D_{12}$  is the interdiffusion coefficient ( $D_{12}=\lambda_{m1}/c_{m1}\rho$ ). The mass-exchange coefficient  $a_m$  is the analog of the heat-exchange coefficient  $a_h$  referred to the difference in the volume enthalpy concentration ( $Nu=a_h l/a_p$ ).

The Péclet number is defined as

$$Pe = \frac{wl}{a} = \frac{w\rho c_p}{\lambda/l}. \quad (3-5-6)$$

The quantity  $c_p \rho w$  is the heat-flux density, for the heat transferred by a moving fluid for a temperature drop of  $1^\circ$ . The ratio  $(\lambda/l)$  is the density of the conductive heat flux in a layer of thickness  $l$ . In both cases the same motivating temperature drive must be taken. Consequently, the Péclet number represents the ratio of the convective heat transfer to the conductive transfer in a moving mixture.

The Prandtl number  $\nu/a$  is the ratio of two molecular-transfer coefficients, the numerator representing momentum transfer by means of internal friction and the denominator representing heat transfer by conduction. Consequently, the Prandtl number also represents the ratio between the velocity and temperature fields. The relation  $Nu=f(Re, Pr)$  may thus be given the following physical interpretation: the amount of transferred heat ( $Nu$ ) depends on the nature of the velocity field and on its coupling to the temperature field ( $Pr$ ).

\* [This definition should be  $Nu_m = \frac{a_m l}{\lambda_{m1}} = \frac{a_m l}{D_{12} c_{m1} \rho}$ .]

For ideal gases the Prandtl number can be calculated using a formula from the kinetic theory of gases, provided the number of atoms is known. Theoretically  $Pr$  for an ideal gas is independent of temperature; for a real gas  $Pr$  must be determined experimentally.

The mass-exchange Prandtl number is sometimes called the Schmidt number ( $Sc \equiv Pr_m$ ), by analogy with the heat-exchange Prandtl number. It characterizes the coupling between the chemical-potential field and the temperature field.

As a result of their mutual independence, any combinations of the above similarity criteria constitute new, perfectly valid criteria. Criteria are chosen from all the many possible combinations on the basis of practical considerations, so that a desired quantity will, if possible, enter into only one criterion. Variables which are not specified according to a condition should be eliminated. An example of such a combination is the Stanton number

$$St = \frac{Nu}{Re Pr} = \frac{\alpha}{c_p \rho w}. \quad (3-5-7)$$

From the system of differential equations describing heat and mass transfer in a binary gaseous mixture, taking into account thermal diffusion and diffusive thermal conduction, two similarity criteria are obtained, the Soret number and the Dufour number. The Soret number, characterizing the thermal-diffusion effect, is equal to the thermal-diffusion constant, a quantity which depends on the conditions and mechanism of interaction between the molecules. The Dufour number is the ratio of the isothermal mass-transfer heat  $Q^*$  to the enthalpy of a unit mass of mixture. Consequently, the Dufour number characterizes the diffusive heat conduction, in comparison with the convective heat transfer, under conditions when the linear velocities of the diffusive and convective transfers are equal.

The dimensionless parameter  $Lu$  indicates the rate of variation of the field of the mass-transfer potential ( $\theta$  or  $p$ ) relative to the variation of the temperature field (the heat-transfer potential). Similarly, the Lewis number represents the rate of variation of the chemical-potential field with respect to the temperature field in a moving fluid flow (gaseous mixture). The difference between  $Lu$  and  $Le$  is analogous to the difference between  $Bi$  and  $Nu$ .  $Lu$  is the ratio of the mass-transfer potential conductivity  $a_m$  or  $a_p$  of a porous body to the thermal diffusivity  $a_q$ , while the Lewis number is the ratio of the diffusion coefficient  $D$  of a mixture to the quantity  $a_q$ . Consequently,  $Lu$  indicates the coupling between heat and mass transfer in a porous body or dispersed medium, while  $Le$  indicates this coupling for a gaseous mixture. The numerical value of  $Le$  is close to unity, whereas  $Lu$  varies from 0 to 2 or 3 and  $Lu_p$  is of the order of 100 or 1000; thus,  $Lu$  varies over a wide range.

The Kossovich number  $Ko$  gives the relation between the amount of heat ( $r\Delta u$ ) consumed during the vaporization of liquid and the heat ( $c_q\Delta t$ ) consumed to heat a moist body.

The Posnov number indicates the drop  $\Delta\theta$  in mass-transfer potential caused by a temperature difference ( $Pn_\theta$ ) or a pressure difference ( $Pn_p$ ).

The Bulygin number is the ratio of the heat consumed during evaporation of the liquid which is removed by seepage transfer to the heat consumed in heating a moist body.

The Kirpichev heat-exchange number  $Ki_q$  indicates the ratio of the heat flux conveyed to the surface of a body to the heat flux drawn off within a body. Similarly, the Kirpichev mass-exchange number  $Ki_m$  is the ratio of the mass-transfer flux to the isothermal mass flux supplied to the surface of a body.

### 3-6. A Dimensionless Form of the Mathematical Model of Heat and Mass Transfer

In the previous sections we considered two methods for obtaining dimensionless numbers. An evaluation of the experimental data in terms of generalized dimensionless variables (using criterial equations) makes it possible to extend the results of a single experiment to a group of similar phenomena. Similarity theory may be applied most fruitfully in cases when it is impossible to find a relation between the variables by solving the corresponding differential equations. In those particular cases when such a relation can be found, the necessity of using similarity theory diminishes. However, even then dimensionless means of describing the mathematical model of the process are of definite advantage.

First, a solution of the heat-transfer and mass-transfer equations in dimensionless form is less difficult than a solution of these equations in dimensional form, since the number of generalized variables is reduced. For this reason the amount of computational work associated with dimensionless solutions is at a minimum.

Second, when the differential equations and limiting conditions are written in dimensionless form, it is possible to make the following step toward a generalization of transfer phenomena: for a large group of coupled transfer phenomena the systems of differential equations are identical and the physical meanings of the corresponding dimensionless numbers are similar. Consequently, the possibility presents itself of setting up models of coupled unsteady-state processes, whereby the conditions of complex and expensive processes may be investigated and analyzed using relatively simple and inexpensive analogs.

In Chapter II we considered the generalized system of differential transfer equations. This system of equations can easily be put into dimensionless form. For example, for a one-dimensional stationary body or for a medium without sources system of equations (2-6-16) in Chapter II becomes

$$\frac{\partial \theta_i}{\partial Fo_i} = \sum_{j=1}^n K_{ij} \left( \frac{\partial^2 \theta_j}{\partial X^2} + \frac{1}{X} \frac{\partial \theta_j}{\partial X} \right), \quad (3-6-1)$$

$(i, j = 1, 2, \dots, n)$

where  $\theta_i = \frac{\theta_i - \theta_*}{(\Delta\theta)_i}$  is the dimensionless transfer potential ( $(\Delta\theta)_i$  is some potential difference chosen appropriately with respect to the conditions of the problem, while  $\theta_*$  is a fixed initial potential);  $K_{ij} = \frac{(\Delta\theta)_j}{(\Delta\theta)_i} \frac{R_{ij}}{a_i}$  is a simple or compound dimensionless number characterizing the transfer of the  $i$ th coupled quantity under the effect of the  $j$ th motive force;  $Fo_i = \frac{a_i \tau}{l^2}$  is the

homochronicity criterion;  $X = \frac{\xi}{l}$  is a dimensionless spatial coordinate; and  $\Gamma$  is a form factor, zero for an infinite plate, unity for an infinite cylinder, and equal to two for an infinite sphere. In particular, the parameters entering into (3-6-1) for the case of a binary gaseous mixture ( $i, j=1, 2$ ) are

$$\left. \begin{aligned} \theta_1 &= t^*, \quad \theta_2 = \rho^*_{10}; \\ K_{11} &= 1 + \text{Du So Le}; \quad K_{12} = \text{Du Le}; \\ K_{21} &= \text{Le So}; \quad K_{22} = \text{Le}, \end{aligned} \right\} \quad (3-6-2)$$

whereas for molecular heat and mass transfer in a dispersed medium, which is coupled with a phase or chemical transformation ( $i, j=1, 2$ ), the parameters are

$$\left. \begin{aligned} \theta_1 &= T; \quad \theta_2 = \theta; \\ K_{11} &= 1 + \text{Ko}^* \text{Pn Lu}; \quad K_{12} = -\text{Ko}^* \text{Lu}; \\ K_{21} &= -\text{Lu Pn}; \quad K_{22} = \text{Lu}. \end{aligned} \right\} \quad (3-6-3)$$

If equation (3-6-1) describes the drying or moistening of a body, then the modified Kossovich number  $\text{Ko}^*$  is actually the product of the phase-transition ratio times the ordinary Kossovich number:  $\text{Ko}^* = \epsilon \text{Ko}$ .

Relations (3-6-2) and (3-6-3) are a good example, provided we keep in mind the previously mentioned physical meanings of the individual similarity criteria, of the deep-seated resemblance existing between the two different transfer processes and the individual similarity criteria. In this case the coupling criteria  $\text{Le}$  and  $\text{Lu}$ , the Soret number and the Posnov number, and the Dufour number and the Kossovich number  $\text{Ko}^*$  form analogous pairs. The difference in the signs of  $K_{12}$  and  $K_{21}$  is due to the choice of the dimensionless potential  $\theta$ . The analogy between phenomena of heat and mass transfer in various media makes it possible to concentrate our main attention in the following chapters on solving and analyzing system of equations (3-6-1) with the dimensionless coefficients expressed by relations such as (3-6-3), namely the system of equations:

$$\begin{aligned} \frac{\partial T(X, \text{Fo})}{\partial \text{Fo}} &= (1 + \text{Ko}^* \text{Pn Lu}) \left[ \frac{\partial^2 T(X, \text{Fo})}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial T(X, \text{Fo})}{\partial X} \right] - \\ &\quad - \text{Ko}^* \text{Lu} \left[ \frac{\partial^2 \theta(X, \text{Fo})}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \theta(X, \text{Fo})}{\partial X} \right]; \end{aligned} \quad (3-6-4)$$

$$\begin{aligned} \frac{\partial \theta(X, \text{Fo})}{\partial \text{Fo}} &= \text{Lu} \left[ \frac{\partial^2 \theta(X, \text{Fo})}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \theta(X, \text{Fo})}{\partial X} \right] - \\ &\quad - \text{Lu Pn} \left[ \frac{\partial^2 T(X, \text{Fo})}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial T(X, \text{Fo})}{\partial X} \right]. \end{aligned} \quad (3-6-5)$$

It will also be possible to relate, in a simple way, problems of coupled transfer to problems from the classical theory of heat conduction and diffusion.

## Chapter IV

### UNSTEADY FIELDS OF HEAT-TRANSFER AND MASS-TRANSFER POTENTIAL WITH BOUNDARY CONDITIONS OF THE FIRST KIND

#### 4-1. The Dimensionless Boundary Conditions

For boundary conditions of the first kind the values of the transfer potentials at the surface of the body or system are given. Such conditions may exist in different types of heat-engineering and chemical equipment while the coefficients of heat and mass transfer are being determined, and in many other cases as well. During the course of the process the values of the potentials at the surface may either be maintained constant or varied in time according to some known law.

In general boundary conditions of the first kind represent a particular case of the more general class of boundary conditions of the third kind. If the heat-exchange and mass-exchange coefficients or the corresponding Biot numbers go to infinity, then the latter conditions become boundary conditions of the first kind. However, there are a number of problems related to the transfer of heat and matter which are described only by boundary conditions of the first kind (for example, heat and mass transfer in an insulated body). In such cases an additional condition is imposed upon the heat and mass transfer, namely that the integral mass-transfer potential (or for a moist body, the average mass content) must be constant. For a one-dimensional classical body this condition may be written as

$$\frac{\Gamma+1}{R^{\Gamma+1}} \int_0^R \xi^\Gamma \theta(\xi, \tau) d\xi = \theta_0 = \text{constant},$$

or in terms of dimensionless quantities as

$$\int_0^1 X^\Gamma \theta(X, Fo) dX = 0, \quad (4-1-1)$$

where  $\theta = (\theta_0 - \theta)/\theta_0$  is the dimensionless mass-transfer potential;  $X = \xi/R$  is the dimensionless spatial coordinate ( $\xi$  is the instantaneous position, and  $R$  is a characteristic dimension, equal to half the thickness of an infinite plate or to the radius of a cylinder or sphere); and  $\Gamma$  is the form factor.

As mentioned above in § 3-6, this and the following chapters will be devoted to solving the following one-dimensional system of heat-transfer and mass-transfer equations, expressed in terms of dimensionless quantities:

$$\begin{aligned} \frac{\partial T(X, Fo)}{\partial Fo} = & (1 + Ko^* Pn Lu) \left[ \frac{\partial^2 T(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial T(X, Fo)}{\partial X} \right] - \\ & - Ko^* Lu \left[ \frac{\partial^2 \theta(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \theta(X, Fo)}{\partial X} \right]; \end{aligned} \quad (4-1-2')$$

$$\begin{aligned} \frac{\partial \theta(X, Fo)}{\partial Fo} = & \text{Lu} \left[ \frac{\partial^2 \theta(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \theta(X, Fo)}{\partial X} \right] - \\ & - \text{Lu Pn} \left[ \frac{\partial^2 T(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial T(X, Fo)}{\partial X} \right]. \end{aligned} \quad (4-1-3)$$

Let us note that, taking into account (4-1-3), equation (4-1-2') may be rewritten as

$$\frac{\partial T(X, Fo)}{\partial Fo} = \frac{\partial^2 T(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial T(X, Fo)}{\partial X} - K_0 \star \frac{\partial \theta(X, Fo)}{\partial Fo}. \quad (4-1-2)$$

System of equations (4-1-2') and (4-1-3) is equivalent to system (4-1-2) and (4-1-3), but the method for solving the latter is somewhat simpler. Therefore, system (4-1-2) and (4-1-3) will be solved here, for a constant value of the dimensionless integral mass-transfer potential (see (4-1-1)) and for a constant value of the dimensionless temperature at the body surface of

$$T(1, Fo) = \frac{t_s - t_0}{t_0} = -1. \quad (4-1-4)$$

The potential differences  $\Delta t$  and  $\Delta \theta$  which enter into the similarity criteria (see Chapter III, § 3-4) will be considered, in this chapter, to be equal to the initial values of the transfer potentials:  $\Delta t = t_0$  and  $\Delta \theta = \theta_0$ . In addition, the problems are assumed to be symmetric, so that

$$\frac{\partial T(0, Fo)}{\partial X} = \frac{\partial \theta(0, Fo)}{\partial X} = 0, \quad T(0, Fo) \neq \infty, \quad \theta(0, Fo) \neq \infty. \quad (4-1-5)$$

In § 4-3 several solutions will be given of problems in which the heat transfer and mass transfer do not affect one another. Problems with variable surface temperature will be deferred until Chapter VII.

Solutions of the system of heat-transfer and mass-transfer equations with boundary conditions of the first kind have been obtained by Kuznetsov, Mikhailov, and Sizyakova. In addition, a detailed analysis of the process has been given by Sizyakova and Mikhailov, while Sizyakova, Oleinikov, Kazanskii, and Gamayunov have verified the correctness of the theoretical model by means of direct experiments. On the basis of analytical solutions, Gamayunov, Kuznetsov, and Sizyakova have proposed methods for an experimental determination of the transfer coefficients. Solutions of the corresponding problems for the separate transfer of heat and matter have been given by Lykov, Carslaw and Jaeger, Crank, and others.

#### 4-2. Unsteady Fields of Heat-Transfer and Mass-Transfer Potential for Constant Integral Mass-Transfer Potential

The general method for solving the system of differential heat-transfer and mass-transfer equations in terms of dimensionless quantities will now be demonstrated, using as an example the determination of the potential fields for heat and mass transfer with constant initial conditions  $T(X, 0) = \theta(X, 0) = 0$  and for bodies of classical shape. The solutions in this section were obtained by means of Laplace transformation.

##### a) Solution for an infinite plate ( $\Gamma = 0$ )



If we write

$$F_L(X, s) = \int_0^\infty F(X, Fo) \exp(-sFo) dFo, \quad (4-2-1)$$

and then apply this integral Laplace transformation to equations (4-1-2) and (4-1-3), taking into account the initial conditions  $T(X, 0) = \theta(X, 0) = 0$ , we obtain

$$sT_L(X, s) = T''_L(X, s) - Ko^*s\theta_L(X, s); \quad (4-2-2)$$

$$s\theta_L(X, s) = Lu\theta''_L(X, s) - LuPnT''_L(X, s). \quad (4-2-3)$$

In the following the parentheses and the independent variables  $X$  and  $s$  will be omitted, provided no misunderstanding is possible.

The differential equation for the dimensionless heat-transfer potential  $T_L$  is found as follows. From equation (4-2-2)  $\theta_L$  and its second derivative  $\theta''_L$  may be determined and the expressions obtained substituted into equation (4-2-3). Then, after simplification, we obtain

$$T_L^{IV} - s \left( 1 + Ko^*Pn + \frac{1}{Lu} \right) T''_L + \frac{s^2}{Lu} T_L = 0. \quad (4-2-4)$$

The general integral of this nonhomogeneous [fourth-order] ordinary differential equation may be determined by means of another Laplace transformation with respect to the coordinate  $X$ . After the necessary operations, we obtain

$$T_L = A_1 \exp(\sqrt{s v_1} X) + A_2 \exp(\sqrt{s v_2} X) + \\ + A_3 \exp(-\sqrt{s v_1} X) + A_4 \exp(-\sqrt{s v_2} X), \quad (4-2-5)$$

whereas the transform of the mass-transfer potential is

$$\theta_L = -\frac{1}{Ko^*} [A_1(1 - v_1^2) \exp(\sqrt{s v_1} X) + A_2(1 - v_2^2) \exp(\sqrt{s v_2} X) + \\ + A_3(1 - v_1^2) \exp(-\sqrt{s v_1} X) + A_4(1 - v_2^2) \exp(-\sqrt{s v_2} X)], \quad (4-2-6)$$

where  $A_k (k=1, 2, 3, 4)$  are arbitrary quantities, constant with respect to the spatial coordinates; and

$$v_i^2 = \frac{1}{2} \left[ \left( 1 + Ko^*Pn + \frac{1}{Lu} \right) + (-1)^i \sqrt{\left( 1 + Ko^*Pn + \frac{1}{Lu} \right)^2 - \frac{4}{Lu}} \right], \\ (i=1, 2). \quad (4-2-7)$$

Values of  $v_i$  ( $i=1, 2$ ) for variations of  $Lu$  from 0 to 100 and for variations of the compound dimensionless number  $Fe = Ko^*Pn$  from 0 to 1.0 are given in Table 4-1.

Symmetry conditions (4-1-5) for the transforms are

$$T'_L(0, s) = \theta'_L(0, s) = 0.$$

It follows from these conditions that  $A_1 = A_3$  and  $A_2 = A_4$ . Therefore, the general solution of the given problem may be rewritten in terms of the transforms as

$$T_L(X, s) = \sum_{i=1}^2 B_i \operatorname{ch} \sqrt{s v_i} X; \quad (4-2-8)$$

$$\theta_L(X, s) = -\frac{1}{Ko^*} \sum_{i=1}^2 B_i (1 - v_i^2) \operatorname{ch} \sqrt{s v_i} X. \quad (4-2-9)$$

TABLE 4-1

Values of  $v_i = v_i(\text{Lu}, \text{Ko}^*\text{Pn})$  ( $i = 1, 2$ )

$\text{Lu} \backslash \text{Ko}^*\text{Pn}$	$v_i$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	$v_1$ $v_2$	1 $\infty$	1 $\infty$	1 $\infty$	1 $\infty$	1 $\infty$	1 $\infty$	1 $\infty$	1 $\infty$	1 $\infty$	1 $\infty$	1 $\infty$
0.01	$v_1$ $v_2$	1 10.0000	0.9995 10.0050	0.9990 10.0100	0.9985 10.0150	0.9980 10.0201	0.9975 10.0251	0.9970 10.0301	0.9965 10.0352	0.9960 10.0403	0.9955 10.0453	0.9950 10.0503
0.02	$v_1$ $v_2$	1 7.0711	0.9990 7.0782	0.9980 7.0854	0.9969 7.0926	0.9960 7.0998	0.9949 7.1070	0.9939 7.1141	0.9929 7.1213	0.9919 7.1285	0.9909 7.1356	0.9899 7.1428
0.04	$v_1$ $v_2$	1 5.0000	0.9979 5.0104	0.9959 5.0208	0.9938 5.0311	0.9918 5.0414	0.9897 5.0517	0.9877 5.0620	0.9858 5.0722	0.9838 5.0825	0.9818 5.0927	0.9798 5.1029
0.06	$v_1$ $v_2$	1 4.0825	0.9968 4.0955	0.9937 4.1084	0.9906 4.1213	0.9875 4.1341	0.9845 4.1469	0.9815 4.1597	0.9784 4.1724	0.9754 4.1850	0.9725 4.1977	0.9696 4.2102
0.08	$v_1$ $v_2$	1 3.5355	0.9957 3.5508	0.9914 3.5660	0.9873 3.5812	0.9831 3.5963	0.9791 3.6113	0.9750 3.6262	0.9710 3.6410	0.9671 3.6557	0.9632 3.6704	0.9595 3.6850
0.10	$v_1$ $v_2$	1 3.1623	0.9945 3.1798	0.9891 3.1971	0.9838 3.2143	0.9786 3.2314	0.9735 3.2484	0.9684 3.2652	0.9635 3.2820	0.9587 3.2986	0.9539 3.3151	0.9492 3.3315
0.20	$v_1$ $v_2$	1 2.2361	0.9878 2.2636	0.9762 2.2907	0.9651 2.3170	0.9544 2.3428	0.9440 2.3681	0.9344 2.3931	0.9250 2.4175	0.9158 2.4415	0.9071 2.4652	0.8986 2.4884
0.30	$v_1$ $v_2$	1 1.8257	0.9796 1.8640	0.9609 1.9000	0.9437 1.9346	0.9278 1.9680	0.9129 2.0000	0.8989 2.0310	0.8858 2.0612	0.8733 2.0906	0.8615 2.1192	0.8503 2.1472
0.40	$v_1$ $v_2$	1 1.5811	0.9694 1.6310	0.9431 1.6764	0.9200 1.7185	0.8993 1.7582	0.8803 1.7953	0.8632 1.8316	0.8473 1.8660	0.8325 1.8992	0.8187 1.9312	0.8057 1.9623
0.60	$v_1$ $v_2$	1 1.2910	0.9415 1.3712	0.9004 1.4339	0.8677 1.4879	0.8402 1.5365	0.8167 1.5811	0.7955 1.6229	0.7766 1.6624	0.7594 1.7000	0.7437 1.7360	0.7292 1.7706
0.80	$v_1$ $v_2$	1 1.1180	0.9020 1.2395	0.8511 1.3136	0.8137 1.3740	0.7835 1.4269	0.7580 1.4750	0.7358 1.5193	0.7162 1.5611	0.6985 1.6007	0.6824 1.6384	0.6676 1.6746

TABLE 4-1 (Continued)

La	Ko* Pn		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
	$y_1$	$y_2$											
1.00	$y_1$	$y_2$	1 1.0000	0.8543 1.1705	0.8011 1.2483	0.7630 1.3107	0.7326 1.3650	0.7071 1.4142	0.6851 1.4597	0.6657 1.5023	0.6482 1.5427	0.6324 1.5811	0.6180 1.6181
2.00	$y_1$	$y_2$	1 0.7071	0.6526 1.0836	0.6150 1.1496	0.5858 1.2070	0.5617 1.2587	0.5409 1.3067	0.5233 1.3513	0.5073 1.3938	0.4931 1.4342	0.4801 1.4730	0.4682 1.5102
4.00	$y_1$	$y_2$	1 0.5000	0.4706 1.0623	0.4472 1.1180	0.4276 1.1693	0.4108 1.2171	0.3960 1.2622	0.3831 1.3050	0.3714 1.3461	0.3609 1.3856	0.3512 1.4236	0.3423 1.4604
6.00	$y_1$	$y_2$	1 0.4083	0.3863 1.0571	0.3680 1.1097	0.3523 1.1587	0.3389 1.2049	0.3267 1.2487	0.3162 1.2910	0.3066 1.3314	0.2979 1.3704	0.2900 1.4080	0.2826 1.4446
8.00	$y_1$	$y_2$	1 0.3536	0.3352 1.0548	0.3198 1.1058	0.3062 1.1544	0.2949 1.1992	0.2846 1.2424	0.2753 1.2842	0.2670 1.3243	0.2594 1.3629	0.2525 1.4004	0.2461 1.4368
10.00	$y_1$	$y_2$	1 0.3162	0.3002 1.0535	0.2865 1.1036	0.2748 1.1509	0.2645 1.1958	0.2556 1.2388	0.2470 1.2802	0.2396 1.3201	0.2328 1.3586	0.2265 1.3960	0.2208 1.4322
20.0	$y_1$	$y_2$	1 0.2236	0.2127 1.0510	0.2034 1.0994	0.1952 1.1454	0.1880 1.1894	0.1817 1.2318	0.1758 1.2724	0.1704 1.3118	0.1657 1.3500	0.1612 1.3870	0.1572 1.4231
30.0	$y_1$	$y_2$	1 0.1825	0.1737 1.0503	0.1661 1.0981	0.1595 1.1436	0.1538 1.1873	0.1488 1.2293	0.1437 1.2699	0.1394 1.3091	0.1355 1.3472	0.1319 1.3841	0.1285 1.4201
40.0	$y_1$	$y_2$	1 0.1581	0.1506 1.0499	0.1439 1.0974	0.1384 1.1427	0.1332 1.1863	0.1285 1.2282	0.1247 1.2686	0.1208 1.3078	0.1174 1.3458	0.1145 1.3827	0.1114 1.4187
50.0	$y_1$	$y_2$	1 0.1414	0.1347 1.0497	0.1288 1.0970	0.1239 1.1422	0.1194 1.1857	0.1153 1.2275	0.1116 1.2679	0.1082 1.3070	0.1051 1.3450	0.1022 1.3819	0.0998 1.4177
60.0	$y_1$	$y_2$	1 0.1292	0.1230 1.0495	0.1179 1.0957	0.1131 1.1419	0.1091 1.1852	0.1054 1.2270	0.1020 1.2674	0.0990 1.3065	0.0962 1.3444	0.0935 1.3812	0.0911 1.4172
80.0	$y_1$	$y_2$	1 0.1118	0.1066 1.0493	0.1020 1.0964	0.0980 1.1415	0.0943 1.1847	0.0909 1.2265	0.0883 1.2668	0.0854 1.3058	0.0831 1.3437	0.0810 1.3805	0.0791 1.4164
100	$y_1$	$y_2$	1 0.1000	0.0954 1.0492	0.0911 1.0962	0.0875 1.1412	0.0843 1.1845	0.0817 1.2261	0.0791 1.2663	0.0765 1.3054	0.0745 1.3433	0.0725 1.3801	0.0707 1.4160

Here  $B_i (i=1, 2)$  are new constants with respect to  $X$ . These may be determined from boundary conditions (4-1-1) and (4-1-4), which in terms of the transforms become

$$T_L(1, s) = -\frac{1}{s}; \quad (4-2-10)$$

$$\int_0^1 \theta_L(X, s) dX = 0. \quad (4-2-11)$$

Solutions (4-2-8) and (4-2-9) may now be combined with boundary conditions (4-2-10) and (4-2-11) to obtain the system of algebraic equations\*

$$\left. \begin{aligned} B_1 \operatorname{ch} \sqrt{s} v_1 + B_2 \operatorname{ch} \sqrt{s} v_2 &= -\frac{1}{s}; \\ B_1 (1 - v_1^2) \frac{\operatorname{sh} \sqrt{s} v_1}{\sqrt{s} v_1} + B_2 (1 - v_2^2) \frac{\operatorname{sh} \sqrt{s} v_2}{\sqrt{s} v_2} &= 0, \end{aligned} \right\} \quad (4-2-12)$$

and then these may be solved for  $B_i$  to give

$$B_1 = -\frac{v_1 (1 - v_2^2) \operatorname{sh} \sqrt{s} v_2}{s [v_1 (1 - v_2^2) \operatorname{sh} \sqrt{s} v_2 \operatorname{ch} \sqrt{s} v_1 - v_2 (1 - v_1^2) \operatorname{sh} \sqrt{s} v_1 \operatorname{ch} \sqrt{s} v_2]},$$

$$B_2 = +\frac{v_2 (1 - v_1^2) \operatorname{sh} \sqrt{s} v_1}{s [v_1 (1 - v_2^2) \operatorname{sh} \sqrt{s} v_2 \operatorname{ch} \sqrt{s} v_1 - v_2 (1 - v_1^2) \operatorname{sh} \sqrt{s} v_1 \operatorname{ch} \sqrt{s} v_2]}.$$

Consequently, the transforms of the distribution functions for the trans-potentials are

$$T_L(X, s) = -\frac{v_1 (1 - v_2^2) \operatorname{sh} \sqrt{s} v_2 \operatorname{ch} \sqrt{s} v_1 X - v_2 (1 - v_1^2) \operatorname{sh} \sqrt{s} v_1 \operatorname{ch} \sqrt{s} v_2 X}{s [v_1 (1 - v_2^2) \operatorname{sh} \sqrt{s} v_2 \operatorname{ch} \sqrt{s} v_1 - v_2 (1 - v_1^2) \operatorname{sh} \sqrt{s} v_1 \operatorname{ch} \sqrt{s} v_2]}; \quad (4-2-13)$$

$$\theta_L(X, s) = \frac{1}{K_0^*} \times$$

$$\times \frac{v_1 (1 - v_2^2) \operatorname{sh} \sqrt{s} v_2 (1 - v_1^2) \operatorname{ch} \sqrt{s} v_1 X - v_2 (1 - v_1^2) \operatorname{sh} \sqrt{s} v_1 (1 - v_2^2) \operatorname{ch} \sqrt{s} v_2 X}{s [v_1 (1 - v_2^2) \operatorname{sh} \sqrt{s} v_2 \operatorname{ch} \sqrt{s} v_1 - v_2 (1 - v_1^2) \operatorname{sh} \sqrt{s} v_1 \operatorname{ch} \sqrt{s} v_2]}. \quad (4-2-14)$$

To obtain the original functions [inverse transforms] the generalized expansion theorem may be used. The expansion theorem is valid for transforms which may be represented as ratios of two generalized polynomials  $\Phi(s) : \psi(s)$ , and also for ratios of nongeneralized polynomials, provided the latter are reduced to the former by multiplication or division by  $s^l$  (where  $|l| \leq 1$ ). It is here assumed that the function  $\psi(s)$  has only simple roots and that the polynomial in the denominator is one degree higher than the polynomial in the numerator.

In our case each of the equations obtained can be rewritten as a ratio of two polynomials:

$$-T_L = \frac{\Phi_1(s)}{\psi(s)}, \quad K_0^* \theta_L = \frac{\Phi_2(s)}{\psi(s)}. \quad (4-2-15)$$

The denominator  $\psi(s) = s [v_1 (1 - v_2^2) \operatorname{sh} \sqrt{s} v_2 \operatorname{ch} \sqrt{s} v_1 - v_2 (1 - v_1^2) \operatorname{sh} \sqrt{s} v_1 \operatorname{ch} \sqrt{s} v_2]$  is the same in both equations of (4-2-15), and the polynomials in the numerator are, respectively,

$$\Phi_1(s) = v_1 (1 - v_2^2) \operatorname{sh} \sqrt{s} v_2 \operatorname{ch} \sqrt{s} v_1 X - v_2 (1 - v_1^2) \operatorname{sh} \sqrt{s} v_1 \operatorname{ch} \sqrt{s} v_2 X;$$

\* [Note that throughout this text  $\operatorname{ch}$  and  $\operatorname{sh}$  refer to the hyperbolic cosine and sine,  $\cosh$  and  $\sinh$ .]

$$\begin{aligned}\Phi_2(s) = & v_1(1-v_2^2) \operatorname{sh} \sqrt{s} v_1(1-v_1^2) \operatorname{ch} \sqrt{s} v_1 X - \\ & - v_2(1-v_1^2) \operatorname{sh} \sqrt{s} v_1(1-v_2^2) \operatorname{ch} \sqrt{s} v_2 X.\end{aligned}$$

An analysis of the problem shows that the conditions required to make the expansion theory applicable are fulfilled.

Let us now find the roots of  $\psi(s)$ . When the function is set equal to zero,  $\psi(s)=0$ , the following roots are obtained:

- 1) the zero root,  $s=0$ ;
- 2) the roots satisfying the equation

$$v_1(1-v_2^2) \operatorname{sh} \sqrt{s} v_1 \operatorname{ch} \sqrt{s} v_1 - v_2(1-v_1^2) \operatorname{sh} \sqrt{s} v_1 \operatorname{ch} \sqrt{s} v_2 = 0.$$

This equation can also be written as

$$v_1(1-v_2^2) \sin v_1 \mu \cos v_1 \mu - v_2(1-v_1^2) \sin v_1 \mu \cos v_2 \mu = 0, \quad (4-2-16)$$

where  $\mu = i\sqrt{s}$ .

Equation (4-2-16) has an infinite number of real simple roots, which are the common roots of the system of equations

$$\left. \begin{aligned} \sin v_1 \mu &= 0; \\ \sin v_2 \mu &= 0, \end{aligned} \right\} \quad (4-2-17)$$

and the roots of the equation

$$\cot v_2 \mu - N \cot v_1 \mu = 0, \quad (4-2-18)$$

where  $N = \frac{v_1}{v_2} \cdot \frac{1-v_2^2}{1-v_1^2}$ . Equations (4-2-17) have common roots only when

$\frac{v_2}{v_1} = \frac{\lambda v_2}{\lambda v_1} = \frac{\alpha}{\beta}$  is a rational fraction, where  $\alpha$  and  $\beta$  are integers. The roots

of characteristic equation (4-2-18) can be obtained using a graphical method or else using familiar numerical methods.

Taking into account the values obtained for the characteristic-equation roots, we may write the expansion theorem as

$$\begin{aligned} L^{-1} \left[ \frac{\Phi_k(s)}{\psi(s)} \right] = & \lim_{s \rightarrow 0} \frac{\Phi_k(s)}{\psi'(s)} + \sum_{i=1}^{\infty} \frac{\Phi_k(s_i)}{\psi'(s_i)} \exp(s_i Fo) + \\ & + \sum_{n=1}^{\infty} \frac{\Phi_k(s_n)}{\psi'(s_n)} \exp(s_n Fo), \end{aligned} \quad (4-2-19)$$

( $k=1, 2$ ),

where the symbol  $L^{-1}$  denotes an inverse Laplace transformation. Let us consider in more detail the method for obtaining the inverse transform for the dimensionless heat-transfer potential.

In order to determine the first term of (4-2-19), let us write the ratio  $\Phi_1(s)/\psi(s)$  as

$$\frac{\Phi_1(s)}{\psi(s)} = \frac{N \operatorname{ch} \sqrt{s} v_1 X - \operatorname{ch} \sqrt{s} v_2 X \cdot \frac{\operatorname{sh} \sqrt{s} v_1}{\operatorname{sh} \sqrt{s} v_2}}{s \left[ N \operatorname{ch} \sqrt{s} v_1 - \operatorname{ch} \sqrt{s} v_2 \cdot \frac{\operatorname{sh} \sqrt{s} v_1}{\operatorname{sh} \sqrt{s} v_2} \right]} =$$

$$\begin{aligned}
& N \left[ 1 + \frac{1}{2!} (\sqrt{s} \bar{v}_1 X)^2 + \dots \right] - \left[ 1 + \frac{1}{2!} (\sqrt{s} \bar{v}_2 X)^2 + \dots \right] \cdot \frac{1 + \frac{1}{3!} (\sqrt{s} \bar{v}_1)^3 + \dots}{1 + \frac{1}{3!} (\sqrt{s} \bar{v}_2)^3 + \dots} \cdot \frac{\bar{v}_1}{\bar{v}_2} \\
& = \frac{s \left\{ N \left[ 1 + \frac{1}{2!} (\sqrt{s} \bar{v}_1)^2 + \dots \right] - \left[ 1 + \frac{1}{2!} (\sqrt{s} \bar{v}_2)^2 + \dots \right] \cdot \frac{1 + \frac{1}{3!} (\sqrt{s} \bar{v}_1)^3 + \dots}{1 + \frac{1}{3!} (\sqrt{s} \bar{v}_2)^3 + \dots} \cdot \frac{\bar{v}_1}{\bar{v}_2} \right\}}{s \left\{ N \left[ 1 + \frac{1}{2!} (\sqrt{s} \bar{v}_1)^2 + \dots \right] - \left[ 1 + \frac{1}{2!} (\sqrt{s} \bar{v}_2)^2 + \dots \right] \cdot \frac{1 + \frac{1}{3!} (\sqrt{s} \bar{v}_1)^3 + \dots}{1 + \frac{1}{3!} (\sqrt{s} \bar{v}_2)^3 + \dots} \cdot \frac{\bar{v}_1}{\bar{v}_2} \right\}}. \quad (4-2-20)
\end{aligned}$$

Now, by differentiating the denominator of (4-2-20) with respect to  $s$  and taking the limit as  $s \rightarrow 0$ , we find that

$$\lim_{s \rightarrow 0} \frac{\Phi_1(s)}{\psi'(s)} = 1.$$

In the second term of (4-2-19) the summation is over the common roots of equations (4-2-17). If we use the notation  $v_l \mu_l = a l \pi$ , with  $l = 1, 2, 3, \dots$ , then the common roots may be written as  $\mu_l = \frac{a}{v_2} l \pi = \frac{\beta}{v_1} l \pi$ . Next, in  $\Phi_1(s_l)$  we may replace the hyperbolic functions by trigonometric functions in accordance with the formulas  $\operatorname{sh} z = -i \sin iz$  and  $\operatorname{ch} z = \cos iz$ , so that we have

$$\begin{aligned}
\Phi_1(s_l) &= -i [v_1 (1 - v_2^2) \sin v_2 \mu_l \cos v_1 \mu_l X - v_2 (1 - v_1^2) \sin v_1 \mu_l \cos v_2 \mu_l X] = \\
&= -i [v_1 (1 - v_2^2) \sin a l \pi \cos \beta l \pi X - v_2 (1 - v_1^2) \sin \beta l \pi \cos a l \pi X] = 0.
\end{aligned}$$

Accordingly,  $\psi'(s_l)$  becomes

$$\psi'(s_l) = -i \frac{\mu_l}{1} (v_1^2 - v_2^2) v_1 v_2 \cos v_1 \mu_l \cos v_2 \mu_l \neq 0,$$

and thus

$$\sum_{l=1}^{\infty} \frac{\Phi_1(s_l)}{\psi'(s_l)} \exp(s_l \operatorname{Fo}) = 0.$$

In the third term of (4-2-19) the summation is over the positive roots of equation (4-2-18), and so

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{\Phi_1(s_n)}{\psi'(s_n)} \exp(s_n \operatorname{Fo}) = \\
& = 2 \sum_{n=1}^{\infty} \frac{\frac{v_1 (1 - v_2^2)}{v_2^2 - v_1^2} \sin v_2 \mu_n \cos v_1 \mu_n X - \frac{v_2 (1 - v_1^2)}{v_2^2 - v_1^2} \sin v_1 \mu_n \cos v_2 \mu_n X}{\mu_n [\sin v_1 \mu_n \sin v_2 \mu_n - v_1 v_2 \cos v_1 \mu_n \cos v_2 \mu_n]} \exp(-\mu_n^2 \operatorname{Fo}).
\end{aligned}$$

The values obtained for the terms may now be substituted into (4-2-19). Then, taking into account (4-2-15), we finally obtain the following distribution of the heat-transfer potential:

$$T(X, \operatorname{Fo}) = - \left[ 1 + \sum_{n=1}^{\infty} \sum_{l=1}^2 C_{nl} \cos v_l \mu_n X \exp(-\mu_n^2 \operatorname{Fo}) \right], \quad (4-2-21)$$

where

$$C_{n1} = \frac{2 \frac{v_1 (1 - v_2^2)}{v_2^2 - v_1^2} \sin v_2 \mu_n}{\mu_n \left[ \sin v_1 \mu_n \sin v_2 \mu_n - \frac{1}{\operatorname{Lu}} \cos v_1 \mu_n \cos v_2 \mu_n \right]}, \quad (4-2-22)$$

TABLE 4-2

Roots of the characteristic equation

$$\cot v_1 \mu - \frac{v_1}{v_2} \cdot \frac{1 - v_2^2}{1 - v_1^2} \cot v_1 \mu = 0.$$

Lu	Ko*Pn=0.2				Ko*Pn=0.4				Ko*Pn=0.6			
	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$
0.01	0.31384	0.62767	0.94150	1.57555	0.31352	0.62709	0.94052	1.56371	0.31320	0.62647	0.93957	1.57988
0.10	0.97917	1.58525	1.97127	2.94849	0.96587	1.59781	1.95715	2.91793	0.95340	1.60861	1.94610	2.89885
0.40	1.50506	2.04333	3.72304	4.94349	1.45380	2.08289	3.56685	4.98678	1.41084	2.11229	3.45100	4.91410
0.80	1.52996	2.78562	4.46096	5.74909	1.49251	2.80082	4.29679	5.73008	1.45796	2.72601	4.17045	5.64072
1.00	1.53295	3.06585	4.59881	6.13173	1.49771	2.99541	4.49313	5.99084	1.46476	2.92948	4.39423	5.85903
4.00	1.53978	4.18061	6.55288	7.49716	1.51003	3.85424	6.24262	7.68325	1.48152	3.64375	5.87087	7.72263

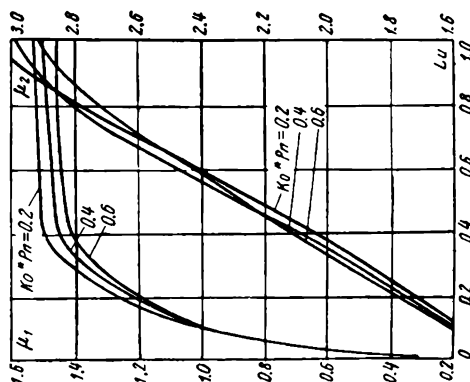


FIGURE 4-1. Roots of characteristic equation as functions of Ko\*Pn and Lu, for an infinite plate

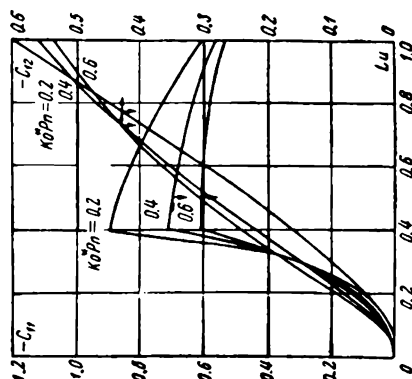


FIGURE 4-2. Coefficients  $C_{n1}$  as functions of Lu and Ko\*Pn, for an infinite plate

$$C_{ni} = - \frac{\frac{v_2(1-v_1^2)}{v_2^2-v_1^2} \sin v_1 \mu_n}{\mu_n \left[ \sin v_1 \mu_n \sin v_2 \mu_n - \frac{1}{Lu} \cos v_1 \mu_n \cos v_2 \mu_n \right]}, \quad (4-2-23)$$

and  $\mu_n$  are the roots of characteristic equation (4-2-18). The first four non-zero values of  $\mu_n$  are given in Table 4-2 and, to aid practical calculations, are plotted on the graph in Figure 4-1.

In a similar way it is possible to obtain the inverse transform for the distribution of the dimensionless mass-transfer potential. In this case the first and second terms of (4-2-19) are zero, while the third term is

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\Phi_2(s_n)}{\psi(s_n)} \exp(s_n Fo) = \\ & = 2 \sum_{n=1}^{\infty} \frac{\frac{v_1(1-v_2^2)}{v_2^2-v_1^2} \sin v_2 \mu_n (1-v_1^2) \cos v_1 \mu_n X - \frac{v_2(1-v_1^2)}{v_2^2-v_1^2} \sin v_1 \mu_n (1-v_2^2) \cos v_2 \mu_n X}{\mu_n [\sin v_1 \mu_n \sin v_2 \mu_n - v_1 v_2 \cos v_1 \mu_n \cos v_2 \mu_n]} \times \\ & \quad \times \exp(-\mu_n^2 Fo). \end{aligned}$$

Therefore, the potential distribution is

$$\theta(X, Fo) = \frac{1}{Ko^*} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1-v_i^2) \cos v_i \mu_n X \exp(-\mu_n^2 Fo). \quad (4-2-24)$$

Coefficients  $C_{ni}$  are defined by (4-2-22) and (4-2-23). Their values for the first two roots of the characteristic equation are given in Table 4-3 and plotted in Figure 4-2.

TABLE 4-3

Values of the constant coefficients for an infinite plate

Lu	Ko*Pn	$C_{ni}$				$D_{\tau n}$	
		11	12	21	22	n=1	n=2
0.01	0.2	0.00000	-0.00004	-0.00011	0.00005	-0.00000	-0.00011
	0.4	-0.00013	-0.00008	0.00008	0.00009	-0.00012	-0.00008
	0.6	-0.00019	-0.00013	0.00008	0.00014	-0.00019	-0.00008
0.10	0.2	-0.01241	-0.00701	-1.24084	0.01006	-0.01058	-0.79322
	0.4	-0.02213	-0.01296	-1.20624	0.01984	-0.01906	-0.77448
	0.6	-0.02991	-0.01811	-1.16761	0.02955	-0.02601	-0.75430
0.40	0.2	-0.89236	-0.16516	-0.30768	0.11179	-0.65945	-0.15878
	0.4	-0.71454	-0.22322	-0.44550	0.15290	-0.57594	-0.24781
	0.6	-0.60836	-0.24782	-0.52225	0.17469	-0.51949	-0.30736
0.80	0.2	-0.73068	-0.45632	-0.21034	0.17365	-0.74653	-0.08529
	0.4	-0.64075	-0.47217	-0.28677	0.22715	-0.69237	-0.15146
	0.6	-0.58402	-0.46448	-0.33053	0.25782	-0.64593	-0.20182
1.00	0.2	-0.59960	-0.59957	-0.20152	0.20152	-0.75495	-0.08528
	0.4	-0.56655	-0.56656	-0.25839	0.25839	-0.70615	-0.14688
	0.6	-0.53692	-0.53687	-0.28852	0.28849	-0.66299	-0.19145
4.00	0.2	-0.19921	-1.02465	-0.05297	0.40543	-0.77231	-0.11376
	0.4	-0.28808	-0.88831	-0.07789	0.39844	-0.73616	-0.13361
	0.6	-0.33494	-0.79618	-0.09610	0.39160	-0.70228	-0.15007



The average value of the dimensionless transfer potential for an infinite plate is defined as

$$\bar{\varphi}(Fo) = \int_0^1 \varphi(X, Fo) dX, \quad (4-2-25)$$

so that the average dimensionless heat-transfer and mass-transfer potentials are

$$\bar{T}(Fo) = - \left[ 1 + \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} \exp(-\mu_n^2 Fo) \right]; \quad (4-2-26)$$

$$\bar{\theta}(Fo) = \frac{1}{K_0^*} \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} (1 - \nu_i^2) \exp(-\mu_n^2 Fo) \quad (4-2-27)$$

or in a somewhat different form

$$\bar{T}(Fo) = - \left[ 1 + \sum_{n=1}^{\infty} D_{Tn} \exp(-\mu_n^2 Fo) \right]; \quad (4-2-26')$$

$$\bar{\theta}(Fo) = \sum_{n=1}^{\infty} D_{\theta n} \exp(-\mu_n^2 Fo). \quad (4-2-27')$$

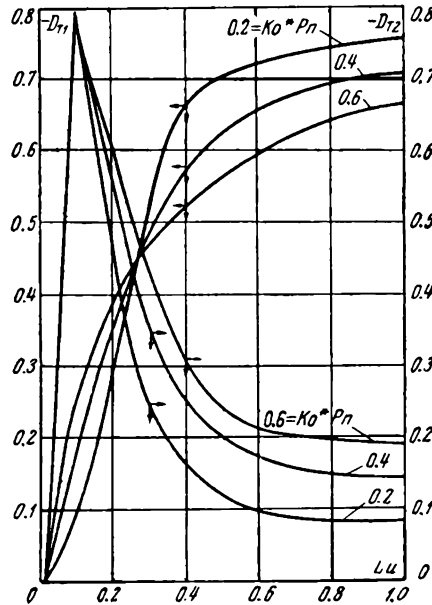


FIGURE 4-3. Coefficients  $D_{Tn}$  as functions of  $Lu$  and  $Ko^*Pn$ , for an infinite plate

Here

$$D_{ni} = C_{ni} \frac{\sin \nu_i \mu_n}{\nu_i \mu_n}; \quad D_{Tn} = \sum_{i=1}^2 D_{ni}; \quad D_{\theta n} = \frac{1}{K_0^*} \sum_{i=1}^2 D_{ni} (1 - \nu_i^2), \quad (4-2-28)$$

where  $C_{ni}$  ( $i=1, 2$ ) are defined by equations (4-2-22) and (4-2-23).

In accordance with the conditions of the problem coefficients  $D_{0n}$  are zero, and so  $\bar{\theta}(Fo) \equiv 0$ . The first two values of  $D_{rn}$  ( $n=1, 2$ ) are listed in Table 4-3 and plotted in Figure 4-3.

An analysis of the above results shows that, due to the rapid increase in the absolute values of successive characteristic-equation roots  $\mu_n$  (see Table 4-2), it will be true that

$$\mu_1 < \mu_2 < \dots \mu_n < \dots$$

Consequently, due to the very rapid decrease in the exponential factors  $\exp(-\mu_n^2 Fo)$ , the infinite sums entering into (4-2-21), (4-2-24), (4-2-26), (4-2-27), (4-2-26'), and (4-2-27') will converge quite rapidly. Therefore, from a certain minimum value of the Fourier number all the terms in the series expansion may be neglected except the first two, and the results will be within some predetermined accuracy range. For example, numerical calculations of the given problem with  $Lu = 0.4$  and  $Ko \cdot Pn = 0.4$  show that, to an accuracy of 0.5%, we may neglect all the terms of the series under the summation sign except the first two, beginning at the following values of the Fourier number: for  $T(0, Fo)$ , from  $Fo = 0.16$ ; for  $T(1, Fo)$ , from  $Fo = 0.08$ ; for  $\theta(0, Fo)$ , from  $Fo = 0.24$ ; and for  $\theta(1, Fo)$ , from  $Fo = 0.22$ .

An increase in  $Lu$  leads to a quite appreciable shift in these "initial" values of  $Fo$  toward lower values. The value of the compound dimensionless number  $Ko \cdot Pn$  affects the "initial" Fourier number only slightly, although a tendency for it to decrease with the latter is observed.

The accomplishment of the above simplifications is of great practical importance, since it reduces considerably the amount of computational work required and makes it possible to represent the general solutions in a form convenient for practical application.

#### b) Solution for an infinite cylinder ( $\Gamma=1$ )

If we apply the integral Laplace transformation (4-2-1) to the system of heat-transfer and mass-transfer equations (4-1-2) and (4-1-3), we obtain

$$sT_L = T''_L + \frac{1}{X} T'_L - Ko \cdot s\theta_L; \quad (4-2-29)$$

$$s\theta_L = Lu \left( \theta'_L + \frac{1}{X} \theta_L \right) - Lu Pn \left( T''_L + \frac{1}{X} T'_L \right). \quad (4-2-30)$$

In the derivation of equations (4-2-29) and (4-2-30) we took into account the zero initial conditions. From (4-2-29)  $\theta_L$  and its second derivative  $\theta''_L$  may be determined and then substituted into (4-2-30) to obtain a nonhomogeneous fourth-order differential equation in  $T_L$ :

$$T_L^{IV} + \frac{2}{X} T_L''' - \frac{1}{X^2} T_L'' + \frac{1}{X^2} T'_L - s \left( 1 + Ko \cdot Pn + \frac{1}{Lu} \right) \left( T''_L + \frac{1}{X} T'_L \right) + \frac{s^2}{Lu} T_L = 0. \quad (4-2-31)$$

It is easy to show that (4-2-31) is satisfied by zero-order Bessel functions of the first and second kinds with imaginary arguments  $I_0(\sqrt{sv}X)$  and  $K_0(\sqrt{sv}X)$ , where the parameter  $v$  is defined by (4-2-27). Consequently, the general integral of equation (4-2-31) is

$$T_L(X, s) = \sum_{i=1}^2 [A_i K_0(\sqrt{sv_i}X) + B_i I_0(\sqrt{sv_i}X)], \quad (4-2-32)$$

and the mass-transfer potential is, analogously,

$$\theta_L(X, s) = -\frac{1}{K_0^*} \sum_{i=1}^2 [A_i(1 - v_i^2) K_0(\sqrt{s v_i} X) + B_i(1 - v_i^2) I_0(\sqrt{s v_i} X)]. \quad (4-2-33)$$

According to conditions (4-1-5) the transfer potentials along the cylinder axis ( $X=0$ ) cannot be infinite. Since for  $X \rightarrow 0$  we have  $K_0(\sqrt{s v_i} X) \rightarrow -\infty$ , the coefficients  $A_i$  must be zero. Thus, solutions (4-2-32) and (4-2-33) simplify to

$$T_L(X, s) = \sum_{i=1}^2 B_i I_0(\sqrt{s v_i} X); \quad (4-2-34)$$

$$\theta_L(X, s) = -\frac{1}{K_0^*} \sum_{i=1}^2 B_i(1 - v_i^2) I_0(\sqrt{s v_i} X). \quad (4-2-35)$$

The constants  $B_i$  are determined from the boundary conditions. For an infinite cylinder these conditions may be written in terms of the transforms as

$$T_L(1, s) = -\frac{1}{s}, \quad (4-2-36)$$

$$\int_0^1 X \theta_L(X, s) dX = 0. \quad (4-2-37)$$

Equations (4-2-34) and (4-2-35) may now be combined with conditions (4-2-36) and (4-2-37) to obtain a system of equations in  $B_i$  ( $i=1, 2$ ):

$$\left. \begin{aligned} \sum_{i=1}^2 B_i I_0(\sqrt{s v_i}) &= -\frac{1}{s}; \\ \sum_{i=1}^2 B_i(1 - v_i^2) \frac{I_1(\sqrt{s v_i})}{\sqrt{s v_i}} &= 0. \end{aligned} \right\} \quad (4-2-38)$$

Equation (4-2-38) may be solved to give

$$B_1 = -\frac{v_1(1 - v_2^2) I_1(\sqrt{s v_2})}{s [v_1(1 - v_2^2) I_1(\sqrt{s v_2}) I_0(\sqrt{s v_1}) - v_2(1 - v_1^2) I_1(\sqrt{s v_1}) I_0(\sqrt{s v_2})]};$$

$$B_2 = \frac{v_2(1 - v_1^2) I_1(\sqrt{s v_1})}{s [v_1(1 - v_2^2) I_1(\sqrt{s v_2}) I_0(\sqrt{s v_1}) - v_2(1 - v_1^2) I_1(\sqrt{s v_1}) I_0(\sqrt{s v_2})]}.$$

Thus, the solution for an infinite cylinder, expressed in terms of transforms, has the following final form:

$$T_L(X, s) = -\frac{v_1(1 - v_2^2) I_1(\sqrt{s v_2}) I_0(\sqrt{s v_1} X) - v_2(1 - v_1^2) I_1(\sqrt{s v_1}) I_0(\sqrt{s v_2} X)}{s [v_1(1 - v_2^2) I_1(\sqrt{s v_2}) I_0(\sqrt{s v_1}) - v_2(1 - v_1^2) I_1(\sqrt{s v_1}) I_0(\sqrt{s v_2})]}; \quad (4-2-39)$$

$$\theta_L(X, s) = \frac{1}{K_0^*} \times$$

$$\times \frac{v_1(1 - v_2^2) I_1(\sqrt{s v_2})(1 - v_1^2) I_0(\sqrt{s v_1} X) - v_2(1 - v_1^2) I_1(\sqrt{s v_1})(1 - v_2^2) I_0(\sqrt{s v_2} X)}{s [v_1(1 - v_2^2) I_1(\sqrt{s v_2}) I_0(\sqrt{s v_1}) - v_2(1 - v_1^2) I_1(\sqrt{s v_1}) I_0(\sqrt{s v_2})]}. \quad (4-2-40)$$

An analysis of equations (4-2-39) and (4-2-40) shows that inverse transforms can be obtained by means of the expansion theorem, just as in the analogous problem for an infinite plate. If we define

$$\psi(s) = s [v_1(1 - v_2^2) I_1(\sqrt{s v_2}) I_0(\sqrt{s v_1}) - v_2(1 - v_1^2) I_1(\sqrt{s v_1}) I_0(\sqrt{s v_2})];$$

$$\Phi_1(s) = v_1(1 - v_2^2) I_1(\sqrt{s} v_2) I_0(\sqrt{s} v_1 X) - v_2(1 - v_1^2) I_1(\sqrt{s} v_1) I_0(\sqrt{s} v_2 X);$$

$$\begin{aligned} \Phi_2(s) = & v_1(1 - v_2^2) I_1(\sqrt{s} v_2) (1 - v_1^2) I_0(\sqrt{s} v_1 X) - \\ & - v_2(1 - v_1^2) I_1(\sqrt{s} v_1) (1 - v_2^2) I_0(\sqrt{s} v_2 X), \end{aligned}$$

then the solution methods for the present problem will be similar to those for the previous one.

Let us find the roots of  $\psi(s)$  by setting it equal to zero:

$$\psi(s) = 0.$$

The following roots are obtained:

- 1)  $s=0$ , the zero root;
- 2)  $s=s_i$ , the infinite number of roots satisfying the system of equations

$$\begin{aligned} J_1(v_1 \mu_i) &= 0; \\ J_1(v_2 \mu_i) &= 0, \end{aligned} \quad (4-2-41)$$

where  $\mu_i = i\sqrt{s_i}$ ;

- 3)  $s=s_n$ , the infinite number of roots satisfying the equation

$$\frac{J_0(v_2 \mu_n)}{J_1(v_2 \mu_n)} - N \frac{J_0(v_1 \mu_n)}{J_1(v_1 \mu_n)} = 0, \quad (4-2-42)$$

where

$$N = \frac{v_1}{v_2} \cdot \frac{1 - v_2^2}{1 - v_1^2} \text{ and } \mu_n = i\sqrt{s_n}.$$

Consequently, the expansion theorem may once again be written in the form of (4-2-19). The first term of (4-2-19) corresponds to the zero root ( $s=0$ ), the second term to the roots common to the system (4-2-41), and the third term to the roots of characteristic equation (4-2-42).

Since the first term  $\frac{\Phi_1(0)}{\psi'(0)} = \frac{0}{0}$  is indeterminate, an indirect method of solution must be used, which gives

$$\begin{aligned} \frac{\Phi_1(s)}{\psi(s)} &= \frac{I_1(\sqrt{s} v_2) v_2 (1 - v_1^2)}{I_1(\sqrt{s} v_2) v_2 (1 - v_1^2)} \frac{N I_0(\sqrt{s} v_1 X) - I_0(\sqrt{s} v_2 X) \frac{I_1(\sqrt{s} v_1)}{I_1(\sqrt{s} v_2)}}{s \left\{ N I_0(\sqrt{s} v_1) - I_0(\sqrt{s} v_2) \frac{I_1(\sqrt{s} v_1)}{I_1(\sqrt{s} v_2)} \right\}} = \\ &= \frac{N \left[ 1 + \frac{1}{2^2} (\sqrt{s} v_1 X)^2 + \frac{1}{2^2 \cdot 4^2} (\sqrt{s} v_1 X)^4 + \dots \right] -}{s \left\{ N \left[ 1 + \frac{1}{2^2} (\sqrt{s} v_1)^2 + \frac{1}{2^2 \cdot 4^2} (\sqrt{s} v_1)^4 + \dots \right] - \right.} \\ &\quad \left. - \left[ 1 + \frac{1}{2^2} (\sqrt{s} v_2 X)^2 + \frac{1}{2^2 \cdot 4^2} (\sqrt{s} v_2 X)^4 + \dots \right] \times \right. \\ &\quad \left. - \left[ 1 + \frac{1}{2^2} (\sqrt{s} v_2)^2 + \frac{1}{2^2 \cdot 4^2} (\sqrt{s} v_2)^4 + \dots \right] \times \right. \\ &\quad \left. \times \frac{\frac{1}{2} + \frac{1}{2^2 \cdot 4} (\sqrt{s} v_1)^2 + \dots}{\frac{1}{2} + \frac{1}{2^2 \cdot 4} (\sqrt{s} v_2)^2 + \dots} \frac{v_1}{v_2} \right. \\ &\quad \left. \times \frac{\frac{1}{2} + \frac{1}{2^2 \cdot 4} (\sqrt{s} v_1)^2 + \dots}{\frac{1}{2} + \frac{1}{2^2 \cdot 4} (\sqrt{s} v_2)^2 + \dots} \frac{v_1}{v_2} \right\}} \end{aligned}$$

Now the denominator of the last expression may be differentiated with

respect to  $s$ , and when we take the limit of the whole expression as  $s \rightarrow 0$  we obtain

$$\lim_{s \rightarrow 0} \frac{\Phi_1(s)}{\psi'(s)} = 1.$$

In the same way it is possible to show that

$$\lim_{s \rightarrow 0} \frac{\Phi_2(s)}{\psi'(s)} = 0.$$

The second term of (4-2-19) is zero in this case, since  $\Phi_2(s_i) = 0$  but  $\psi'(s_i) \neq 0$ .

We next take into account the condition for the characteristic-equation roots  $s_n$

$$v_1(1 - v_2^2) I_1(\sqrt{s} v_1) I_0(\sqrt{s} v_1) - v_2(1 - v_1^2) I_1(\sqrt{s} v_2) I_0(\sqrt{s} v_2) = 0$$

and the recurrence formula

$$I_1(\sqrt{s} v_i) = I_0(\sqrt{s} v_i) - \frac{I_1(\sqrt{s} v_i)}{\sqrt{s} v_i}$$

to obtain the third term of (4-2-19) in the forms:  
for heat transfer,

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\Phi_1(s_n)}{\psi'(s_n)} \exp(s_n Fo) = \\ & = 2 \sum_{n=1}^{\infty} \frac{\left[ \frac{v_1(1 - v_2^2)}{v_2^2 - v_1^2} J_1(v_1 \mu_n) J_0(v_1 \mu_n X) - \frac{v_2(1 - v_1^2)}{v_2^2 - v_1^2} J_2(v_1 \mu_n) J_0(v_2 \mu_n X) \right] \exp(-\mu_n^2 Fo)}{\mu_n [J_1(v_1 \mu_n) J_1(v_2 \mu_n) - v_1 v_2 J_0(v_1 \mu_n) J_0(v_2 \mu_n)]}, \end{aligned}$$

for mass transfer,

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\Phi_2(s_n)}{\psi'(s_n)} \exp(s_n Fo) = \\ & = 2 \sum_{n=1}^{\infty} \frac{\frac{v_1(1 - v_2^2)}{v_2^2 - v_1^2} J_1(v_1 \mu_n) (1 - v_1^2) J_0(v_1 \mu_n X) - \frac{v_2(1 - v_1^2)}{v_2^2 - v_1^2} J_2(v_1 \mu_n) (1 - v_2^2) J_0(v_2 \mu_n X)}{\mu_n [J_1(v_1 \mu_n) J_1(v_2 \mu_n) - v_1 v_2 J_0(v_1 \mu_n) J_0(v_2 \mu_n)]} \times \\ & \quad \times \exp(-\mu_n^2 Fo). \end{aligned}$$

When the above expressions are substituted into (4-2-19) and (4-2-15) is taken into account, we finally obtain the following distributions for the heat-transfer and mass-transfer potentials in an infinite cylinder:

$$T(X, Fo) = - \left[ 1 + \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} J_0(v_i \mu_n X) \exp(-\mu_n^2 Fo) \right], \quad (4-2-43)$$

and

$$\theta(X, Fo) = \frac{1}{Ko} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2) J_0(v_i \mu_n X) \exp(-\mu_n^2 Fo), \quad (4-2-44)$$

where

$$C_{n1} = \frac{2 \frac{v_1(1 - v_2^2)}{v_2^2 - v_1^2} J_1(v_1 \mu_n)}{\mu_n \left[ J_1(v_1 \mu_n) J_1(v_2 \mu_n) - \frac{1}{Lu} J_0(v_1 \mu_n) J_0(v_2 \mu_n) \right]}, \quad (4-2-45)$$

TABLE 4 - 4  
Roots of the characteristic equation

$$\frac{J_0(\nu_1 \mu_1)}{J_1(\nu_1 \mu_1)} - \frac{\nu_1}{\nu_2} \frac{1 - \nu_2^2}{1 - \nu_1^2} \frac{J_0(\nu_1 \mu_1)}{J_1(\nu_1 \mu_1)} = 0$$

Lu	Ko*Pn=0.2				Ko*Pn=0.4				Ko*Pn=0.6			
	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$
0.01	0.38278	0.70084	1.01631	1.33101	0.38239	0.70013	1.01527	1.32965	0.38200	0.69942	1.01424	1.32830
0.10	1.19675	2.18428	2.43955	3.18424	1.18253	2.15500	2.46897	3.15314	1.16895	2.12853	2.49514	3.12380
0.40	2.12667	2.69756	4.17785	5.69732	2.01396	2.79819	4.01394	5.59569	1.93096	2.85974	3.88837	5.43083
0.80	2.28049	3.48433	5.11379	6.53401	2.18164	3.49863	4.90949	6.47110	2.09883	3.48285	4.77922	6.31300
1.00	2.29571	3.81128	5.30785	6.91608	2.20391	3.76066	5.17142	6.73615	2.12454	3.69281	5.07024	6.55770
4.00	2.32802	4.95764	7.53257	8.70427	2.25314	4.61332	7.01836	8.94404	2.18567	4.36003	6.60664	8.72472

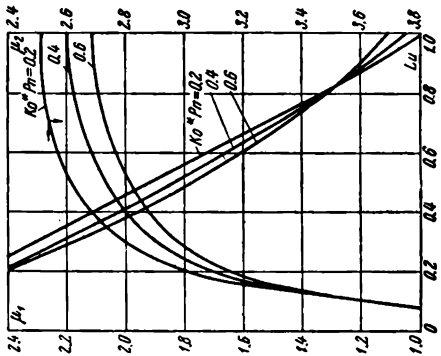


FIGURE 4-4. The characteristic-equation roots as functions of Lu and Ko\*Pn, for an infinite cylinder

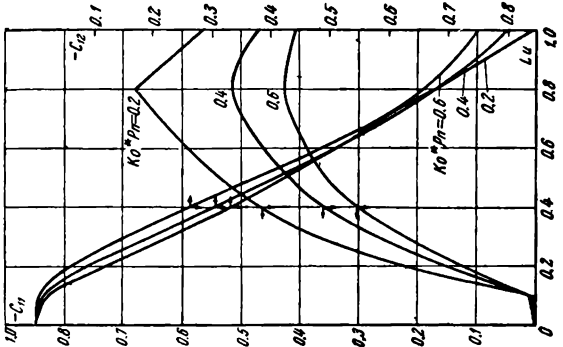


FIGURE 4-5. Coefficients  $C_{n1}$  as functions of Lu and Ko\*Pn, for an infinite cylinder

TABLE 4-5  
Values of the constant coefficients for an infinite cylinder\*

Lu	KoPm	$C_{n1}$				$D_{n1}$				$D_{n2}$	
		11	12	21	22	11	12	21	22	$n=1$	$n=2$
0.01	0.2	-0.00002	-0.00005	-0.00002	0.00007	-0.00002	0.00000	-0.00002	0.00000	-0.00002	-0.00002
	0.4	-0.00004	-0.00010	-0.00005	0.00014	-0.00004	0.00000	-0.00005	0.00000	-0.00004	-0.00005
	0.6	-0.00006	-0.00015	-0.00007	0.00021	-0.00006	0.00000	-0.00007	0.00000	-0.00006	-0.00007
0.10	0.2	-0.00424	-0.00715	-0.06662	0.02942	-0.00354	-0.00001	-0.03462	-0.00008	-0.00355	-0.03470
	0.4	-0.00783	-0.01345	-0.08703	0.04692	-0.00659	-0.00003	-0.04683	-0.00021	-0.00662	-0.04704
	0.6	-0.01088	-0.01903	-0.09370	0.05902	-0.00923	-0.00006	-0.05200	-0.00033	-0.00929	-0.05234
0.40	0.2	-0.46265	-0.26287	-1.04058	0.23137	-0.26588	-0.01622	-0.39743	-0.02425	-0.28210	-0.42168
	0.4	-0.36006	-0.31158	-1.07025	0.29716	-0.23128	-0.02115	-0.41936	-0.03836	-0.25243	-0.45771
	0.6	-0.30367	-0.32906	-1.04913	0.36269	-0.20973	-0.02269	-0.42896	-0.04642	-0.23242	-0.47637
0.80	0.2	-0.67715	-0.67573	-0.51642	0.42440	-0.40458	-0.15368	-0.12253	-0.04654	-0.55826	-0.16307
	0.4	-0.51439	-0.68232	-0.61845	0.55076	-0.34804	-0.12969	-0.19345	-0.07208	-0.47773	-0.26554
	0.6	-0.42786	-0.65663	-0.62450	0.62832	-0.31236	-0.10946	-0.23438	-0.08213	-0.42181	-0.31652
1.00	0.2	-0.56315	-0.84579	-0.45685	0.50136	-0.35635	-0.22869	-0.09546	-0.06126	-0.58504	-0.15672
	0.4	-0.46863	-0.79870	-0.51793	0.64737	-0.33235	-0.17836	-0.15956	-0.08563	-0.51071	-0.24520
	0.6	-0.40889	-0.75024	-0.50737	0.71882	-0.30976	-0.14538	-0.19636	-0.09216	-0.45515	-0.28853
4.00	0.2	-0.17277	-1.32990	-0.07745	1.01226	-0.15043	-0.48138	-0.03867	-0.12374	-0.63181	-0.16341
	0.4	-0.23684	-1.17386	-0.10978	0.98045	-0.21237	-0.36682	-0.06734	-0.11631	-0.57919	-0.18364
	0.6	-0.26320	-1.06298	-0.12573	0.94924	-0.24080	-0.29216	-0.08945	-0.10853	-0.53296	-0.19798

\* The coefficient  $D_{0n}$  is zero.

$$C_{n2} = - \frac{2 \frac{v_2(1-v_1^2)}{v_2^2-v_1^2} J_1(v_1\mu_n)}{\mu_n \left[ J_1(v_1\mu_n) J_1(v_2\mu_n) - \frac{1}{Lu} J_0(v_1\mu_n) J_0(v_2\mu_n) \right]}, \quad (4-2-46)$$

and  $\mu_n$  are the roots of characteristic equation (4-2-42). The first four values of these roots are listed in Table 4-4 and plotted in Figure 4-4. The values of coefficients  $C_{ni}$  are listed in Table 4-5 and plotted in Figure 4-5.

The average value of a dimensionless potential for an infinite cylinder is

$$\bar{\varphi}(Fo) = 2 \int_0^1 X \varphi(X, Fo) dX. \quad (4-2-47)$$

Thus, after applying the necessary operations to equations (4-2-43) and (4-2-44), we obtain

$$\bar{T}(Fo) = - \left[ 1 + \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} \exp(-\mu_n^2 Fo) \right]; \quad (4-2-48)$$

$$\bar{\theta}(Fo) = \frac{1}{Ko^*} \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} (1 - v_i^2) \exp(-\mu_n^2 Fo). \quad (4-2-49)$$

For the given problem

$$D_{ni} = 2C_{ni} \frac{J_1(v_i\mu_n)}{v_i\mu_n}, \quad (4-2-50)$$

where  $C_{ni}$  ( $i=1, 2$ ) are defined by (4-2-45) and (4-2-46). Coefficients  $C_{ni}$ ,  $D_{ni}$ , and  $D_{\tau n} = \sum_{i=1}^2 D_{ni}$  are all listed in Table 4-5 for the first two values of the characteristic-equation roots  $\mu_n$ , and the corresponding curves for  $D_{\tau n}$  are given in Figure 4-6.

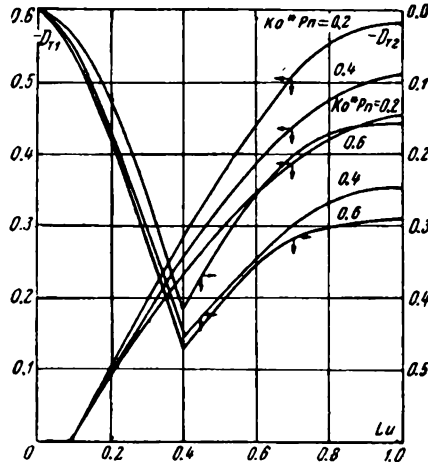


FIGURE 4-6. Coefficients  $D_{\tau n}$  as functions of  $Lu$  and  $Ko^*Pn$ , for an infinite cylinder



Just as in the case of an infinite plate, beginning with a certain value of the Fourier number we may neglect, in practical calculations, all but the first two terms of the infinite series entering into equations (4-2-43), (4-2-44), (4-2-48), and (4-2-49). However, on the whole, the values of these "initial" Fourier numbers for an infinite cylinder are smaller than those for an infinite plate.

c) **Solution for a sphere ( $\Gamma=2$ )**

System (4-1-2) and (4-1-3) of heat-transfer and mass-transfer equations may be rewritten as follows for a sphere ( $\Gamma=2$ ):

$$\begin{aligned}\frac{\partial T^*}{\partial Fo} &= \frac{\partial^2 T^*}{\partial X^2} - Ko^* \frac{\partial \theta^*}{\partial Fo}; \\ \frac{\partial \theta^*}{\partial Fo} &= Lu \frac{\partial^2 \theta^*}{\partial X^2} - LuPn \frac{\partial^2 T^*}{\partial X^2},\end{aligned}$$

where  $T^* = XT$  and  $\theta^* = X\theta$ . Since this way of writing the system of transfer equations makes it identical to that for an infinite plate, its solution in terms of transforms will be

$$\begin{aligned}T^*_L &= A_1 \exp(\sqrt{s} \mathbf{v}_1 X) + A_2 \exp(\sqrt{s} \mathbf{v}_2 X) + A_3 \exp(-\sqrt{s} \mathbf{v}_1 X) + \\ &+ A_4 \exp(-\sqrt{s} \mathbf{v}_2 X); \quad (4-2-51)\end{aligned}$$

$$\begin{aligned}\theta^*_L &= -\frac{1}{Ko^*} [A_1 (1 - \mathbf{v}_1^2) \exp(\sqrt{s} \mathbf{v}_1 X) + A_2 (1 - \mathbf{v}_2^2) \exp(\sqrt{s} \mathbf{v}_2 X) + \\ &+ A_3 (1 - \mathbf{v}_1^2) \exp(-\sqrt{s} \mathbf{v}_1 X) + A_4 (1 - \mathbf{v}_2^2) \exp(-\sqrt{s} \mathbf{v}_2 X)]. \quad (4-2-52)\end{aligned}$$

Here  $A_k$  ( $k=1, 2, 3, 4$ ) are arbitrary constants and  $\mathbf{v}_i$  ( $i=1, 2$ ) are parameters defined by (4-2-7).

Let us find the symmetry conditions for the modified potentials  $T^*$  and  $\theta^*$ , by first writing

$$\frac{dT^*}{dX} = X \frac{dT}{dX} + T = X \frac{dT}{dX} + \frac{T^*}{X},$$

from which

$$T^* = X \frac{dT^*}{dX} - X^2 \frac{dT^*}{dX}.$$

For  $X \rightarrow 0$  we have  $T^* \rightarrow 0$ , since  $T(0, Fo) \neq \infty$ . It can be shown similarly that for  $X \rightarrow 0$ , we have  $\theta^* \rightarrow 0$ . Consequently, in terms of the transforms the symmetry conditions are

$$T^*_L(0, s) = 0 \quad \text{and} \quad \theta^*_L(0, s) = 0. \quad (4-2-53)$$

It follows from condition (4-2-53) that

$$A_1 = -A_3 \quad \text{and} \quad A_2 = -A_4.$$

This means that solutions (4-2-51) and (4-2-52) may be simplified. After some minor transformations we obtain

$$T^*_L(X, s) = \sum_{i=1}^2 B_i \operatorname{sh} \sqrt{s} \mathbf{v}_i X; \quad (4-2-54)$$

$$\theta^*_L(X, s) = -\frac{1}{Ko^*} \sum_{i=1}^2 B_i (1 - \mathbf{v}_i^2) \operatorname{sh} \sqrt{s} \mathbf{v}_i X, \quad (4-2-55)$$

or

$$T_L(X, s) = \sum_{i=1}^2 B_i \frac{\text{sh} \sqrt{s} v_i X}{X}; \quad (4-2-56)$$

$$\theta_L(X, s) = -\frac{1}{\text{Ko}^*} \sum_{i=1}^2 B_i (1 - v_i^2) \frac{\text{sh} \sqrt{s} v_i X}{X}. \quad (4-2-57)$$

The boundary conditions for the given problem, stated in terms of the transforms, are

$$T_L(1, s) = -\frac{1}{s}; \quad (4-2-58)$$

$$\int_0^1 X^2 \theta(X, s) dX = 0. \quad (4-2-59)$$

Equations (4-2-56) and (4-2-57) may now be combined with boundary conditions (4-2-58) and (4-2-59) to obtain a system of equations in  $B_i$  ( $i=1, 2$ ), the solution of which gives

$$B_1 = -\frac{v_1^2(1-v_2^2)[\sqrt{s} v_2 \text{ch} \sqrt{s} v_2 - \text{sh} \sqrt{s} v_2] \text{sh} \sqrt{s} v_1 - \text{sh} \sqrt{s} v_2}{s \{v_1^2(1-v_2^2)[\sqrt{s} v_2 \text{ch} \sqrt{s} v_2 - \text{sh} \sqrt{s} v_2] \text{sh} \sqrt{s} v_1 - \text{sh} \sqrt{s} v_2\}}; \\ B_2 = +\frac{v_2^2(1-v_1^2)[\sqrt{s} v_1 \text{ch} \sqrt{s} v_1 - \text{sh} \sqrt{s} v_1] \text{sh} \sqrt{s} v_2 - \text{sh} \sqrt{s} v_1}{s \{v_1^2(1-v_2^2)[\sqrt{s} v_2 \text{ch} \sqrt{s} v_2 - \text{sh} \sqrt{s} v_2] \text{sh} \sqrt{s} v_1 - \text{sh} \sqrt{s} v_1\}}.$$

Consequently, the transform solution has the final form

$$T^*_L = -\frac{v_1^2(1-v_2^2)[\sqrt{s} v_2 \text{ch} \sqrt{s} v_2 - \text{sh} \sqrt{s} v_2] \text{sh} \sqrt{s} v_1 X - \text{sh} \sqrt{s} v_2}{s \{v_1^2(1-v_2^2)[\sqrt{s} v_2 \text{ch} \sqrt{s} v_2 - \text{sh} \sqrt{s} v_2] \text{sh} \sqrt{s} v_1 - \text{sh} \sqrt{s} v_2\}}; \quad (4-2-60)$$

$$\theta^*_L = \frac{1}{\text{Ko}^*} \frac{v_1^2(1-v_2^2)[\sqrt{s} v_2 \text{ch} \sqrt{s} v_2 - \text{sh} \sqrt{s} v_2] (1-v_1^2) \text{sh} \sqrt{s} v_1 X - \text{sh} \sqrt{s} v_2}{s \{v_1^2(1-v_2^2)[\sqrt{s} v_2 \text{ch} \sqrt{s} v_2 - \text{sh} \sqrt{s} v_2] \text{sh} \sqrt{s} v_1 - \text{sh} \sqrt{s} v_2\}}. \quad (4-2-61)$$

To obtain operational [transform] solutions (4-2-60) and (4-2-61) in terms of the inverse transforms  $T^*$  and  $\theta^*$ , the expansion theorem is used. First, (4-2-60) and (4-2-61) are written as

$$-T^*_L = \frac{\Phi_1(s)}{\psi(s)} \quad \text{and} \quad \text{Ko}^* \theta^*_L = \frac{\Phi_2(s)}{\psi(s)},$$

where the values of  $\Phi_1(s)$ ,  $\Phi_2(s)$ , and  $\psi(s)$  are found from the above equations. Once again, the ratios  $\Phi_1(s)/\psi(s)$  and  $\Phi_2(s)/\psi(s)$  may be regarded as ratios of generalized polynomials in  $s$ , to which the expansion theorem is applicable. The common denominator  $\psi(s)$  has the following roots:

- the zero root,  $s=0$ ;
- the roots of the equation

$$v_1^2(1-v_2^2)[\sqrt{s} v_2 \text{ch} \sqrt{s} v_2 - \text{sh} \sqrt{s} v_2] \text{sh} \sqrt{s} v_1 - \text{sh} \sqrt{s} v_2 = 0,$$

which may also be written as

$$\begin{aligned} & \nu_1^2 (1 - \nu_2^2) (\sin \nu_2 \mu - \nu_2 \mu \cos \nu_2 \mu) \sin \nu_1 \mu - \nu_2^2 (1 - \nu_1^2) \times \\ & \times (\sin \nu_1 \mu - \nu_1 \mu \cos \nu_1 \mu) \sin \nu_2 \mu = 0, \end{aligned}$$

where

$$\mu = i \sqrt{s}.$$

The last equation is satisfied by the roots common to the system of equations

$$\left. \begin{aligned} \sin \nu_1 \mu &= 0; \\ \sin \nu_2 \mu &= 0 \end{aligned} \right\} \quad (4-2-62)$$

(this system has common roots  $\mu_l$  provided that  $\frac{\nu_2}{\nu_1}$  is a rational fraction), and by the roots  $\mu_n$  of the equation

$$\frac{\nu_1^2 (1 - \nu_2^2)}{\nu_2^2 (1 - \nu_1^2)} \nu_2 \mu \cot \nu_2 \mu - \nu_1 \mu \cot \nu_1 \mu = \frac{\nu_1^2 (1 - \nu_2^2)}{\nu_2^2 (1 - \nu_1^2)} - 1. \quad (4-2-63)$$

Consequently, the expansion theorem once again has the form of (4-2-19), in which the first term corresponds to the zero root, the second term to the roots of system of equations (4-2-62), and the third term to the roots of equation (4-2-63).

In order to obtain the zero-root term let us first expand the hyperbolic functions contained in  $\Phi_h(s)$  and  $\psi(s)$  as infinite series. Then, after cancellation, we obtain

$$\begin{aligned} \frac{\Phi_1(s)}{\psi(s)} &= \frac{\nu_1^2 (1 - \nu_2^2)}{\nu_2^2 (1 - \nu_1^2)} \left[ \left( \frac{1}{2!} - \frac{1}{3!} \right) + \left( \frac{1}{4!} - \frac{1}{5!} \right) (V \bar{s} \nu_2)^2 + \dots \right] - \\ &= s \left[ \frac{\nu_1^2 (1 - \nu_2^2)}{\nu_2^2 (1 - \nu_1^2)} \left[ \left( \frac{1}{2!} - \frac{1}{3!} \right) + \left( \frac{1}{4!} - \frac{1}{5!} \right) (V \bar{s} \nu_2)^2 + \dots \right] - \right. \\ &\quad \left. - \frac{\nu_1}{\nu_2} \left[ \left( \frac{1}{2!} - \frac{1}{3!} \right) + \left( \frac{1}{4!} - \frac{1}{5!} \right) (V \bar{s} \nu_1)^2 + \dots \right] \times \right. \\ &\quad \left. - \frac{\nu_1}{\nu_2} \left[ \left( \frac{1}{2!} - \frac{1}{3!} \right) + \left( \frac{1}{4!} - \frac{1}{5!} \right) (V \bar{s} \nu_1)^2 + \dots \right] \times \right. \\ &\quad \left. 1 + \frac{1}{3!} (V \bar{s} \nu_2 X)^2 + \dots \right. \\ &\quad \times \frac{1 + \frac{1}{3!} (V \bar{s} \nu_1 X)^2 + \dots}{1 + \frac{1}{3!} (V \bar{s} \nu_1 X)^2 + \dots} \left. 1 + \frac{1}{3!} (V \bar{s} \nu_1 X)^2 + \dots \right. \\ &\quad \times \frac{1 + \frac{1}{3!} (V \bar{s} \nu_2)^2 + \dots}{1 + \frac{1}{3!} (V \bar{s} \nu_1)^2 + \dots} \left. 1 + \frac{1}{3!} (V \bar{s} \nu_1)^2 + \dots \right] X. \end{aligned}$$

The denominator of this ratio is now differentiated with respect to  $s$ , and when we take the limit of the ratio  $\Phi_1(s)/\psi'(s)$  as  $s \rightarrow 0$  the first term of (4-2-19) is found to be

$$\lim_{s \rightarrow 0} \frac{\Phi_1(s)}{\psi'(s)} = X.$$

Similarly, the first term of the mass-transfer equation is

$$\lim_{s \rightarrow 0} \frac{\Phi_2(s)}{\psi'(s)} = 0.$$

Let us find the second terms of (4-2-19). The roots common to system of equations (4-2-62) have the form  $\mu_l = \frac{\beta}{\nu_1} l \pi = \frac{\alpha}{\nu_2} l \pi$  ( $l = 1, 2, 3, \dots$ ), provided  $\frac{\nu_2}{\nu_1} = \frac{\alpha}{\beta}$

is a rational fraction. By means of a series of transformations it is easy to show that

$$\begin{aligned}\psi'(\mu_i) &= \frac{1}{2} v_1 v_2 (v_2^2 - v_1^2) \mu_i^2 \cos v_1 \mu_i \cos v_2 \mu_i; \\ \Phi_1(\mu_i) &= -[v_1^2(1 - v_2^2) v_2 \mu_i \cos v_1 \mu_i \sin v_1 \mu_i X - \\ &\quad - v_2^2(1 - v_1^2) v_1 \mu_i \cos v_1 \mu_i \sin v_2 \mu_i X]; \\ \Phi_2(\mu_i) &= -[v_1^2(1 - v_2^2) v_2 \mu_i \cos v_2 \mu_i (1 - v_1^2) \sin v_1 \mu_i X - \\ &\quad - v_2^2(1 - v_1^2) v_1 \mu_i \cos v_1 \mu_i (1 - v_2^2) \sin v_2 \mu_i X].\end{aligned}$$

Since  $\cos v_1 \mu_i = \cos \beta l \pi = (-1)^{\beta l}$  and  $\cos v_2 \mu_i = \cos \alpha l \pi = (-1)^{\alpha l}$ , we obtain the following expressions for the second terms:

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{\Phi_1(s_i)}{\psi'(s_i)} \exp(s_i F_0) &= - \sum_{i=1}^{\infty} \sum_{l=1}^2 E_{il} \sin v_i \mu_i \exp(-\mu_i^2 F_0); \\ \sum_{i=1}^8 \frac{\Phi_2(s_i)}{\psi'(s_i)} \exp(s_i F_0) &= - \sum_{i=1}^{\infty} \sum_{l=1}^2 E_{il} (1 - v_i^2) \sin v_i \mu_i X \exp(-\mu_i^2 F_0),\end{aligned}$$

where

$$E_{i1} = \frac{2}{\mu_i \cos v_1 \mu_i} \cdot \frac{v_1(1 - v_2^2)}{v_2^2 - v_1^2} = \frac{2}{\beta l \pi (-1)^{\beta l}} \cdot \frac{v_1^2(1 - v_2^2)}{v_2^2 - v_1^2}; \quad (4-2-64)$$

$$E_{i2} = -\frac{2}{\mu_i \cos v_2 \mu_i} \cdot \frac{v_2(1 - v_1^2)}{v_2^2 - v_1^2} = -\frac{2}{\alpha l \pi (-1)^{\alpha l}} \cdot \frac{v_2^2(1 - v_1^2)}{v_2^2 - v_1^2}. \quad (4-2-65)$$

The third terms of (4-2-19) are correspondingly

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\Phi_1(s_n)}{\psi'(s_n)} \exp(s_n F_0) &= \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \sin v_i \mu_n X \exp(-\mu_n^2 F_0); \\ \sum_{n=1}^{\infty} \frac{\Phi_2(s_n)}{\psi'(s_n)} \exp(s_n F_0) &= \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2) \sin v_i \mu_n X \exp(-\mu_n^2 F_0),\end{aligned}$$

where

$$C_{n1} = 2 \frac{v_1(1 - v_2^2)}{v_2(v_2^2 - v_1^2)} \cdot \frac{\sin v_2 \mu_n - v_2 \mu_n \cos v_2 \mu_n}{\mu_n \phi_n}; \quad (4-2-66)$$

$$C_{n2} = -2 \frac{v_2(1 - v_1^2)}{v_1(v_2^2 - v_1^2)} \cdot \frac{\sin v_1 \mu_n - v_1 \mu_n \cos v_1 \mu_n}{\mu_n \phi_n}; \quad (4-2-67)$$

$$\begin{aligned}\phi_n &= \mu_n \left( \cos v_1 \mu_n \cos v_2 \mu_n - \frac{1}{L\alpha} \sin v_1 \mu_n \sin v_2 \mu_n \right) + \\ &+ \frac{v_1^2(1 - v_2^2)}{v_2(v_2^2 - v_1^2)} \cos v_1 \mu_n \sin v_2 \mu_n - \frac{v_2^2(1 - v_1^2)}{v_1(v_2^2 - v_1^2)} \cos v_2 \mu_n \sin v_1 \mu_n;\end{aligned} \quad (4-2-67)$$

and  $\mu_n = i\sqrt{s_n}$  are the roots of characteristic equation (4-2-63).

Thus, the solution of the system of heat-transfer and mass-transfer equations for a sphere has the following final form:

$$T^*(X, Fo) = - \left[ X - \sum_{i=1}^{\infty} \sum_{n=1}^2 E_{in} \sin \nu_i \mu_n X \exp(-\mu_n^2 Fo) + \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \sin \nu_i \mu_n X \exp(-\mu_n^2 Fo) \right]; \quad (4-2-68)$$

$$\theta^*(X, Fo) = - \frac{1}{Ko^*} \left[ \sum_{i=1}^{\infty} \sum_{n=1}^2 E_{in} (1 - \nu_i^2) \sin \nu_i \mu_n X \exp(-\mu_n^2 Fo) - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - \nu_i^2) \sin \nu_i \mu_n X \exp(-\mu_n^2 Fo) \right]; \quad (4-2-69)$$

or

$$T(X, Fo) = - \left[ 1 - \sum_{i=1}^{\infty} \sum_{n=1}^2 E_{in} \frac{\sin \nu_i \mu_n X}{X} \exp(-\mu_n^2 Fo) + \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \frac{\sin \nu_i \mu_n X}{X} \exp(-\mu_n^2 Fo) \right]; \quad (4-2-70)$$

$$\theta(X, Fo) = - \frac{1}{Ko^*} \left[ \sum_{i=1}^{\infty} \sum_{n=1}^2 E_{in} (1 - \nu_i^2) \frac{\sin \nu_i \mu_n X}{X} \exp(-\mu_n^2 Fo) - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - \nu_i^2) \frac{\sin \nu_i \mu_n X}{X} \exp(-\mu_n^2 Fo) \right]. \quad (4-2-71)$$

Coefficients  $E_{in}$  and  $C_{ni}$  are defined, respectively, by equations (4-2-64), (4-2-65), (4-2-66), and (4-2-67).

The first four values of the roots  $\mu_n$  of characteristic equation (4-2-63) are given in Table 4-6 and the corresponding curves are plotted in Figure 4-7. The coefficients  $C_{ni}$  ( $n=1, 2$ ) as functions of  $Lu$  and  $Ko^*Pn$  are given in Table 4-7 and Figure 4-8.

TABLE 4-6

Roots of the characteristic equation

$$\frac{\nu_1^2(1-\nu_2^2)}{\nu_2^2(1-\nu_1^2)} \nu_2 \mu \cot \nu_2 \mu - \nu_1 \mu \cot \nu_1 \mu = \frac{\nu_1^2(1-\nu_2^2)}{\nu_2^2(1-\nu_1^2)} - 1$$

Lu	Ko*Pn=0.2				Ko*Pn=0.4	
	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_1$	$\mu_2$
0.01	0.44889	0.77175	1.08931	1.40520	0.44844	0.77095
0.10	1.40422	2.41278	3.41961	4.39999	1.38815	2.38438
0.40	2.58257	3.39820	4.61758	6.31544	2.44279	3.51388
0.80	2.90726	4.17897	5.72420	8.53647	2.74648	4.21945
1.00	2.94283	4.53810	5.97498	7.66203	2.79165	4.48808
4.00	3.00935	5.67967	8.33880	9.97824	2.89051	5.29700

TABLE 4-6 (continued)

Lu	Ko*Pn=0.4		Ko*Pn=0.6			
	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$
0.01	1.08822	1.40379	0.44798	0.77018	1.08711	1.40237
0.10	3.39787	4.35412	1.37283	2.35730	3.19986	3.38590
0.40	4.45563	6.08955	2.33850	3.56722	4.36068	5.88200
0.80	5.50524	7.14043	2.62055	4.18826	5.39633	6.91892
1.00	5.83378	7.42284	2.66812	4.39044	5.75122	7.20244
4.00	9.98242	11.18359	2.78363	5.01133	11.28541	12.53915

It should be noted that the terms in the above solutions which contain the characteristic-equation roots  $\mu_i$  are negligible in comparison with the other terms. Therefore, in an overwhelming majority of practical cases these

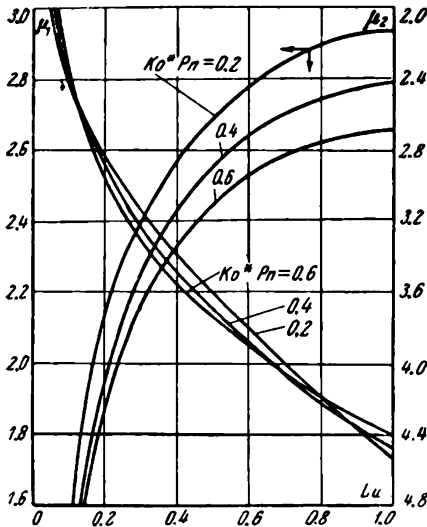
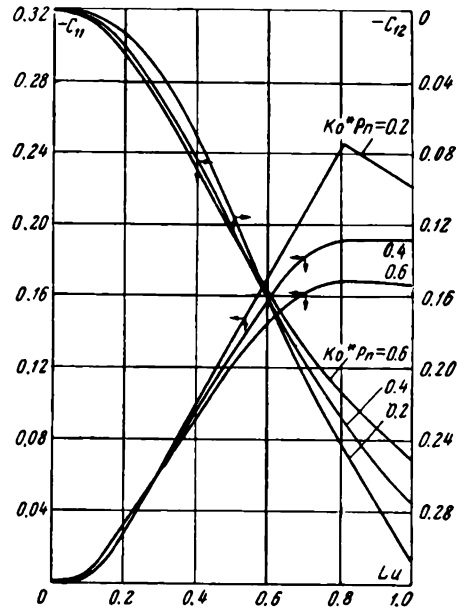


FIGURE 4-7. The characteristic-equation roots as functions of Lu and Ko\*Pn, for a sphere

FIGURE 4-8. Coefficients  $C_{n\epsilon}$  as functions of Lu and Ko\*Pn, for a sphere

sums may be neglected. The same is true concerning the corresponding terms in the expressions for the average dimensionless transfer potentials. The average transfer potential for a body of spherical shape is

$$\bar{\varphi}(Fo) = 3 \int_0^1 X^2 \varphi(X, Fo) dX. \quad (4-2-72)$$

After carrying out the necessary transformations, we obtain

$$\bar{T}(Fo) = - \left[ 1 - \sum_{i=1}^{\infty} \sum_{l=1}^2 F_{li} \exp(-\mu_i^2 Fo) + \sum_{n=1}^{\infty} \sum_{l=1}^2 D_{nl} \exp(-\mu_n^2 Fo) \right]; \quad (4-2-73)$$

$$\bar{\theta}(Fo) = -\frac{1}{Ko^*} \left[ \sum_{i=1}^{\infty} \sum_{l=1}^2 F_{li} (1 - v_i^2) \exp(-v_i^2 Fo) - \sum_{n=1}^{\infty} \sum_{l=1}^2 D_{nl} (1 - v_l^2) \exp(-v_l^2 Fo) \right], \quad (4-2-74)$$

TABLE 4-7  
Values of the constant coefficients for a sphere

Lu	Ko*Pn	C <sub>n1</sub>		C <sub>n2</sub>		D <sub>Tn</sub>	
		n=1	n=2	n=1	n=2	n=1	n=2
0.01	0.2	-0.00006	-0.00002	-0.00001	0.00001	-0.00003	-0.00001
	0.4	-0.00004	-0.00007	-0.00003	0.00003	-0.00002	-0.00005
	0.6	-0.00008	-0.00009	-0.00004	0.00004	-0.00004	-0.00007
0.10	0.2	-0.00181	-0.00426	-0.00186	0.00293	-0.00206	-0.00546
	0.4	-0.00353	-0.00723	-0.00351	0.00531	-0.00398	-0.00934
	0.6	-0.00495	-0.00954	-0.00499	0.00730	-0.00552	-0.01243
0.40	0.2	-0.09840	-0.50408	-0.06884	0.05756	-0.13212	-0.48981
	0.4	-0.09589	-0.45788	-0.08498	0.08046	-0.13646	-0.47160
	0.6	-0.09205	-0.41807	-0.09122	0.10504	-0.13335	-0.45967
0.80	0.2	-0.24377	-0.24837	-0.24066	0.14047	-0.42237	-0.23175
	0.4	-0.19195	-0.27492	-0.22879	0.17332	-0.34433	-0.32090
	0.6	-0.16916	-0.25430	-0.21290	0.19031	-0.29704	-0.34023
1.00	0.2	-0.22073	-0.20487	-0.30727	0.16754	-0.46462	-0.20819
	0.4	-0.19196	-0.24959	-0.27535	0.20479	-0.38587	-0.29094
	0.6	-0.16674	-0.20180	-0.24993	0.21481	-0.33387	-0.30619
4.00	0.2	-0.11418	-0.03515	-0.50348	0.29840	-0.53591	-0.18267
	0.4	-0.16991	-0.05595	-0.42860	0.28197	-0.47640	-0.19917
	0.6	-0.20314	-0.07195	-0.37710	0.26640	-0.42718	-0.20695

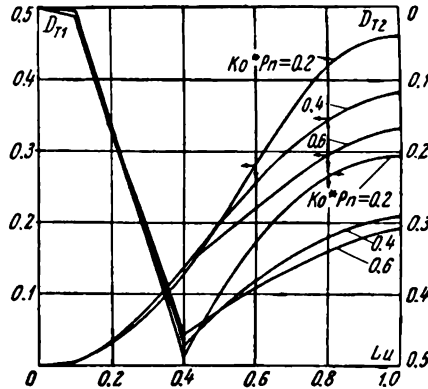


FIGURE 4-9. Coefficients  $D_{Tn}$  as functions of Lu and  $Ko^*Pn$ , for a sphere

where

$$F_{li} = 3E_{li} \frac{\sin v_i \mu_l - v_i \mu_l \cos v_i \mu_l}{v_i^2 \mu_l^2}, \quad (4-2-75)$$

$$D_{ni} = 3C_{ni} \frac{\sin v_i t_n - v_i t_n \cos v_i t_n}{v_i^2 t_n^2}. \quad (4-2-76)$$

The first two values of coefficients  $D_{Tn} = \sum_{i=1}^2 D_{ni}$  are given in Table 4-7 and Figure 4-9; coefficients  $D_{\theta n} = \frac{1}{Ko^*} \sum_{i=1}^2 D_{ni} (1 - v_i^2)$  are zero. The values of the Fourier number beginning with which the obtained solutions may be simplified are approximately equal to the corresponding values of Fo cited for an infinite plate.

### 4-3. Unsteady Fields of Heat-Transfer and Mass-Transfer Potential in the Absence of Phase Transitions ( $Ko^* = 0$ ) and for Constant Integral Mass-Transfer Potential

Unsteady fields of heat-transfer and mass-transfer potential in the absence of phase transitions ( $Ko^* = 0$ ) or thermal-gradient transfer ( $Pn = 0$ ) may be described using the solutions of the corresponding problems in the previous section. This is done by taking the limits of the appropriate quantities in these solutions. This result follows from a direct solution of the "incomplete" system of differential heat-transfer and mass-transfer equations, namely system of equations (4-1-2) and (4-1-3) with  $Ko^*$  or  $Pn$  set equal to zero.

As an illustration let us find the potential distributions in a finite porous rod in the absence of phase transitions. For bodies with other shapes only the final results will be given.

a) Heat and mass transfer in a finite rod ( $\Gamma = 0$ ;  $Ko^* = 0$ ;  $T(X, 0) = 0$ ;  $\Theta(X, 0) = 0$ ).

Let us assume that during the experiment one end of a rod, located at the coordinate origin  $X=0$ , is maintained at a constant heater temperature  $t_n$ :

$$T(0, Fo) = \frac{t_n - t_0}{t_0} = T_n = \text{constant}. \quad (4-3-1)$$

The other end of the rod ( $X=1$ ) has, during the experiment, the temperature of the surroundings  $t_0 = t_0$ , so that

$$T(1, Fo) = 0. \quad (4-3-2)$$

The lateral surfaces of the rod are insulated to prevent heat and mass transfer and in addition condition (4-1-1) is satisfied.

Let us first apply Laplace transformation (4-2-1) to the initial system of differential equations. Then, taking into account the zero initial conditions, we obtain

$$sT_L = T''_L; \quad (4-3-3)$$

$$s\Theta_L = Lu \Theta''_L - Lu Pn T''_L. \quad (4-3-4)$$

Solution (4-3-3) in terms of the transform is

$$T_L(X, s) = \frac{T_n \operatorname{sh} \sqrt{s}(1-X)}{s \operatorname{sh} \sqrt{s}}. \quad (4-3-5)$$

Next we may substitute (4-3-5) into equation (4-3-4) and solve it using a second application of the Laplace transformation with respect to coordinate  $X$ . This gives

$$\begin{aligned} \Theta_L(X, s) = & \frac{T_n Pn}{(1 - 1/Lu) s \operatorname{sh} \sqrt{s}} \left[ \operatorname{sh} \sqrt{s}(1-X) - \operatorname{ch} \sqrt{\frac{s}{Lu}} X \operatorname{sh} \sqrt{s} + \right. \\ & \left. + \sqrt{Lu} \operatorname{ch} \sqrt{s} \operatorname{sh} \sqrt{\frac{s}{Lu}} X \right] + \frac{1}{s} \operatorname{ch} \sqrt{\frac{s}{Lu}} X - \left( A_1 \operatorname{ch} \sqrt{\frac{s}{Lu}} X + A_2 \operatorname{sh} \sqrt{\frac{s}{Lu}} X \right). \end{aligned} \quad (4-3-6)$$



To determine constants  $A_1$  and  $A_2$  we have left only condition (4-1-1), since (4-3-1) and (4-3-2) were used to derive solution (4-3-5). Therefore, let us specify a second condition, characterizing the absence of mass flow through the end face of the rod:

$$\frac{\partial \Theta(0, Fo)}{\partial X} - Pn \frac{\partial T(0, Fo)}{\partial X} = 0,$$

or in terms of transforms

$$\Theta'_L(0, s) - Pn T'_L(0, s) = 0. \quad (4-3-7)$$

Conditions (4-1-1) and (4-3-7) may now be combined with (4-3-6) to obtain, after a number of transformations,

$$\begin{aligned} \Theta_L(X, s) = & \frac{T_\pi Lu Pn}{Lu - 1} \times \\ & \times \left[ \frac{\text{sh} \sqrt{\frac{s}{Lu}} \text{sh} \sqrt{s}(1-X) + \frac{1}{\sqrt{Lu}} \left( 1 - \text{ch} \sqrt{s} \text{ch} \sqrt{\frac{s}{Lu}} \right) \text{ch} \sqrt{\frac{s}{Lu}} X +}{s \text{sh} \sqrt{s} \times} \right. \\ & \left. + \frac{1}{\sqrt{Lu}} \text{ch} \sqrt{s} \text{sh} \sqrt{\frac{s}{Lu}} \text{sh} \sqrt{\frac{s}{Lu}} X \right] \times \\ & \times \frac{\text{sh} \sqrt{s/Lu}}{\text{sh} \sqrt{s/Lu}}. \end{aligned} \quad (4-3-8)$$

The inverse transforms for (4-3-5) and (4-3-8) may be found using the generalized expansion formula. After the necessary transformations, we obtain the required solution of the given problem, written as follows:

$$\begin{aligned} T(X, Fo) = & \frac{t - t_0}{t_0} = T_\pi \left[ (1-X) + \sum_{n=1}^{\infty} (-1)^n \frac{2}{\mu_n} \sin \mu_n (1-X) \exp(-\mu_n^2 Fo) \right]; \quad (4-3-9) \\ \Theta(X, Fo) = & \frac{\theta - \theta_0}{\theta_0} = Pn \left( \frac{1}{2} - X \right) + \\ & + \frac{Lu Pn}{Lu - 1} \left\{ \sum_{n=1}^{\infty} \left( \frac{2}{\sqrt{Lu} \mu_n \sin \frac{\mu_n}{\sqrt{Lu}}} \left[ \cos \frac{\mu_n}{\sqrt{Lu}} (1-X) - (-1)^n \cos \frac{\mu_n}{\sqrt{Lu}} X \right] - \frac{2}{\mu_n} \sin \mu_n X \right) \times \right. \\ & \left. \times \exp(-\mu_n^2 Fo) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} + \cos \sqrt{Lu} \mu_m}{\sqrt{Lu} \mu_m \sin \sqrt{Lu} \mu_m} \cos \mu_m X \exp(-\mu_m^2 Lu Fo) \right\}. \end{aligned} \quad (4-3-10)$$

where  $\mu_n$  and  $\mu_m$  are defined as

$$\mu_n = n\pi \quad \text{and} \quad \mu_m = m\pi \quad (n = 1, 2, 3 \dots) \\ (m = 1, 2, 3 \dots)$$

The infinite series in equations (4-3-9) and (4-3-10) converge rapidly so that for  $Fo > 0.1$  it is sufficient to retain only the first terms. However, it is difficult to determine the transfer potentials using the above solutions when  $Fo < 0.1$ , since many terms of the series must be used, the number of terms increasing rapidly as the Fourier number decreases. Thus we must also seek a solution in a form which is convenient for low Fourier numbers.

Let us write equations (4-3-5) and (4-3-8) in a somewhat different form. The quantity  $1/\text{sh } z$  may be expanded to  $\frac{1}{\text{sh } z} = 2 \sum_{k=1}^{\infty} \exp[-(2k-1)z]$ . Then, since we know that

$$\text{sh } z = \frac{1}{2}(e^z - e^{-z}) \quad \text{and} \quad \text{ch } z = \frac{1}{2}(e^z + e^{-z}),$$

equations (4-3-5) and (4-3-8) may be rewritten as

$$\begin{aligned} T_L = & T_\pi \frac{e^{\sqrt{s}(1-X)} - e^{-\sqrt{s}(1-X)}}{s} \sum_{n=1}^{\infty} e^{-(2n-1)\sqrt{s}} = \frac{T_\pi}{s} \times \\ & \times \sum_{n=1}^{\infty} [e^{-[(2n-1)-(1-X)]\sqrt{s}} - e^{-[(2n-1)+(1-X)]\sqrt{s}}]; \end{aligned}$$

$$\begin{aligned}
\Theta_L &= \frac{Lu Pn}{Lu-1} \frac{1}{s} \left\{ \frac{\text{sh } \sqrt{s}(1-X)}{\text{sh } \sqrt{s}} + \frac{1}{\sqrt{Lu}} \frac{\text{ch } \sqrt{\frac{s}{Lu}} X - \text{ch } \sqrt{s} \text{ch } \sqrt{\frac{s}{Lu}}(1-X)}{\text{sh } \sqrt{s} \text{sh } \sqrt{\frac{s}{Lu}}} \right\} - \\
&= \frac{Lu Pn}{Lu-1} \frac{1}{s} \left\{ [e^{\sqrt{s}(1-X)} - e^{-\sqrt{s}(1-X)}] \sum_{n=1}^{\infty} e^{-(2n-1)\sqrt{s}} + \frac{1}{\sqrt{Lu}} \times \right. \\
&\quad \times \left( e^{\sqrt{\frac{s}{Lu}} X} + e^{-\sqrt{\frac{s}{Lu}} X} \right) 2 \sum_{n=1}^{\infty} e^{-(2n-1)\sqrt{s}} \sum_{m=1}^{\infty} e^{-(2m-1)\sqrt{\frac{s}{Lu}}} - \\
&\quad - \frac{1}{\sqrt{Lu}} (e^{\sqrt{s}} - e^{-\sqrt{s}}) \left[ e^{\sqrt{\frac{s}{Lu}}(1-X)} + e^{-\sqrt{\frac{s}{Lu}}(1-X)} \right] \sum_{n=1}^{\infty} e^{-(2n-1)\sqrt{s}} \times \\
&\quad \times \sum_{m=1}^{\infty} e^{-(2m-1)\sqrt{\frac{s}{Lu}}} \left. \right\} = \frac{Lu Pn}{Lu-1} \frac{1}{s} \left\{ e^{-\sqrt{s} X} + \sum_{n=1}^{\infty} [e^{-(2n+X)\sqrt{s}} - e^{-(2n-X)\sqrt{s}}] - \right. \\
&\quad - \frac{1}{\sqrt{Lu}} e^{\sqrt{\frac{s}{Lu}} X} - \frac{1}{\sqrt{Lu}} \sum_{m=1}^{\infty} \left[ e^{-(2m+X)\sqrt{\frac{s}{Lu}}} + e^{-(2m-X)\sqrt{\frac{s}{Lu}}} \right] + \\
&\quad + \frac{2}{\sqrt{Lu}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n-1} \left[ e^{-\left[n + \frac{1}{\sqrt{Lu}}(m+X)\right]\sqrt{s}} + e^{-\left[n + \frac{1}{\sqrt{Lu}}(m-X)\right]\sqrt{s}} \right] - \\
&\quad \left. - \frac{2}{\sqrt{Lu}} \sum_{n=1}^{\infty} e^{-\left(2n + \frac{1}{\sqrt{Lu}} X\right)\sqrt{s}} \right\}.
\end{aligned}$$

The inverse transforms for each of the terms may be found from tables, using the following inverse-transformation formula [erfc denotes "error function"]:

$$L^{-1} \left[ \frac{1}{s} e^{-k\sqrt{s}} \right] = \text{erfc} \frac{k}{2\sqrt{Fo}},$$

where

$$\text{erfc } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt;$$

$$\text{erfc } 0 = 0, \quad \text{erfc } \infty = 1, \quad \text{erfc}(-x) = -\text{erfc } x, \quad \text{erfc } x = 1 - \text{erf } x.$$

Thus, the potential distributions for heat and mass transfer are

$$T(X, Fo) = T_{\infty} \left\{ \text{erfc} \frac{X}{2\sqrt{Fo}} + \sum_{n=1}^{\infty} \left[ \text{erfc} \frac{2n+X}{2\sqrt{Fo}} - \text{erfc} \frac{2n-X}{2\sqrt{Fo}} \right] \right\}; \quad (4-3-11)$$

$$\begin{aligned}
\Theta(X, Fo) &= \frac{Lu Pn}{Lu-1} \left\{ \text{erfc} \frac{X}{2\sqrt{Fo}} + \sum_{n=1}^{\infty} \left[ \text{erfc} \frac{2n+X}{2\sqrt{Fo}} - \text{erfc} \frac{2n-X}{2\sqrt{Fo}} \right] - \right. \\
&\quad - \frac{1}{\sqrt{Lu}} \text{erfc} \frac{X}{2\sqrt{Lu Fo}} - \frac{1}{\sqrt{Lu}} \sum_{m=1}^{\infty} \left[ \text{erfc} \frac{2m+X}{2\sqrt{Lu Fo}} + \text{erfc} \frac{2m-X}{2\sqrt{Lu Fo}} \right] - \\
&\quad \left. - \frac{2}{\sqrt{Lu}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^n \left[ \text{erfc} \frac{n + \frac{m+X}{\sqrt{Lu}}}{2\sqrt{Fo}} + \text{erfc} \frac{n - \frac{m-X}{\sqrt{Lu}}}{2\sqrt{Fo}} \right] - \frac{2}{\sqrt{Lu}} \sum_{n=1}^{\infty} \text{erfc} \frac{2n + \frac{1}{\sqrt{Lu}} X}{2\sqrt{Fo}} \right\}. \quad (4-3-12)
\end{aligned}$$

For low  $Fo$  all terms in the series except the first term are vanishingly small, since the function  $\text{erfc } z$  decreases rapidly as its argument  $z$  increases; for example, for  $z = 2.7$  we have  $\text{erfc } z \approx 0.0001$  (practically zero). The function  $\text{erfc } z = 1 - \text{erf } z$  is tabulated, so that solutions based on formulas (4-3-11) and (4-3-12) are quite convenient for practical calculations.

Consequently, for low values of the Fourier number it is necessary to use solutions (4-3-11) and (4-3-12), while for high values of  $Fo$  solutions (4-3-9) and (4-3-10) may be used. This specific example indicates the

great advantage of the operational method, a method which makes it possible to solve the problem at hand in two ways, giving one result which is convenient for calculations at low Fo and another result for high Fo.

b) Heat and mass transfer in an infinite plate ( $\Gamma=0$ ;  $Ko^*=0$ ;  $T(X, 0)=\Theta(X, 0)=0$ ).

The potential distributions for heat and mass transfer are in this case

$$T(X, Fo) = \frac{t - t_0}{t_0} = -1 + \sum_{n=1}^{\infty} A_n \cos \mu_n X \exp(-\mu_n^2 Fo) \quad (4-3-13)$$

$$\Theta(X, Fo) = \frac{\theta_s - \theta}{\theta_s} = \frac{Lu Pn}{Lu - 1} \left\{ \sum_{n=1}^{\infty} \left[ A_n \cos \mu_n X - B_n \cos \frac{\mu_n}{\sqrt{Lu}} X \right] \exp(-\mu_n^2 Fo) + \right. \\ \left. + \sum_{m=1}^{\infty} A_m \cos \mu_m X \exp(-\mu_m^2 Lu Fo) \right\}, \quad (4-3-14)$$

where

$$A_n = (-1)^{n+1} \frac{2}{\mu_n}; \quad B_n = \frac{2}{\sqrt{Lu} \mu_n \sin \mu_n / \sqrt{Lu}}; \quad A_m = \frac{2 \tan \sqrt{Lu} \mu_m}{(-1)^m \sqrt{Lu} \mu_m};$$

$$\mu_n = (2n-1)\pi/2 \quad (n=1, 2, 3 \dots)$$

$$\mu_m = m\pi \quad (m=1, 2, 3 \dots)$$

The first six values of the roots  $\mu_n$  and  $\mu_m$  of the characteristic equation, as well as of coefficients  $A_n$ ,  $B_n$ , and  $A_m$ , are given in Table 4-8.

TABLE 4-8

Roots of characteristic equation and constant coefficients for an infinite plate (heat and mass transfer not complicated by phase transitions)

	Parameter		n or m					
			1	2	3	4	5	6
1	$\mu_n$		1.5708	4.7124	7.8540	10.9956	14.1372	17.2788
2	$\mu_m$		3.1416	6.2832	9.4248	12.5664	15.7080	18.8496
3	$A_n$		1.2732	-0.4244	0.2546	-0.1819	0.1415	-0.1157
4	$B_n$	Lu	0.10	-4.1614	1.8624	-2.7785	-2.6771	0.6731
			0.25	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$
			0.50	2.2629	1.6134	-0.3624	1.6382	0.2199
			0.75	1.5152	-0.6572	0.8448	1.6080	-0.2787
			1.00	1.2732	-0.4244	0.2546	-0.1819	0.1415
5	$A_m$	Lu	0.10	-3.0904	-2.2794	0.1093	0.5524	1.5468
			0.25	4681.2	0.0000	-366.818	0.0000	13865
			0.50	1.1828	1.6292	-0.1203	-0.1346	1.6154
			0.75	0.3292	-0.4126	0.7704	1.6182	-0.2486
			1.00	0.0000	0.0000	0.0000	0.0000	0.0000

An analysis of the above solutions shows that, starting from Fo = 0.1 for equation (4-3-13) and starting from Fo = 0.15 for equation (4-3-14), it is sufficient to retain only the first terms of the infinite series during practical calculations.

Let us consider in more detail the solution for the heat-transfer potential. This solution can be applied to many practical problems in which the heat transfer is not complicated by mass transfer or phase transition, and it is also useful in the study of diffusion processes. In the latter case the concentration corresponds to the dimensionless temperature, while the diffusion coefficient corresponds to the thermal diffusivity (which enters into the Fourier number).

If during the transfer process the surface temperature  $t_s$  (or surface concentration) differs from the initial distribution  $t_0$ , then solution (4-3-13) becomes

$$T = \frac{t - t_s}{t_0 - t_s} = \sum_{n=1}^{\infty} A_n \cos \mu_n X \exp(-\mu_n^2 Fo) \quad (4-3-15)$$

For low  $Fo$  it is more convenient to use the other solution (see preceding subsection):

$$T = 1 - \sum_{n=1}^{\infty} (-1)^{n+1} \left\{ \operatorname{erfc} \frac{(2n-1) - X}{2\sqrt{Fo}} + \operatorname{erfc} \frac{(2n-1) + X}{2\sqrt{Fo}} \right\}. \quad (4-3-16)$$

Figure 4-10 shows curves for the dimensionless-temperature distributions at various values of  $Fo$  (from 0.005 to 1.5), obtained using formulas (4-3-15) and (4-3-16). The figure shows that during cooling ( $t_0 > t_s$ ) the temperature at the middle of the plate decreases considerably for  $Fo > 0.06$ . The cooling process is terminated at about  $Fo > 1.5$ . Figure 4-10 can also be used as a nomogram for practical calculations.

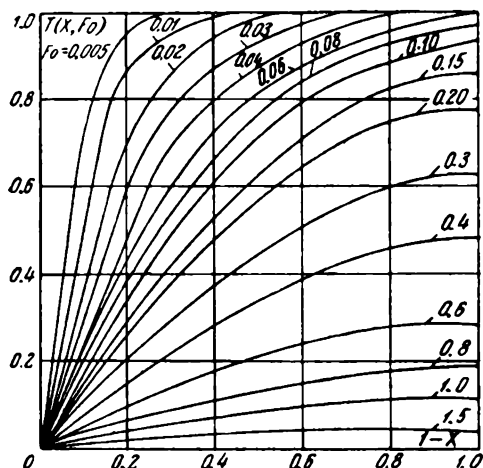


FIGURE 4-10. The dimensionless-temperature distribution in an infinite plate

In order to determine the specific heat and mass flow rates it is necessary to know the average values of the dimensionless transfer potentials; these can be obtained using formula (4-2-25). The average value, over the spatial coordinate, of equation (4-3-15) is

$$\bar{T}(Fo) = \frac{\bar{t} - t_s}{t_0 - t_s} = \sum_{n=1}^{\infty} D_n \exp(-\mu_n^2 Fo),$$

where

$$D_n = A_n \frac{\sin \mu_n}{\mu_n} = \frac{2}{\mu_n^2} = \frac{8}{(2n-1)^2 \pi^2}.$$

For small  $Fo$  this equation becomes

$$\bar{T}(Fo) = 1 - 2\sqrt{\frac{Fo}{\pi}} + 4\sqrt{Fo} \sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{erfc} \frac{n}{\sqrt{Fo}}.$$

Here

$$i \operatorname{erfc} z = \frac{1}{\sqrt{\pi}} e^{-z^2} - z \operatorname{erfc} z.$$

Finally, as an approximation for low  $Fo$ , it is also possible to use the formula

$$\bar{T} = 1 - 2\sqrt{\frac{Fo}{\pi}}.$$

The relation between the average dimensionless temperature and the Fourier number for an infinite plate is shown in Figure 4-11.

For an arbitrary initial transfer-potential distribution  $T(X, 0) = \frac{t - t_*}{t_*} = F(X)$ , solution (4-3-13) becomes

$$T = \frac{t - t_*}{t_*} = -1 + \sum_{n=1}^{\infty} \cos \mu_n X \exp(-\mu_n^2 Fo) 2 \int_0^1 F(X') \cos \mu_n X' dX'.$$

In contrast to (4-3-13) the dimensionless distribution  $T(X, Fo)$  is here the ratio of the dimensional temperature difference  $t - t_*$  to some fixed dimensional temperature  $t_*$ . The choice of  $t_*$  is dictated only by convenience in calculation. The same is true of the function  $F(X)$ , which is the ratio of the initial distribution of the dimensional temperature to the fixed temperature value which is selected:  $F(X) = f(x)/t_*$ .

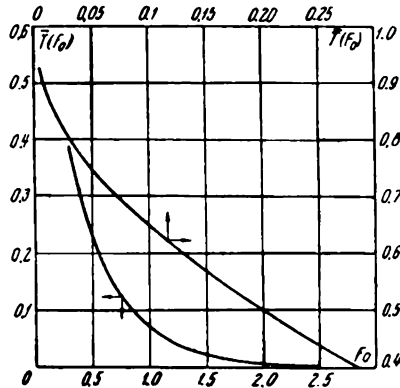


FIGURE 4-11. Relation between average dimensionless temperature and Fourier number for an infinite plate

Let us note in conclusion that problems involving a finite rod with no heat exchange at the surface may be classified with problems involving an infinite plate. In most cases such problems are unsymmetric. For example, for an arbitrary initial transfer-potential distribution with different constant temperatures at the end faces of the rod,  $T(0, Fo) = T_1$  and  $T(l, Fo) = T_2$ , we obtain :

$$T(X, Fo) = T_1 + (T_2 - T_1)X + \sum_{k=1}^{\infty} \frac{2}{\mu_k} [(T_1 + 1) \cos \mu_k - (1 + T_2)] \sin \mu_k X \exp(-\mu_k^2 Fo) + \\ + \sum_{k=1}^{\infty} \sin \mu_k X \exp(-\mu_k^2 Fo) 2 \int_0^1 F(X') \sin \mu_k X' dX',$$

where  $X = \frac{x}{l}$ , and where  $l$  is either the distance between the ends of the rod or else the thickness of the plate.

c) Heat and mass transfer in an infinite cylinder ( $\Gamma = 1$ ;  $Ko^* = 0$ ,  $T(X, 0) = \Theta(X, 0) = 0$ ).

The expressions for the dimensionless transfer potentials are

$$T(X, Fo) = -1 + \sum_{n=1}^{\infty} A_n J_0(\mu_n X) \exp(-\mu_n^2 Fo) \quad (4-3-17)$$

$$\Theta(X, Fo) = \frac{Lu Pn}{Lu - 1} \left\{ \sum_{n=1}^{\infty} \left[ A_n J_0(\mu_n X) - B_n J_0\left(\frac{\mu_n}{\sqrt{Lu}} X\right) \right] \exp(-\mu_n^2 Fo) + \right. \\ \left. + \sum_{m=1}^{\infty} A_m J_0(\mu_m X) \exp(-\mu_m^2 Lu Fo) \right\}. \quad (4-3-18)$$

where  $\mu_n$  and  $\mu_m$  are the roots of the characteristic equations

$$J_0(\mu_n) = 0 \quad \text{and} \quad J_1(\mu_m) = 0, \quad \text{and where} \quad A_n = \frac{2}{\mu_n J_1(\mu_n)},$$

$$B_n = \frac{2}{\sqrt{Lu} \mu_n J_1(\mu_n / \sqrt{Lu})}, \quad A_m = \frac{2 J_1(\sqrt{Lu} \mu_m)}{\sqrt{Lu} \mu_m J_0(\sqrt{Lu} \mu_m) J_0(\mu_m)}.$$

The first six values of the characteristic-equation roots and the coefficients are given in Table 4-9.

TABLE 4-9

Roots of the characteristic equations and constant coefficients for an infinite cylinder  
(heat and mass transfer not complicated by phase transitions)

No.	Parameter			n or m					
				1	2	3	4	5	6
1	$\mu_n$			2.4048	5.5201	8.6537	11.7915	14.9309	18.0711
2	$\mu_m$			3.8317	7.0156	10.1735	13.3237	16.4706	19.6159
3	$A_n$			1.6018	-1.0647	0.8516	-0.7295	0.6487	-0.5896
4	$B_n$	$Lu$	0.10	16.3934	-7.1917	4.8181	-4.4366	5.7921	-18.9933
			0.25	-5.5371	-3.9643	-3.2483	-2.8177	-2.5224	-2.3036
			0.50	6.5789	2.5304	-1.6205	-6.0551	1.0937	-2.9120
			0.75	2.3012	-2.2183	5.8841	3.1461	-1.1577	0.7708
			1.00	1.6022	-1.0647	0.8516	-0.7971	0.6487	-0.5896
5	$A_m$	$Lu$	0.10	-4.0973	16.5659	1.9524	0.8300	-6.2473	-2.0694
			0.25	-5.5226	-0.6678	-3.5574	-0.5216	-2.8525	-0.4419
			0.50	5.5044	2.2796	-0.1983	-0.9558	-6.0026	0.4763
			0.75	0.9162	-1.6737	5.7093	-3.0997	-0.9315	0.3672
			1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

As in the preceding subsection, let us analyze solution (4-3-17) in more detail. If the value of the surface potential differs from its initial distribution, then (4-3-17) becomes

$$T = \frac{t - t_s}{t_o - t_s} = \sum_{n=1}^{\infty} A_n J_0(\mu_n X) \exp(-\mu_n^2 Fo), \quad (4-3-19)$$

for  $Fo > 0.1$ , and the more complex

$$T = \frac{t - t_s}{t_o - t_s} = 1 - \frac{1}{\sqrt{X}} \operatorname{erfc} \frac{1-X}{2\sqrt{Fo}} + \frac{1}{4} \frac{1-X}{X} \cdot \frac{Fo}{\sqrt{X}} \operatorname{erfc} \frac{1-X}{2\sqrt{Fo}} +$$

$$+ \frac{1}{32} \left( \frac{9-2X-7X^2}{X^3} \right) \frac{Fo}{\sqrt{X}} \operatorname{erfc} \frac{1-X}{2\sqrt{Fo}} + \dots \quad (4-3-20)$$

for  $Fo < 0.1$ . For approximate calculations it is sufficient to reduce (4-3-20) to the approximate formula

$$T = 1 - \frac{1}{\sqrt{X}} \operatorname{erfc} \frac{1-X}{2\sqrt{Fo}}.$$

Figure 4-12 shows the distributions, over the dimensionless spatial coordinate, of the dimensionless transfer potential, for various values of  $Fo$  from 0.005 to 0.8.

The transfer potential may be averaged in accordance with formula (4-2-47) to give

$$\bar{T}(Fo) = \frac{\bar{t} - t_s}{t_o - t_s} = \sum_{n=1}^{\infty} D_n \exp(-\mu_n^2 Fo),$$

where

$$D_n = \frac{4}{\mu_n^2}.$$

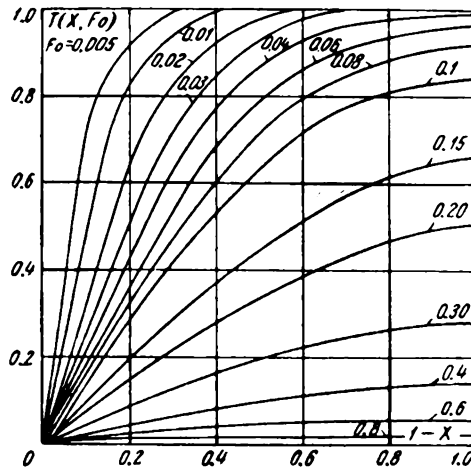


FIGURE 4-12. The dimensionless-temperature distribution in an infinite cylinder

The relation between  $\bar{T}$  and the Fourier number is shown in Figure 4-13, which may serve as a calculation graph.

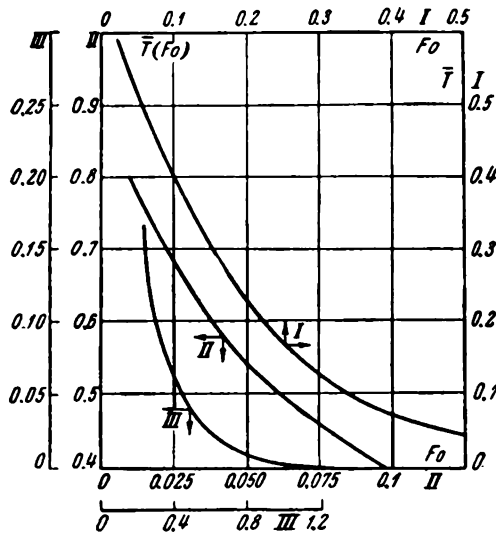


FIGURE 4-13. Relation between the average dimensionless temperature and the Fourier number, for an infinite cylinder

If the initial distribution of the transfer potential is an arbitrary function of the spatial coordinates, then expression (4-3-17) or (4-3-19) becomes

$$T(X, Fo) = \frac{t - t_*}{t_*} = -1 + \sum_{n=1}^{\infty} \frac{2J_0(\mu_n X)}{J_1^2(\mu_n)} \exp(-\mu_n^2 Fo) \int_0^1 X f(X) J_0(\mu_n X) dX.$$

If the infinite cylinder is hollow and  $\kappa = \frac{R_2}{R_1}$  is the ratio of the radii of the external and internal surfaces, whose temperatures are constant and equal respectively to  $t_* = \frac{t_2}{t_*}$  and  $t_* = \frac{t_1}{t_*}$ , and if the initial distribution

is a function of the dimensionless coordinate  $X = \frac{r}{R_1}$ , then the solution of the problem will have the form

$$\begin{aligned} \frac{t}{t_1} = & \frac{\pi^2}{2t_1} \sum_{k=1}^{\infty} \frac{\mu_k^2 J_0^2(\mu_k)}{J_0^2(\mu_k) - J_0^2(x\mu_k)} V_0(\mu_k X) \exp(-\mu_k^2 Fo) \int_1^x X f(X) V_0(\mu_k X) dX - \\ & - \pi \sum_{k=1}^{\infty} \frac{\frac{t_2}{t_1} J_0(\mu_k) - J_0(x\mu_k)}{J_0^2(\mu_k) - J_0^2(x\mu_k)} J_0(\mu_k) V_0(\mu_k X) \exp(-\mu_k^2 Fo) + \\ & + \frac{1}{\ln x} \left[ \ln \frac{x}{X} + \frac{t_2}{t_1} \ln X \right], \end{aligned} \quad (4-3-21)$$

where  $\mu_k$  are the roots of the characteristic equation

$$\begin{aligned} J_0(\mu) Y_0(x\mu) - J_0(x\mu) Y_0(\mu) &= 0; \\ V_0(\mu_k X) &= J_0(\mu_k X) Y_0(x\mu_k) - J_0(x\mu_k) Y_0(\mu_k X). \end{aligned}$$

The first five values of  $\mu_k$  are given in Table 4-10;  $Y_0$  is a zero-order Bessel function of the second kind.

TABLE 4-10

Roots  $\mu_k$  of the characteristic equation  $J_0(\mu) Y_0(x\mu) - Y_0(\mu) J_0(x\mu) = 0$

x	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_5$	$\mu_6$
1.2	15.7014	31.4126	47.1217	62.8302	78.5385	94.2467
1.5	6.2702	12.5598	18.8451	25.1294	31.4133	37.6969
2.0	3.1917	6.3116	9.4446	12.5614	15.7040	18.8462
2.5	2.0732	4.1773	6.2754	8.3717	10.4672	
3.0	1.5485	3.1291	4.7038	6.2767	7.8487	
3.5	1.2339	2.5002	3.7608	5.0196	6.2776	
4.0	1.0244	2.0809	3.1322	4.1816	5.2301	

When the surface temperatures are equal,  $t_2 = t_1 = t_s$ , and the initial temperature distribution is uniform, solution (4-3-21) simplifies to

$$T = \frac{t - t_s}{t_0 - t_s} = \pi \sum_{k=1}^{\infty} \frac{J_0(\mu_k) V_0(\mu_k X)}{J_0(\mu_k) + J_0(x\mu_k)} \exp(-\mu_k^2 Fo)$$

d) Heat and mass transfer in a sphere ( $\Gamma=2$ ;  $Ko^*=0$ ;  $T(X, 0) = \Theta(X, 0)=0$ ). The dimensionless transfer potentials are

$$T(X, Fo) = -1 + \sum_{n=1}^{\infty} A_n \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 Fo) \quad (4-3-22)$$

$$\begin{aligned} \Theta(X, Fo) = & \frac{Lu Pn}{Lu_c - 1} \left\{ \sum_{n=1}^{\infty} \left[ A_n \frac{\sin \mu_n X}{X} - B_n \frac{\sin \frac{\mu_n}{\sqrt{Lu}} X}{X} \right] \exp(-\mu_n^2 Fo) + \right. \\ & \left. + \sum_{m=1}^{\infty} A_m \frac{\sin \mu_m X}{X} \exp(-\mu_m^2 Lu Fo) \right\} \end{aligned} \quad (4-3-23)$$

where  $\mu_n = n\pi$  ( $n=1, 2, 3, \dots$ ) and where  $\mu_m$  are the roots of the characteristic equation

$$\tan \mu_m = \mu_m;$$

and where

$$\begin{aligned} A_n &= (-1)^{n+1} \frac{2}{\mu_n}; \quad B_n = \frac{2}{Lu \left( \sin \frac{\mu_n}{\sqrt{Lu}} - \frac{\mu_n}{\sqrt{Lu}} \cos \frac{\mu_n}{\sqrt{Lu}} \right)}; \\ A_m &= \frac{2(\sin \sqrt{Lu} \mu_m - \sqrt{Lu} \mu_m \cos \sqrt{Lu} \mu_m)}{Lu \mu_m^2 \sin \sqrt{Lu} \mu_m}. \end{aligned}$$



The first six nonzero values of the characteristic-equation roots and the coefficients are given in Table 4-11. For  $Fo > 0.15$  the infinite series which enter into (4-3-22) and (4-3-23) converge quite rapidly, so that during calculation only the first terms of the expansion need be retained. For small values of the dimensionless time ( $Fo < 0.15$ ) several terms of the series must be taken.

TABLE 4-11

Roots of the characteristic equation and constant coefficients for a sphere (heat and mass transfer not complicated by phase transitions)

No.	Parameter			n or m					
				1	2	3	4	5	6
1	$\mu_n$			3.1416	6.2832	9.4248	12.5664	15.7080	18.8496
2	$\mu_m$			4.4934	7.7253	10.9041	14.0662	17.2208	20.3714
3	$A_n$			0.6366	-0.3183	0.2122	-0.1591	0.1273	-0.1061
4	$B_n$	Lu	0.10	2.4450	-2.1000	114.5475	1.0567	-0.4775	0.3383
			0.25	-1.2732	-0.6366	-0.4244	-0.3183	-0.2546	-0.2124
			0.50	18.3318	0.4918	-0.4470	-0.4308	0.1866	-4.0841
			0.75	0.9740	-0.8174	11.7855	0.4290	0.1830	0.1251
			1.00	0.6366	-0.3183	0.2122	-0.1591	0.1273	-0.1061
5	$A_m$	Lu	0.10	-0.2520	0.4175	0.5288	-0.0066	-0.1260	-0.4543
			0.25	-0.5687	-0.2293	-0.2019	-0.1328	-0.1233	-0.0936
			0.50	12.8411	0.2911	0.0028	-0.2339	-0.2928	0.0337
			0.75	0.4304	-0.5665	10.0866	0.3633	-0.1220	0.0429
			1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

If the distribution  $t(1, Fo) = t_s$  of the surface potential differs from the initial potential distribution  $t(X, 0) = t_0$ , then solution (4-3-22) becomes

$$T = \frac{t - t_s}{t_0 - t_s} = \sum_{n=1}^{\infty} A_n \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 Fo). \quad (4-3-24)$$

Solution (4-3-24) is most useful for large values of  $Fo$ . For small values of  $Fo$  the solution has the form

$$T = \frac{t - t_s}{t_0 - t_s} = 1 - \frac{1}{\sqrt{X}} \sum_{n=1}^{\infty} \left( \operatorname{erfc} \frac{(2n-1) - X}{2\sqrt{Fo}} - \operatorname{erfc} \frac{(2n-1) + X}{2\sqrt{Fo}} \right). \quad (4-3-25)$$

Solution (4-3-25) is valid for  $X > 0$ ; when  $X = 0$  we have

$$\frac{t(0, Fo) - t_s}{t_0 - t_s} = 1 - 2\sqrt{\frac{1}{\pi Fo}} \sum_{n=1}^{\infty} \exp\left[-\frac{(2n-1)^2}{4Fo}\right]. \quad (4-3-25')$$

Figure 4-14 shows the distribution of dimensionless potential over the coordinate  $X$  for various values of  $Fo$ , calculated using formulas (4-3-24), (4-3-25), and (4-3-25'). It is evident from Figure 4-14 that for  $Fo = 0.005$  the dimensionless temperatures at depths of  $X \leq 0.6$  differ only slightly from the initial temperature. Only for  $Fo > 0.04$  is the variation in potential at the center of the sphere appreciable. The cooling (heating) process is almost terminated for  $Fo \geq 0.4$ .

The average value of the potential is found using formula (4-2-72). For  $Fo > 0.1$  the most convenient solution is

$$\bar{T}(Fo) = \frac{t - t_s}{t_0 - t_s} = \sum_{n=1}^{\infty} D_n \exp(-\mu_n^2 Fo),$$

where  $D_n = 6/\mu_n^2$ . For  $Fo < 0.1$  the best solution is

$$\bar{T}(Fo) = 1 + 3Fo - 6\sqrt{Fo} \left[ \frac{1}{\sqrt{\pi}} + 2 \sum_{n=1}^{\infty} i \operatorname{erfc} \frac{n}{\sqrt{Fo}} \right].$$

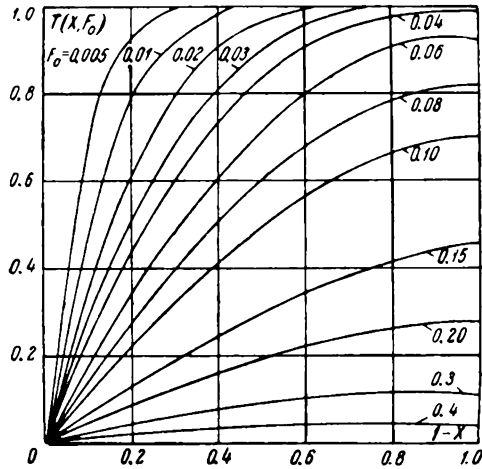


FIGURE 4-14. The dimensionless- temperature distribution in a sphere

The functions  $T=T(Fo)$  corresponding to the last two formulas are plotted in Figure 4-15.

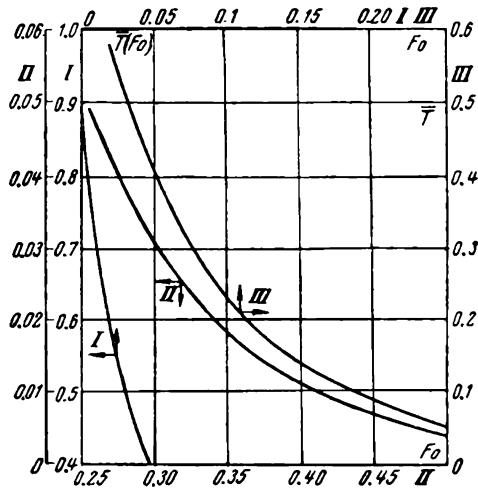


FIGURE 4-15. Average dimensionless temperature as a function of Fourier number, for a sphere

If the initial distribution of the transfer potential is a function of the spatial coordinates, then expressions (4-3-22) and (4-3-24) become

$$T = \frac{t - t_*}{t_*} = -1 + \sum_{n=1}^{\infty} \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 Fo) 2 \int_0^1 X f(X) \sin \mu_n X dX.$$

Finally, if we have a hollow sphere with inner and outer radii of  $R_1$  and  $R_2$ , then for a constant initial potential distribution  $t_0$  we obtain the following solutions:

1. At the beginning of the process the dimensionless potentials at the inner and outer surfaces are respectively

$$T_1 = \frac{t_1 - t_0}{t_0} = \text{constant} \quad \text{and} \quad T_2 = \frac{t_2 - t_0}{t_0} = \text{constant},$$

the characteristic dimension being the inner radius  $R_1$ . The potential distribution is thus

$$T = \frac{t - t_0}{t_0} = \frac{(x - X)T_1 + x(X - 1)T_2}{(x - 1)X} - \sum_{n=1}^{\infty} A_n \frac{1}{X} \left( T_1 \sin \frac{x - X}{x - 1} \mu_n + x T_2 \sin \frac{X - 1}{x - 1} \mu_n \right) \exp \left[ -\mu_n^2 \frac{1}{(x - 1)^2} Fo \right],$$

where  $x = \frac{R_2}{R_1}$ . If in this solution we set  $R_1 = 0$  and  $R = R_2$ , then we arrive at solution (4-3-24) for a solid sphere.

2. The inner surface is heated up to  $T_1$  and maintained constant, while the external surface is insulated  $\frac{\partial T(x, Fo)}{\partial X} = 0$ . The potential distribution is

$$T = \frac{t - t_0}{t_0} = T_1 \left\{ 1 - \sum_{k=1}^{\infty} \frac{A_k}{X} \left( x \cos \frac{x - X}{x - 1} \mu_k - \frac{x - 1}{\mu_k} \sin \frac{x - X}{x - 1} \mu_k \right) \exp \left[ -\mu_k^2 \frac{1}{(x - 1)^2} Fo \right] \right\},$$

where  $\mu_k$  are the roots of the characteristic equation

$$\tan \mu = -\frac{\mu}{1/x - 1};$$

$$A_k = (-1)^{k+1} \frac{2x \sqrt{\mu_k^2 + (1 - 1/x)^2}}{\mu_k^2 + (1/x)^2 - 1/x}.$$

The values of  $\mu_k$  and  $A_k$  are given in Tables 6-46 and 6-47. In using these tables and graphs it is necessary to substitute  $1/x$  for the Biot number  $Bi_0$ .

#### 4-4. Heat and Mass Transfer for a Constant Potential $\theta$

The transfer-potential distributions in the material, as well as the time variations of these in the absence of mass exchange between a body [system] and its surroundings, are determined in general (see § 4-2) by the Fourier number  $Fo$ , by the dimensionless spatial coordinate  $X$ , and by the parameters  $v_1$  and  $v_2$ . In addition, the distribution of mass-transfer potential is a function of the modified Kossovich number  $Ko^*$ , which characterizes the heat consumed in phase transition inside the body, in the heat units which went to its heating. In other words,  $Ko^*$  indicates the effect of phase transitions on mass transfer.

The parameters  $v_i$  ( $i=1, 2$ ) must now be considered in more detail, since with boundary conditions of the first kind these parameters determine directly the close coupling between the phenomena of heat and mass transfer. Equation (4-2-7) shows that  $v_i$  are functions of  $Lu$  and of the compound dimensionless number  $Fe = Ko^* Pn$ . Both of these similarity criteria characterize the coupling between the transfer phenomena. The dimensionless number  $Lu$  represents coupling between the inertial properties of the fields of mass-transfer and heat-transfer (temperature) potential. The compound number  $Fe$  contains in it the Posnov number  $Pn$ , the presence of which is a result of the superposition of thermal processes onto diffusion processes. These considerations thus require that parameters  $v_i$  be interpreted as coupling parameters. The same result, as will be shown in Chapter V, follows from considerations related to the possibility of replacing the system of coupled equations for transfer by two noncoupled heat-conduction-type equations, in terms of the combined "potential"  $Z_i = \rho_i T + q_i \theta$ .

The parameter  $v_1$  (see Table 4-1) varies only slightly with  $Lu$  and  $Fe$ ; as each of the latter numbers increases, it decreases. For  $Lu = 0$ , we have  $v_1$  equal to unity, while for  $Lu > 1.0$  the parameter approaches zero. As  $Fe$  increases, the parameter  $v_2$  increases slightly, whereas as  $Lu$  increases, this parameter decreases approximately logarithmically.

Mass transfer affects heat transfer by means of phase transition, as expressed by the Fedorov number (or rather by  $Ko^*$ ). This follows from the fact that the absence of a phase-transition term in heat-conduction equation (4-1-2) leads to a solution of the system of equations which is characteristic of "pure" heat conduction (see § 4-3), whereas the distribution of the mass-transfer potentials is still a function of the temperature fields, as expressed through  $Lu$  and  $Pn$ .

In the absence of phase transitions in the material the temperature field is similar to the temperature field of a "dry" substance. The dimensionless heat-transfer potential is a function of the Fourier number and the spatial coordinate:

$$T(X, Fo) = f(Fo, X). \quad (4-4-1)$$

Since the characteristic-equation roots increase with an increase in  $n$ , for  $Fo > Fo_1$  (that is, in the normal-condition range), all the terms of the series except the first may be neglected. In this case relation (4-4-1) may be written as

$$T = \frac{t - t_c}{t_0 - t_c} = A_1 \Phi(\mu_1, X) \exp(-\mu_1^2 Fo).$$

Here  $A_1$  is the initial thermal amplitude, equal to  $4/\pi$  for an infinite plate, to  $2/2.405 J_1$  (2.405) for an infinite cylinder, and to  $2/\pi$  for a sphere;  $\Phi(\mu_1, X)$  is a function describing the temperature distribution (it is  $\cos \frac{\pi}{2} X$  for an infinite plate,  $J_0(2.405 X)$  for an infinite cylinder, and  $\sin \pi X$  for a sphere); and  $\mu_1$  is the first root of the corresponding characteristic equation.

For steady-state conditions (that is, when  $Fo \rightarrow \infty$ ) the dimensionless temperature for a plate, cylinder, or sphere is zero, while for an infinite rod it is some linear function of  $X$ .

Under these conditions, namely in the absence of phase transition in the material, the field of the dimensionless mass-transfer potential is a function not only of  $Fo$  and  $X$  but also of  $Lu$  and  $Pn$ .

The analytical solutions for  $\Theta(X, Fo)$  constitute a sum of two series, the first of which represents the modified temperature field and the second of which represents the modified field of mass-transfer potential.

As the dimensionless time increases, the potential  $\Theta$  approaches zero for an infinite plate and cylinder and for a sphere, whereas for an infinite rod

$$\Theta(X, \infty) = Pn(1/2 - X).$$

Sizyakova's calculations, based on the formulas in § 4-3a for the transfer-potential fields in a moist finite rod, have shown (Figure 4-16) that under the effect of a variable temperature field a damped wave of mass-transfer (or mass-content) potential is established in the material. For small values of  $Lu$  the wave becomes attenuated very rapidly. With increasing  $Lu$  it penetrates more deeply into the material,

propagates more quickly, and the absolute value of its maximum increases (in contrast to equation (4-3-14), here  $\theta$  has the opposite

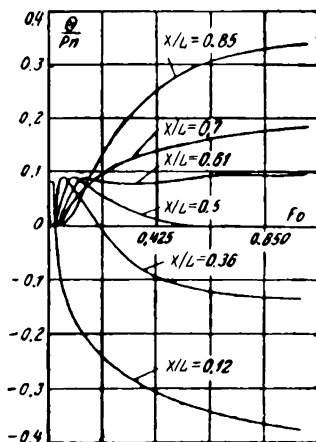


FIGURE 4-16.  $\Theta(X, Fo)/Pn$  as a function of the Fourier number and the dimensionless coordinates

sign:  $\theta_1 = \frac{\theta - \theta_0}{\theta_s - \theta_0}$ , in order to facilitate a comparison between the calculations and the experimental results).

The rate of variation of mass-transfer potential at each point increases greatly as  $Lu$  becomes larger, and a linear steady-state distribution of this potential is established much more rapidly. For example, for  $Lu = 0.8$  and  $Fo = 0.5$  the distribution of dimensionless mass-transfer potential in a finite rod may already be considered steady. The local temperature (Figure 4-17) increases rapidly at first, and then gradually approaches the constant level corresponding to the linear steady-state temperature distribution for the specimen. A comparison of the curves for the variations of the dimensionless transfer potentials at identical moments shows that the most intensive variation in mass-transfer potential occurs during the period of most intensive variation in heat-transfer potential. For small  $Lu$  the variations in  $\theta_1$

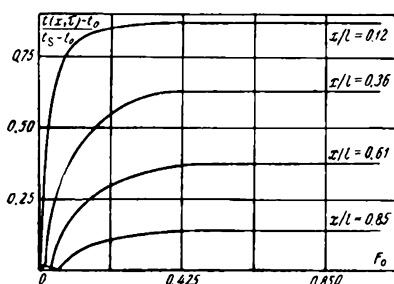


FIGURE 4-17.  $T = \frac{t(x, \tau) - t_0}{t_s - t_0}$  as a function of the Fourier number

take place considerably more gradually than the variations in  $T$ . A steady-state temperature distribution is established much earlier than a steady distribution of mass-transfer potential. Calculations also show that the curves for the distribution of  $\theta_1$  over time and space have maxima. The value of the maximum for a given value of  $Lu$  is the same for all coordinate values, but the maximum increases with  $Lu$ .

Experimental checks, using quartz sand, of the analytical solutions for a finite rod ( $Ko^* = 0$ ) were carried out by Sizyakova and also by Oleinikov and Kazanskii. These experiments confirmed the correctness of the mathematical model of heat and mass transfer. At the same time, a comparison of the experimental curves for the variations in mass content and temperature (Figures 4-18 and 4-19) with the analytical curves (Figures 4-16 and 4-17) shows that the experimental maxima are not as sharp as the calculated ones, that they are reached much later, and that the mass-content wave penetrated more deeply into the material. The experimental temperature increase takes place more smoothly, and the steady-state distribution is established much later, than those predicted by the analytical curves. All these factors indicate the presence of a considerable amount of phase transition (vaporization) in the material. The maximum mass content is attained much later, since in a moist body part of the moisture which has vaporized goes away into more remote layers. Therefore, the mass-content wave also

penetrates deeper into the material. Due to the vaporization the rate of temperature rise under real conditions is lower than that in the analytical curves. Phase transition in the material is entirely possible, since the [experimental] heater temperature was sufficiently high ( $t_c = 72$  to  $163^\circ\text{C}$ ). The presence of phase transition was demonstrated directly by Gamayunov, using the radioactive-tracer method.

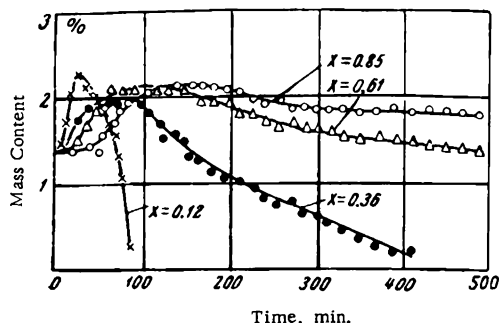


FIGURE 4-18. Experimental curves for the time variation of the mass content

In a closed insulated system in which there is mass transfer, moisture circulation occurs. Moisture in the form of liquid is brought to the hotter part of the system, where it evaporates and moves as vapor into the interior. Then, after condensing in the colder regions, the moisture is again transported as liquid to the hot end. The transfer of moisture in the form

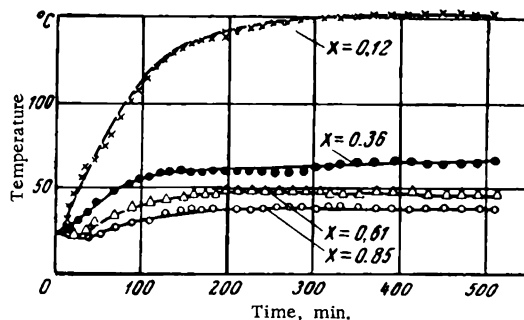


FIGURE 4-19. Experimental curves for the temperature variation

of vapor has been observed in many experiments carried out in a temperature range from  $30$  to  $60^\circ\text{C}$ . Consequently, experimental studies lead us to conclude that the process must be represented mathematically by the complete system of differential heat-transfer and mass-transfer equations (4-1-2) and (4-1-3).

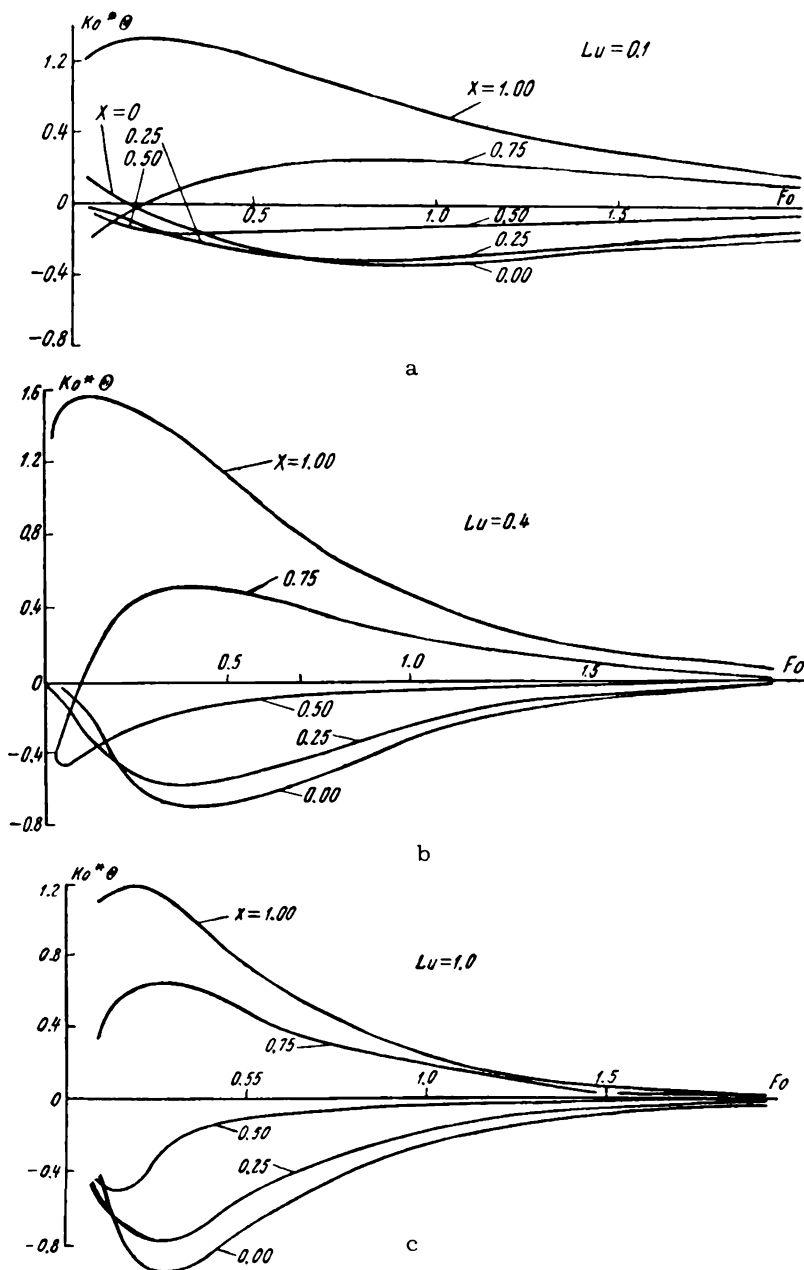


FIGURE 4-20. Time development of the  $Ko \cdot \Theta$  fields for various values of  $Lu$ , for a symmetrical infinite plate  
a)  $Lu = 0.1$ ; b)  $Lu = 0.4$ ; c)  $Lu = 1.0$

During the heating of a finite rod (or, which is the same thing, an infinite plate whose surfaces have different temperatures) extrema of the same sign are observed, the absolute values of which are practically the same. However, during the cooling of a plate with equal surface temperatures (§§ 4-2, a and 4-3, b), the nature of the transfer-potential distribution is somewhat different. Along with minima of the mass-transfer potential maxima are also observed, and the magnitudes of the extrema at various points of the body are different.

As  $Lu$  increases (Figure 4-20), the extrema of dimensionless mass-transfer potential become more pronounced, their amplitudes increase, and the moments of their formation are shifted toward smaller  $Fo$ . Also, the development of the fields of mass-transfer potential becomes more intensive and the steady state is established more rapidly. For small values of the Fourier number, the potential  $\Theta$  increases with  $Lu$  (Figure 4-21), but for  $Fo \sim 0.4$  an inflection point begins to develop on the curves. Then, for higher  $Fo$ , the inflection point becomes displaced toward lower  $Lu$ . After Fourier numbers of 1.2 to 1.3 this point reaches the ordinate axis, as a result of which a new profile of the mass-transfer-potential curve is formed:  $\Theta$  decreases with an increase in  $Lu$ , but the main variation in potential occurs up to  $Lu = 1.0$ . For  $Lu > 1.0$ , the potential  $\Theta$  becomes self-similar with respect to  $Lu$ .

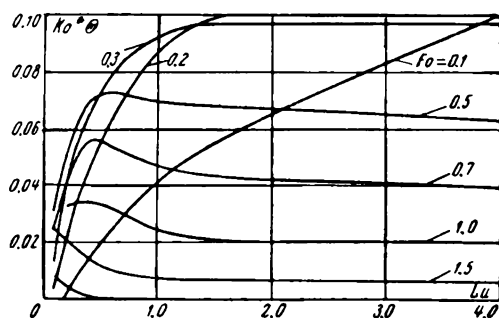


FIGURE 4-21.  $Ko * \Theta$  as a function of  $Lu$

The temperature field is practically independent of  $Lu$ . A slight decrease in  $T$  is observed only for small values of this dimensionless number. Certain individual features of the temperature and mass fields have led to the suggestion that unsteady-state methods be used to determine the transfer coefficients.



## Chapter V

### UNSTEADY FIELDS OF HEAT-TRANSFER AND MASS-TRANSFER POTENTIAL WITH BOUNDARY CONDITIONS OF THE SECOND KIND

#### 5-1. The Dimensionless Boundary Conditions

In many high-temperature processes the specific heat flux and mass flux at the surface of a body are complicated functions of the transfer potentials. For example, when a body is irradiated the thermal flux is proportional to the difference between the fourth powers of the absolute temperatures of the surfaces involved in the heat exchange. Complete solutions of the system of differential heat-transfer and mass-transfer equations are in this case very difficult to obtain. However, such solutions may be obtained more easily, provided that appropriately selected functions in which the fluxes depend only on time are specified in the boundary conditions. The boundary conditions are then of the second kind.

In this case the dimensionless boundary conditions for a one-dimensional body may be written as

$$-\frac{\partial T(l, Fo)}{\partial X} + Ki_q(Fo) = 0; \quad (5-1-1)$$

$$-\frac{\partial \theta(l, Fo)}{\partial X} + Pn \frac{\partial T(l, Fo)}{\partial X} + Ki_m(Fo) = 0, \quad (5-1-2)$$

where  $Ki_q = \frac{j_q(\tau)R}{\lambda_q \tau_0}$  and  $Ki_m = \frac{j_m(\tau)R}{\lambda_m \theta_0}$  are respectively the Kirpichev numbers for heat exchange and mass exchange. The initial conditions for the given system of equations may be written as

$$T(X, 0) = \frac{f_1(x) - t_*}{t_*} = F_1(X) \quad \text{and} \quad \theta(X, 0) = \frac{\theta_* - f_2(x)}{\theta_*} = F_2(X). \quad (5-1-3)$$

If, in addition to the heat source, there is also a heat sink at the surface of the body, as is observed during the dynamic cooling of various objects which are being irradiated, or during the radiation drying of moist dispersed media, then condition (5-1-1) should be generalized. For example, for the above situations this condition takes the form

$$-\frac{\partial T(l, Fo)}{\partial X} + Ki_q(Fo) - (1 - \epsilon) Lu Ko Ki_m(Fo) = 0. \quad (5-1-4)$$

Solutions of the system of heat-transfer and mass-transfer equations with boundary conditions (5-1-4) and (5-1-2) will be considered in § 5-3. In these problems the modified Kossovich number  $Ko^*$  will be introduced as the product of the phase-transition ratio ( $\epsilon$ ) times the [ordinary] Kossovich number ( $Ko$ ) used in studies of drying processes. Such a definition of  $Ko^*$

is very convenient, in view of the difference between the phase-transition rates at the surface of the body and in its interior.

Particular solutions of the system of heat-transfer and mass-transfer equations for boundary conditions of the second kind have been obtained by Gamayunov, Koslova, Lykov, Mikhailov, Plyat, and Prudnikov. Solutions of the system of equations in the absence of thermal-gradient transfer have been given by Smirnov. Various cases of uncoupled transfer have been considered by many Soviet and other authors. Some problems of this type will be considered in § 5-5.

## 5-2. Basic Problems of Coupled Heat and Mass Transfer

In this section we will consider the solutions of system of equations (4-1-2) and (4-1-3) for an infinite plate, an infinite cylinder, and a sphere, with boundary conditions (5-1-1) and (5-1-2), symmetry conditions (4-1-5), and initial conditions (5-1-3).

a) Solution for an infinite plate ( $\Gamma=0$ ). To solve this problem let us use the method of combined Fourier and Laplace integral transformations.

First, with respect to variable  $X$  the finite Fourier cosine transformation

$$\{\varphi(\mu, Fo)\}_c = \int_0^1 \varphi(X, Fo) \cos \mu X dX, \quad (5-2-1)$$

is applied, where  $\mu = n\pi$  ( $n=0, 1, 2, \dots$ ) [and where the subscript  $c$  denotes the cosine transformation]. The second-order derivatives may be transformed as follows:

$$\int_0^1 \frac{\partial^2 \varphi(X, Fo)}{\partial X^2} \cos \mu X dX = (-1)^n \frac{\partial \varphi(1, Fo)}{\partial X} - \frac{\partial \varphi(0, Fo)}{\partial X} - \mu^2 \{\varphi(\mu, Fo)\}_c. \quad (5-2-2)$$

The inversion from the transform  $\{\varphi(\mu, Fo)\}_c$  to the original function is carried out using the formula

$$\varphi(X, Fo) = \{\varphi(0, Fo)\}_c + 2 \sum_{n=1}^{\infty} \{\varphi(\mu_n, Fo)\}_c \cos \mu_n X. \quad (5-2-3)$$

Thus, the application of a Fourier cosine transformation, with boundary conditions (5-1-1) and (5-1-2) and symmetry conditions (4-1-5), to the second derivatives for the case at hand gives

$$\int_0^1 \frac{\partial^2 T}{\partial X^2} \cos \mu X dX = (-1)^n Ki_q(Fo) - \mu^2 T_c; \quad (5-2-4)$$

$$\int_0^1 \frac{\partial^2 \theta}{\partial X^2} \cos \mu X dX = (-1)^n [Pn Ki_q(Fo) + Ki_m(Fo)] - \mu^2 \theta_c. \quad (5-2-5)$$

It will be assumed here and in the following that the Fourier-transformation operator and the differentiation operator  $\partial/\partial Fo$  are commutative. Then it is possible, after multiplying all the terms in equations (4-1-2) and

(4-1-3) by  $\cos \mu_n X$ , integrating with respect to  $X$  from 0 to 1, and taking into account (5-2-4) and (5-2-5), to transform the original system of partial differential equations into a system of two ordinary differential equations:

$$\left. \begin{aligned} \frac{dT_e}{dF_0} &= (-1)^n K_{iq}(F_0) - \mu^2 T_e - K_0 \frac{d\theta_e}{dF_0}; \\ \frac{d\theta_e}{dF_0} &= (-1)^n Lu K_{im}(F_0) - Lu \mu^2 \theta_e + Lu P_n \mu^2 T_e. \end{aligned} \right\} \quad (5-2-6)$$

Here initial conditions (5-1-3) are transformed, after application of transformation (5-2-1), into the conditions

$$T_{ec} = T_e(\mu, 0) = \int_0^1 F_1(X) \cos \mu X dX, \quad \theta_{ec} = \theta_e(\mu, 0) = \int_0^1 F_2(X) \cos \mu X dX. \quad (5-2-7)$$

Let us now apply the Laplace integral transformation

$$\{\varphi(X, s)\}_L = \int_0^\infty \varphi(X, F_0) \exp(-s F_0) dF_0. \quad (5-2-8)$$

Then, if we take (5-2-7) into account, equations (5-2-6) become the following system of algebraic equations:

$$\left. \begin{aligned} (s + \mu^2) T_{eL} + s K_0 \theta_{eL} &= (-1)^n \{K_{iq}\}_L + T_{ec} + K_0 \theta_{ec}; \\ -Lu P_n \mu^2 T_{eL} + (s + Lu \mu^2) \theta_{eL} &= (-1)^n Lu \{K_{im}\}_L + \theta_{ec}, \end{aligned} \right\}$$

where

$$\{K_{ik}\}_L = \int_0^\infty K_{ik}(F_0) \exp(-s F_0) dF_0 \quad \text{and} \quad k = \begin{cases} 1 & \text{or } q \\ 2 & \text{or } m \end{cases}.$$

Now, the latter system may be solved for  $T_{eL}(\mu, s)$  and  $\theta_{eL}(\mu, s)$ , and when we invert with respect to the parameter  $s$  to find the inverse transforms of these potentials we obtain

$$T_e(\mu, F_0) = \sum_{k, i=1}^2 [A_{ki}^q \{P_{ki}\}_0 + B_{ki}^q \{Q_{ki}\}_0]; \quad (5-2-9)$$

$$\theta_e(\mu, F_0) = \sum_{k, i=1}^2 [A_{ki}^m \{P_{ki}\}_0 + B_{ki}^m \{Q_{ki}\}_0], \quad (5-2-10)$$

where

$$\left. \begin{aligned} A_{1i}^q &= (-1)^i \frac{1 - v_i^2}{v_1^2 - v_2^2}; & A_{1i}^m &= B_{1i}^m = (-1)^i \frac{P_n}{v_1^2 - v_2^2}; \\ A_{2i}^q &= (-1)^i \frac{K_0}{v_1^2 - v_2^2}; & A_{2i}^m &= (-1)^i \frac{1/Lu - v_i^2 + K_0 P_n}{v_1^2 - v_2^2}; \\ B_{1i}^q &= (-1)^i \frac{1 - v_i^2}{v_1^2 - v_2^2}; & B_{2i}^m &= (-1)^i \frac{Lu(1/Lu - v_i^2)}{v_1^2 - v_2^2}; \\ B_{2i}^q &= (-1)^i \frac{K_0 Lu v_i^2}{v_1^2 - v_2^2}; \end{aligned} \right\} \quad (5-2-11)$$

TABLE 5 - 1  
Values of coefficients  $A_{1i}^q$ ,  $A_{2i}^m$ , and  $B_{2i}^m$

Lu	t	Ko*Pn = 0.1			Ko*Pn = 0.4			Ko*Pn = 0.7			Ko*Pn = 1.0		
		$A_{1i}^q$	$A_{2i}^m$	$B_{2i}^m$	$A_{1i}^q$	$A_{2i}^m$	$B_{2i}^m$	$A_{1i}^q$	$A_{2i}^m$	$B_{2i}^m$	$A_{1i}^q$	$A_{2i}^m$	$B_{2i}^m$
0.20	1	0.0058	0.9942	0.1940	0.0195	0.9806	0.1786	0.0289	0.9711	0.1662	0.0358	0.9643	0.1557
	2	0.9943	0.0058	0.0060	0.9803	0.0194	0.0213	0.9708	0.0289	0.0339	0.9642	0.0357	0.0443
0.40	1	0.0350	0.9650	0.3618	0.0838	0.9162	0.2964	0.1021	0.8980	0.2579	0.1096	0.8905	0.2313
	2	0.9649	0.0350	0.0373	0.9162	0.0838	0.1036	0.8980	0.1020	0.1421	0.8905	0.1095	0.1687
0.60	1	0.1143	0.8858	0.4711	0.1777	0.8223	0.3485	0.1837	0.8163	0.2953	0.1799	0.8201	0.2412
	2	0.8857	0.1142	0.1289	0.8223	0.1777	0.2517	0.8162	0.1838	0.3046	0.8200	0.1799	0.3385
0.80	1	0.2579	0.7421	0.4830	0.2715	0.7285	0.3578	0.2531	0.7469	0.3065	0.2350	0.7650	0.2728
	2	0.7421	0.2533	0.3170	0.7285	0.2714	0.4422	0.7468	0.2531	0.4935	0.7650	0.2350	0.5272
1.00	1	0.4220	0.5782	0.4220	0.3493	0.6508	0.3493	0.3070	0.6935	0.3070	0.2764	0.7236	0.2764
	2	0.5780	0.4218	0.5780	0.6508	0.3492	0.6507	0.6930	0.3070	0.6930	0.7237	0.2764	0.7236

$$\left. \begin{aligned} \{P_{1i}\}_c &= T_{0c} \exp [-\mu^2 v_i^2 \text{Lu Fo}]; \\ \{P_{2i}\}_c &= \theta_{0c} \exp [-\mu^2 v_i^2 \text{Lu Fo}]; \\ \{Q_{1i}\}_c &= (-1)^n \int_0^{\text{Fo}} K i_q (\text{Fo}^*) \exp [-\mu^2 v_i^2 \text{Lu} (\text{Fo}-\text{Fo}^*)] d\text{Fo}^*; \\ \{Q_{2i}\}_c &= (-1)^n \int_0^{\text{Fo}} K i_m (\text{Fo}^*) \exp [-\mu^2 v_i^2 \text{Lu} (\text{Fo}-\text{Fo}^*)] d\text{Fo}^*; \end{aligned} \right\} \quad (5-2-12)$$

$$v_i^2 = \frac{1}{2} \left[ \left( 1 + K o^* P n + \frac{1}{\text{Lu}} \right) + (-1)^i \sqrt{\left( 1 + K o^* P n + \frac{1}{\text{Lu}} \right)^2 - \frac{4}{\text{Lu}}} \right], \quad (5-2-13)$$

$$(i=1, 2), \left( k = \begin{matrix} 1 & \text{or} & q \\ 2 & \text{or} & m \end{matrix} \right).$$

From these relations it is evident that the subscripts  $ki$  of the constant coefficients  $A$  and  $B$  indicate that these coefficients pertain to the corresponding quantities  $P$  and  $Q$ .

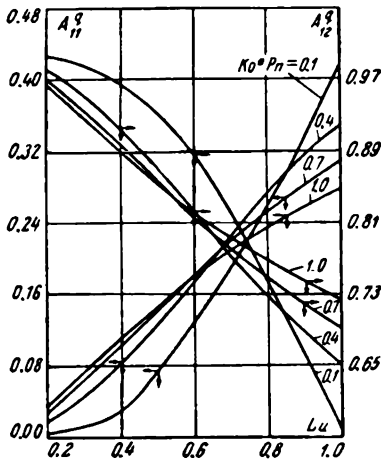


FIGURE 5-1. Coefficients  $A_{1i}^q$  as functions of  $\text{Lu}$ , for various values of  $K o^* P n$

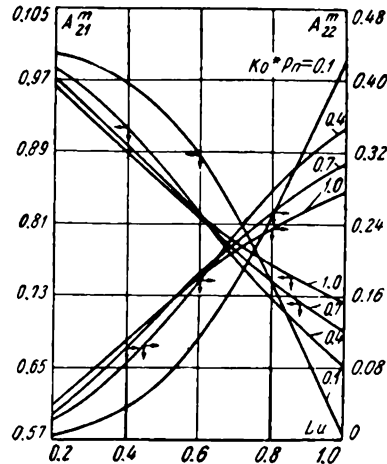


FIGURE 5-2. Coefficients  $A_{2i}^m$  as functions of  $\text{Lu}$ , for various values of  $K o^* P n$

Coefficients  $A_{ki}$  and  $B_{ki}$  as functions of  $K o^* P n$  and  $\text{Lu}$  are listed in Tables 5-1 and 5-2 and plotted in Figures 5-1 through 5-6. For convenience in calculation coefficients  $A_{2i}^q$  and  $B_{2i}^m$  are referred to the Posnov number:

$$A_{2i}^q = (-1)^i \frac{K o^* P n}{v_1^2 - v_2^2} \cdot \frac{1}{P n},$$

$$B_{2i}^m = (-1)^i \frac{K o^* P n \text{Lu} v_i^2}{v_1^2 - v_2^2} \cdot \frac{1}{P n}.$$

The values of these coefficients for various values of  $K o^*$  can easily be determined provided we take into account the single-valued correspondence between  $P n$  and  $K o^*$  for a given value of  $\text{Fe} = K o^* P n$ . The relation between these dimensionless numbers is shown by the nomogram in Figure 5-7.

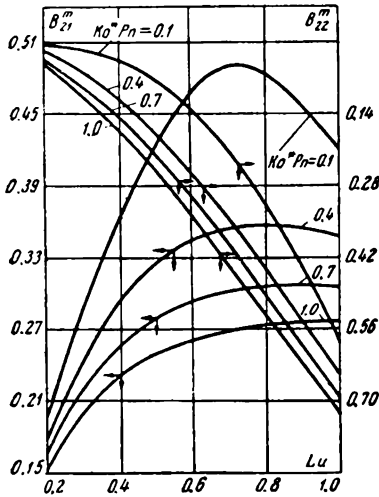


FIGURE 5-3. Coefficients  $B_{2i}^m$  as functions of  $Lu$ , for various values of  $Ko \cdot Pn$

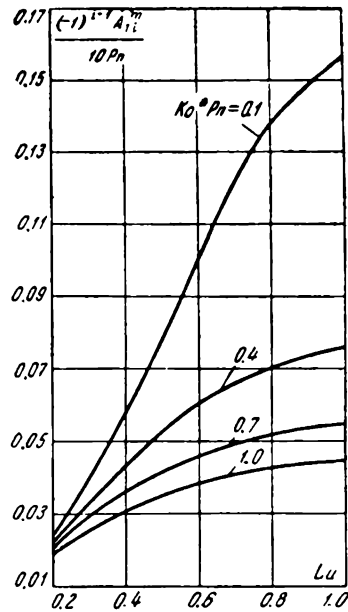


FIGURE 5-4. Coefficient  $\frac{1}{Pn} A_{1i}^m$  as a function of  $Lu$ , for various values of  $Ko \cdot Pn$

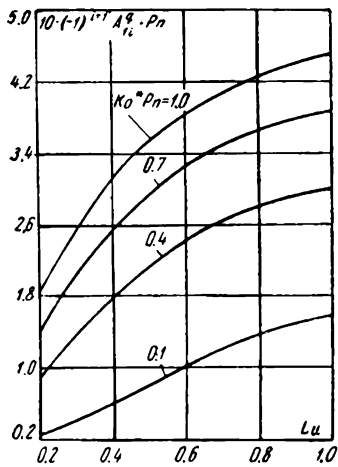


FIGURE 5-5. Coefficients  $Pn A_{1i}^q$  as functions of  $Lu$ , for various values of  $Ko \cdot Pn$

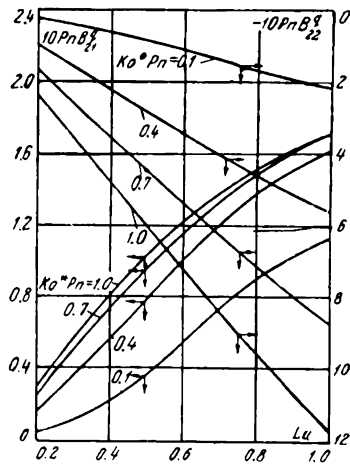


FIGURE 5-6. Coefficients  $Pn Bi_{2i}^q$  as functions of  $Lu$ , for various values of  $Ko \cdot Pn$

TABLE 5 - 2

Values of coefficients  $A_{II}^m$ ,  $A_{II}^q$ , and  $B_{II}^q$ .

Lu	Pn	i	Ko*Pn = 0.1			Ko*Pn = 0.4			Ko*Pn = 0.7			Ko*Pn = 1.0		
			$A_{II}^m$	$A_{II}^q$	$B_{II}^q$	$A_{II}^m$	$A_{II}^q$	$B_{II}^q$	$A_{II}^m$	$A_{II}^q$	$B_{II}^q$	$A_{II}^m$	$A_{II}^q$	$B_{II}^q$
0.2	0.1	1	0.0241	0.2412	0.0470	0.0218	0.8736	0.1592	0.0200	1.4038	0.2402	0.0186	1.8572	0.2999
	2	2	-0.0241	-0.2412	-0.0471	-0.0218	-0.8736	-0.9589	-0.0200	-1.4038	-1.6409	-0.0186	-1.8572	-2.2999
	0.4	1	0.0964	0.0603	0.0118	0.0874	0.2184	0.0398	0.0309	0.3509	0.0743	0.0743	0.4643	0.0750
	2	2	-0.0964	-0.0603	-0.0618	-0.0874	-0.2184	-0.2937	-0.0802	-0.3509	-0.4102	-0.0743	-0.4643	-0.5750
0.7	0.1	1	0.1688	0.0344	0.0067	0.1529	0.1248	0.0227	0.1403	0.2005	0.0343	0.1300	0.2653	0.0429
	2	2	-0.1688	-0.0344	-0.0352	-0.1529	-0.1248	-0.1370	-0.1403	-0.2005	-0.2344	-0.1300	-0.2653	-0.3285
	0.4	1	0.0581	0.5812	0.2185	0.0438	1.7524	0.5669	0.0362	2.5326	0.7274	0.0312	3.1242	0.8110
	2	2	-0.0581	-0.5812	-0.6184	-0.0438	-1.7524	-2.1668	-0.0362	-2.5326	-3.5274	-0.0312	-3.1242	-4.8119
0.4	0.1	1	0.2325	0.1453	0.0546	0.1752	0.4381	0.1417	0.1447	0.6331	0.1818	0.1250	0.7810	0.2027
	2	2	-0.2325	-0.1453	-0.1546	-0.1752	-0.4381	-0.5417	-0.1447	-0.6331	-0.8818	-0.1250	-0.7810	-1.2029
	0.4	1	0.4068	0.0830	0.0311	0.3067	0.2504	0.0810	0.2533	0.3618	0.1039	0.2187	0.4463	0.1159
	2	2	-0.4068	-0.0830	-0.0883	-0.3067	-0.2504	-0.3096	-0.2533	-0.3618	-0.5039	-0.2187	-0.4463	-0.6874
0.6	0.1	1	0.1006	1.0062	0.5351	0.0604	2.4172	1.0237	0.0463	3.2396	1.1724	0.0384	3.8413	1.2254
	2	2	-0.1006	-1.0062	-1.1351	-0.0604	-2.4172	-3.4240	-0.0463	-3.2396	-5.3719	-0.0384	-3.8413	-7.2255
	0.4	1	0.4025	0.2515	0.1334	0.2417	0.6043	0.2559	0.1851	0.8099	0.2931	0.1536	0.9603	0.3063
	2	2	-0.4025	-0.2515	-0.2837	-0.2417	-0.6043	-0.8560	-0.1851	-0.8099	-1.3430	-0.1536	-0.9603	-1.8063
0.7	0.1	1	0.7043	0.1437	0.0764	0.4230	0.3453	0.1462	0.3240	0.4628	0.1675	0.2689	0.5488	0.1751
	2	2	-0.7043	-0.1437	-0.1621	-0.4230	-0.3453	-0.4891	-0.3240	-0.4628	-0.7624	-0.2689	-0.5488	-1.0323
	0.1	1	0.1384	1.3835	0.9005	0.0703	2.8128	1.3814	0.0519	3.6379	1.4926	0.0424	4.2401	1.5116
	2	2	-0.1384	-1.3835	-1.7005	-0.0703	-2.8128	-4.5815	-0.0519	-3.6379	-7.0924	-0.0424	-4.2401	-9.5122
0.8	0.1	1	0.5534	0.3459	0.2255	0.2813	0.7032	0.3453	0.2079	0.9095	0.3732	0.1696	1.0600	0.3779
	2	2	-0.5434	-0.3459	-0.4251	-0.2813	-0.7032	-1.1454	-0.2079	-0.9095	-1.7732	-0.1696	-1.0600	-2.3780
	0.4	1	0.9684	0.1976	0.1286	0.4922	0.4018	0.1973	0.3638	0.5197	0.2132	0.2968	0.6057	0.2159
	2	2	-0.9684	-0.1976	-0.2429	-0.4922	-0.4018	-0.6545	-0.3638	-0.5197	-0.1032	-0.2968	-0.6057	-1.3588
1.0	0.1	1	0.1562	1.5618	1.1395	0.0754	3.0156	1.6185	0.0551	3.8598	1.7107	0.0447	4.4721	1.7079
	2	2	-0.1562	-1.5618	-2.1398	-0.0754	-3.0156	-5.6187	-0.0551	-3.8598	-8.7112	-0.0447	-4.4721	-11.7088
	0.4	1	0.6247	0.3904	0.2849	0.3016	0.7539	0.4046	0.2206	0.9649	0.4276	0.1789	1.1180	0.4270
	2	2	-0.6248	-0.3904	-0.5348	-0.3016	-0.7539	-1.4047	-0.2206	-0.9649	-2.1777	-0.1789	-1.1180	-2.9271
0.7	0.1	1	1.0933	0.2231	0.1628	0.5277	0.4308	0.2312	0.3860	0.5514	0.2444	0.3130	0.6388	0.2439
	2	2	-1.0933	-0.2231	-0.3057	-0.5277	-0.4308	-0.8027	-0.3860	-0.5514	-1.2444	-0.3130	-0.6388	-1.6725

It is easy to show that transforms  $T_c(\mu, Fo)$  and  $\Theta_c(\mu, Fo)$ , which are defined by expressions (5-2-9) and (5-2-10), satisfy initial conditions (5-2-7). Let us find now the inverse transforms, using formula (5-2-3). After some simple transformations we obtain

$$T(X, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 (A_{ki}^q P_{ki} + B_{ki}^q Q_{ki}); \quad (5-2-14)$$

$$\Theta(X, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 (A_{ki}^m P_{ki} + B_{ki}^m Q_{ki}), \quad (5-2-15)$$

where

$$P_{ki} = \int_0^1 F_k(X) dX + 2 \sum_{n=1}^{\infty} \cos \mu_n X \exp[-\mu_n^2 \nu_i^2 Lu Fo] \int_0^1 F_k(X) \cos \mu_n X dX; \quad (5-2-16)$$

$$Q_{ki} = \int_0^{Fo} Ki_k(Fo^*) dFo^* + 2 \sum_{n=1}^{\infty} (-1)^n \cos \mu_n X \exp[-\mu_n^2 \nu_i^2 Lu Fo] \times \\ \times \int_0^{Fo} Ki_k(Fo^*) \exp[\mu_n^2 \nu_i^2 Lu Fo^*] dFo^*, \quad (5-2-17)$$

$$\mu_n = n\pi (n = 1, 2, 3, \dots); \quad i = 1, 2; \quad k = \begin{cases} 1 & \text{for } F, \text{ and } q \text{ for } Ki \\ 2 & \text{for } F, \text{ and } m \text{ for } Ki \end{cases}.$$

On the basis of general solution (5-2-14) and (5-2-15) it is possible to obtain a series of particular solutions.

1. If the initial transfer-potential distributions are constant over the cross section of the material and have the fixed values  $t_* = t_0$  and  $\theta_* = \theta_0$ , then

$$F_1(X) = F_2(X) = 0.$$

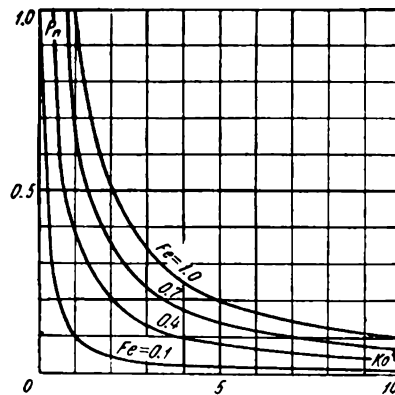


FIGURE 5-7. Relation between  $P_n$  and  $Ko^*$ , for various values of  $Fe = Ko^* P_n$

Next, since  $\sin \mu_n = 0$ , (5-2-16) becomes

$$P_{ki} = 0.$$



Consequently,

$$T(X, Fo) = \frac{t - t_0}{t_0} - \sum_{k=1}^2 \sum_{i=1}^2 B_{ki}^q Q_{ki}; \quad (5-2-18)$$

$$\Theta(X, Fo) = \frac{\theta_0 - \theta}{\theta_0} - \sum_{k=1}^2 \sum_{i=1}^2 B_{ki}^m Q_{ki}. \quad (5-2-19)$$

2. If the heat and mass fluxes at the surface of the body are constant ( $Kl_q$  and  $Kl_m$  constant), then solutions (5-2-14) and (5-2-15) become

$$T(X, Fo) = (Kl_q - Ko^* Lu Kl_m) Fo - \frac{1}{6} (1 - 3X^2) Kl_q + \sum_{k=1}^2 \sum_{i=1}^2 A_{ki}^q P_{ki} - \frac{1}{Lu} \sum_{n=1}^{\infty} \left\{ (-1)^n \frac{2}{\mu_n^2} \cos \mu_n X \sum_{k=1}^2 \sum_{i=1}^2 Kl_k \frac{B_{ki}^q}{v_i^2} \exp(-\mu_n^2 v_i^2 Lu Fo) \right\}; \quad (5-2-20)$$

$$\Theta(X, Fo) = Kl_m Lu Fo - \frac{1}{6} (1 - 3X^2) (Pn Kl_q + Kl_m) + \sum_{k=1}^2 \sum_{i=1}^2 A_{ki}^m P_{ki} - \frac{1}{Lu} \sum_{n=1}^{\infty} \left\{ (-1)^n \frac{2}{\mu_n^2} \cos \mu_n X \sum_{k=1}^2 \sum_{i=1}^2 Kl_k \frac{B_{ki}^m}{v_i^2} \exp(-\mu_n^2 v_i^2 Lu Fo) \right\}. \quad (5-2-21)$$

To derive (5-2-20) and (5-2-21) the relation

$$2 \sum_{n=1}^{\infty} (-1)^n \frac{1}{\mu_n^2} \cos \mu_n X = -\frac{1}{6} (1 - 3X^2)$$

was used, and the sums of individual groups of coefficients  $A$  and  $B$  were evaluated.

3. If  $F_1(X) = F_2(X) = 0$  and if  $Kl_q$  and  $Kl_m$  are constant, then taking into account equations (5-2-18) through (5-2-21) the solution of the problem, after expanding coefficients  $B_{ki}/v_i^2$ , becomes

$$T(X, Fo) = (Kl_q - Ko^* Lu Kl_m) Fo - \frac{1}{6} (1 - 3X^2) Kl_q - \sum_{n=1}^{\infty} \sum_{i=1}^2 (-1)^n \frac{2}{\mu_n^2} C_i^q \cos \mu_n X \exp(-\mu_n^2 v_i^2 Lu Fo);$$

$$\Theta(X, Fo) = Kl_m Lu Fo - \frac{1}{6} (1 - 3X^2) (Pn Kl_q + Kl_m) - \sum_{n=1}^{\infty} \sum_{i=1}^2 (-1)^n \frac{2}{\mu_n^2} C_i^m \cos \mu_n X \exp(-\mu_n^2 v_i^2 Lu Fo),$$

where

$$\left. \begin{aligned} C_i^q &= \frac{Kl_q (v_2^2 - 1/Lu) + Ko^* Kl_m}{v_2^2 - v_1^2}; & C_2^q &= -\frac{Kl_q (v_1^2 - 1/Lu) + Ko^* Kl_m}{v_2^2 - v_1^2}; \\ C_i^m &= \frac{Pn Kl_q v_2^2 - (1 - v_2^2) Kl_m}{v_2^2 - v_1^2}; & C_2^m &= -\frac{Pn Kl_q v_1^2 - (1 - v_1^2) Kl_m}{v_2^2 - v_1^2}. \end{aligned} \right\} \quad (5-2-22)$$

Values of coefficients  $C_i^q$  and  $C_i^m$  for various values of the related similarity criteria are given in Tables 5-3 through 5-5. Coefficients  $(-1)^n \frac{2}{\mu_n^2}$  are listed in Table 4-8, row 3.

4. If no thermal-gradient mass transfer takes place in the material, so that  $Pn=0$ , then solutions (5-2-14) and (5-2-15) become

$$T = P_{11} + Q_{11} + \frac{Ko^* Lu}{Lu - 1} (P_{21} - P_{22} + Q_{21} - Lu Q_{22});$$

$$\Theta = P_{21} + Lu Q_{21}$$

TABLE 5 - 3

Coefficients  $C_i^q = C_i^q(\text{Lu}, \text{Ko}^*\text{Pn}, \text{Kl}_q)$  and  $C_i^m = C_i^m(\text{Lu}, \text{Ko}^*\text{Pn}, \text{Kl}_q)$  (for  $\text{Kl}_m = 0.5$  and  $\text{Pn} = 0.5$ )

Lu	$\text{Kl}_q$	$\text{Ko}^*\text{Pn} = 0.1$						$\text{Ko}^*\text{Pn} = 0.4$						$\text{Ko}^*\text{Pn} = 0.7$						$\text{Ko}^*\text{Pn} = 1.0$					
		$C_i^q$		$C_i^m$		$C_i^q$		$C_i^m$		$C_i^q$		$C_i^m$		$C_i^q$		$C_i^m$		$C_i^q$		$C_i^m$		$C_i^q$		$C_i^m$	
		$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$
0.2	0.1	0.0271	0.0729	0.5588	-0.0088	0.0981	0.0019	0.5502	-0.0002	0.1572	-0.0572	0.5442	0.0059	0.2078	-0.1079	0.5396	0.0104	0.2078	-0.1079	0.5396	0.0104	0.2078	-0.1079	0.5396	0.0104
	0.5	0.0390	0.4609	0.8058	-0.0559	0.1407	0.3592	0.7900	-0.0400	0.2249	0.2751	0.7784	-0.0284	0.2964	0.2036	0.7696	-0.0196	0.2964	0.2036	0.7696	-0.0196	0.2964	0.2036	0.7696	-0.0196
	1.0	0.0540	0.9460	1.1146	-0.1123	0.1941	0.8059	1.0897	-0.0898	0.3095	0.6904	1.0713	-0.0713	0.4071	0.5929	1.0571	-0.0571	0.4071	0.5929	1.0571	-0.0571	0.4071	0.5929	1.0571	-0.0571
	1.5	0.0689	1.4311	1.4235	-0.1735	0.2474	1.2525	1.3895	-0.1395	0.3942	1.1058	1.3643	-0.1142	0.5177	0.9822	1.3441	-0.0946	0.5177	0.9822	1.3441	-0.0946	0.5177	0.9822	1.3441	-0.0946
	2.0	0.0838	1.9162	1.7323	-0.2323	0.3009	1.6991	1.6892	-0.1892	0.4788	1.5212	1.6570	-0.1570	0.6284	1.3715	1.6321	-0.1321	0.6284	1.3715	1.6321	-0.1321	0.6284	1.3715	1.6321	-0.1321
0.6	0.1	0.1221	-0.0221	0.5374	0.0126	0.2836	-0.1836	0.4824	0.0675	0.3778	-0.2748	0.4721	0.0779	0.4405	-0.3405	0.4702	0.0797	0.4405	-0.3405	0.4702	0.0797	0.4405	-0.3405	0.4702	0.0797
	0.5	0.2080	0.2919	0.9158	-0.1658	0.4514	0.0486	0.7678	-0.0178	0.5778	-0.0778	0.7279	0.0221	0.6661	-0.1661	0.7111	0.0389	0.6661	-0.1661	0.7111	0.0389	0.6661	-0.1661	0.7111	0.0389
	1.0	0.3155	0.6845	1.3888	-0.3888	0.6611	0.3389	1.1244	-0.1244	0.8317	0.1683	1.0477	-0.0477	0.9481	0.0519	1.0122	-0.0122	0.9481	0.0519	1.0122	-0.0122	0.9481	0.0519	1.0122	-0.0122
	1.5	0.4228	1.0771	1.8617	-0.6118	0.8708	0.6292	1.4810	-0.2311	1.0855	0.4144	1.3675	-0.1175	1.2301	0.2698	1.3132	-0.0632	1.2301	0.2698	1.3132	-0.0632	1.2301	0.2698	1.3132	-0.0632
	2.0	0.5303	1.4697	2.3347	-0.8348	1.0805	0.9194	1.8377	-0.3377	1.3394	0.6606	1.6873	-0.1873	1.5121	0.4878	1.6143	-0.1143	1.5121	0.4878	1.6143	-0.1143	1.5121	0.4878	1.6143	-0.1143
1.0	0.1	0.2140	-0.1140	0.3959	0.1540	0.3666	-0.2666	0.3956	0.1544	0.4552	-0.3552	0.4087	0.1413	0.7236	-0.3970	0.4203	0.1297	0.7236	-0.3970	0.4203	0.1297	0.7236	-0.3970	0.4203	0.1297
	0.5	0.4452	0.0548	0.8238	-0.0739	0.6269	-0.1269	0.6765	0.0735	0.7324	-0.2324	0.6576	0.0924	0.8090	-0.3090	0.6545	0.0955	0.8090	-0.3090	0.6545	0.0955	0.8090	-0.3090	0.6545	0.0955
	1.0	0.7342	0.2658	1.3589	-0.3589	0.9523	0.0477	1.0277	-0.0277	1.0789	-0.0789	0.9687	0.0313	1.1708	-0.1708	0.9472	0.0528	1.1708	-0.1708	0.9472	0.0528	1.1708	-0.1708	0.9472	0.0528
	1.5	1.0231	0.4768	1.8938	-0.6438	1.2776	0.2223	1.3788	-0.1288	1.4254	0.0745	1.2797	-0.0298	1.5326	-0.0326	1.2399	0.0101	1.5326	-0.0326	1.2399	0.0101	1.5326	-0.0326	1.2399	0.0101
	2.0	1.3122	0.6878	2.4287	-0.9288	1.6030	0.3970	1.7299	-0.2300	1.7719	0.2280	1.5908	-0.0909	1.8944	0.1056	1.5326	-0.0326	1.8944	0.1056	1.5326	-0.0326	1.8944	0.1056	1.5326	-0.0326

TABLE 5 - 4

Coefficients  $C_l^q = C_l^q(\text{Lu}, \text{Ko}^*\text{Pn}, \text{Kl}_m)$  and  $C_l^m = C_l^m(\text{Lu}, \text{Ko}^*\text{Pn}, \text{Kl}_m)$  (for  $\text{Kl}_q = 0.5$  and  $\text{Kl}_m = 0.5$ )

Lu	Kl <sub>m</sub>	Ko*Pn = 0.1				Ko*Pn = 0.4				Ko*Pn = 0.7				Ko*Pn = 1.0			
		$C_l^q$		$C_l^m$		$C_l^q$		$C_l^m$		$C_l^q$		$C_l^m$		$C_l^q$		$C_l^m$	
		i=1	i=2	i=1	i=2	i=1	i=2	i=1	i=2	i=1	i=2	i=1	i=2	i=1	i=2	i=1	i=2
0.2	0.1	0.0198	0.4802	0.4082	-0.0582	0.0708	0.4291	0.3978	-0.0478	0.1127	0.3873	0.3900	-0.0400	0.1478	0.3522	0.3839	-0.0339
	0.5	0.0390	0.4609	0.8059	-0.0559	0.1407	0.3592	0.7900	-0.0400	0.2249	0.2751	0.7784	-0.0284	0.2964	0.2036	0.7696	-0.0196
	1.0	0.0632	0.4368	1.3029	-0.0529	0.2281	0.2718	1.2828	-0.0303	0.3652	0.1347	1.2639	-0.0189	0.4821	0.0179	1.2517	-0.0017
	1.5	0.0872	0.4127	1.8000	-0.0500	0.3156	0.1845	1.7705	-0.0195	0.5056	-0.0056	1.7494	-0.0005	0.6678	-0.1678	1.7338	0.0161
	2.0	0.1114	0.3886	2.2971	-0.0471	0.4029	0.0971	2.2608	-0.0086	0.6459	-0.1459	2.2349	0.0150	0.8535	-0.3535	2.2159	0.0340
0.6	0.1	0.1275	0.3724	0.4818	-0.2115	0.2580	0.2419	0.4389	-0.0889	0.3187	0.1813	0.4014	0.0514	0.3588	0.1412	0.3830	-0.0331
	0.5	0.2080	0.2919	0.9158	-0.1658	0.4514	0.0486	0.7678	-0.0178	0.5779	-0.0778	0.7279	0.0220	0.6661	-0.1661	0.7111	0.0389
	1.0	0.3087	0.1913	1.3586	-0.1077	0.6931	-0.1931	1.1789	0.0711	0.9019	-0.4018	1.1361	0.1139	1.0503	-0.5503	1.1211	0.1288
	1.5	0.4093	0.0907	1.8014	-0.0515	0.9348	-0.4348	1.5901	0.1599	1.2259	-0.7258	1.5442	0.2057	1.4344	-0.9344	1.5312	0.2188
	2.0	0.5099	0.0100	2.2443	-0.0056	1.1765	-0.6765	2.0012	0.2488	1.5499	-1.0498	1.9524	0.2976	1.8185	-1.3185	1.9413	0.3087
1.0	0.1	0.3202	0.1797	0.5927	-0.2427	0.3857	0.1143	0.4162	-0.0653	0.4237	0.0763	0.3804	-0.0304	0.4512	0.0488	0.3650	-0.0151
	0.5	0.4452	0.0648	0.8238	-0.0739	0.6269	-0.1269	0.6765	0.0735	0.7324	-0.2324	0.6575	0.9241	0.8090	-0.3090	0.6545	0.0955
	1.0	0.6013	-0.1014	1.1129	0.1371	0.9284	-0.4285	1.0019	0.2481	1.1184	-0.6735	1.0041	0.2459	1.2561	-0.7561	1.0627	0.2337
	1.5	0.7575	-0.2575	1.4018	0.3481	1.2300	-0.7300	1.3272	0.4227	1.5043	-1.0043	1.3505	0.3994	1.7033	-1.2033	1.3781	0.3719
	2.0	0.9137	-0.4137	1.6909	0.5591	1.5315	-1.0315	1.6526	0.5974	1.8903	-1.3903	1.6971	0.5529	2.1505	-1.6505	1.7399	0.5101

TABLE 5-5

Coefficients  $C_l^i = C_l^i(\text{Lu}, \text{Ko}^*\text{Pn}, \text{Kl}_q, \text{Pn})$  (for  $\text{Kl}_q = 0.5$  and  $\text{Kl}_m = 0.5$ )

Lu	Pn	Ko*Pn = 0.1				Ko*Pn = 0.4				Ko*Pn = 0.7				Ko*Pn = 1.0			
		$C_l^i$		$C_m^i$		$C_l^i$		$C_m^i$		$C_l^i$		$C_m^i$		$C_l^i$		$C_m^i$	
		$l=1$	$l=2$	$l=1$	$l=2$	$l=1$	$l=2$	$l=1$	$l=2$	$l=1$	$l=2$	$l=1$	$l=2$	$l=1$	$l=2$	$l=1$	$l=2$
0.2	0.1	0.1355	0.3645	0.5588	0.0088	0.4902	0.0972	0.5502	-0.0002	0.7862	-0.2862	0.5441	0.0059	1.0383	-0.5383	0.5386	0.0104
	0.25	0.0632	0.4369	0.6517	-0.0265	0.2281	0.2718	0.6401	-0.0151	0.5657	0.1347	0.6319	-0.0070	0.4821	0.0179	0.6258	-0.0009
	0.50	0.0390	0.4610	0.8058	-0.0559	0.1407	0.3592	0.7900	-0.0400	0.2249	0.2751	0.7784	-0.0284	0.2964	0.2036	0.7697	-0.0196
	0.75	0.0310	0.4690	0.9602	-0.0853	0.1094	0.3884	0.9399	-0.0649	0.1781	0.3218	0.9248	-0.0498	0.2346	0.2655	0.9133	0.0384
0.6	1.00	0.0270	0.4790	1.1146	-0.1147	0.0970	0.4029	1.0897	-0.0898	0.1548	0.3452	1.0713	-0.0713	0.2035	0.2964	1.0571	0.0571
	0.1	0.6105	0.1106	0.5374	0.0126	1.4182	-0.9182	0.4824	0.0675	1.8739	-1.3738	0.4721	0.0779	2.2026	-1.7026	0.4771	0.0797
	0.25	0.3087	0.1913	0.6793	-0.0543	0.6931	-0.2294	0.5895	0.0355	0.9019	-0.4018	0.5680	0.0569	1.0502	-0.5503	0.5606	0.0644
	0.50	0.2080	0.2919	0.9158	-0.1658	0.4514	0.0486	0.7678	-0.0178	0.5779	0.0778	0.7279	0.0291	0.6661	-0.1661	0.7111	0.0389
1.0	0.75	0.1745	0.3255	1.1523	-0.2773	0.3708	0.1292	0.9461	-0.0711	0.4698	0.0302	0.8878	-0.0128	0.5381	-0.0381	0.8616	0.0134
	1.00	0.1577	0.3422	1.3888	-0.3888	0.3306	0.1694	1.1244	-0.1244	0.4159	0.0415	1.0477	-0.0477	0.4740	0.0259	1.0122	-0.0122
	0.1	1.0699	-0.5699	0.3959	0.1540	1.8331	-1.3331	0.3956	0.1544	2.2762	-1.7762	0.4087	0.1413	2.5976	-2.0976	0.4203	0.1296
	0.25	0.6013	-0.1014	0.5563	0.0686	0.9284	-0.4285	0.5009	0.1240	1.1184	-0.6184	0.5020	0.1229	1.2561	-0.7561	0.5082	0.1168
1.0	0.50	0.4452	0.0548	0.8238	-0.0739	0.6269	-0.1269	0.6765	0.0735	0.7324	-0.2324	0.6575	0.0924	0.8090	-0.3089	0.6545	0.0955
	0.75	0.3931	0.1069	1.0912	-0.2165	0.5263	-0.0264	0.8521	0.0229	0.6037	-0.1038	0.8131	0.0619	0.6599	-0.1599	0.8008	0.0741
	1.00	0.3671	0.1329	1.3431	-0.3589	0.4761	-0.0238	1.0277	0.0277	0.5394	-0.0395	0.9686	0.0309	0.5854	-0.0854	0.9472	0.0528

or in expanded form

$$\begin{aligned}
 T(X, Fo) = & \int_0^1 F_1(X) dX + \int_0^{Fo} [Kl_q(Fo^*) - Ko^* Lu Kl_m(Fo^*)] dFo^* + \\
 & + 2 \sum_{n=1}^{\infty} \cos \mu_n X \left\{ \exp(-\mu_n^2 Fo) \int_0^1 F_1(X) \cos \mu_n X dX + \right. \\
 & + \frac{Ko^* Lu}{Lu - 1} [\exp(-\mu_n^2 Fo) - \exp(-\mu_n^2 Lu Fo)] \int_0^1 F_2(X) \cos \mu_n X dX \Big\} + \\
 & + 2 \sum_{n=1}^{\infty} (-1)^n \cos \mu_n X \left\{ \exp(-\mu_n^2 Fo) \int_0^{Fo} Kl_q(Fo^*) \exp(\mu_n^2 Fo^*) dFo^* + \right. \\
 & + \frac{Ko^* Lu}{Lu - 1} [\exp(-\mu_n^2 Fo) Kl_m(Fo^*) \exp(\mu_n^2 Fo^*) - \\
 & - Lu \exp(-\mu_n^2 Lu Fo) \int_0^{Fo} Kl_m(Fo^*) \exp(\mu_n^2 Lu Fo^*) dFo^*] \Big\}; \quad (5-2-23)
 \end{aligned}$$

$$\begin{aligned}
 \Theta(X, Fo) = & \int_0^1 F_2(X) dX + Lu \int_0^{Fo} Kl_m(Fo^*) dFo^* + \\
 & + 2 \sum_{n=1}^{\infty} \cos \mu_n X \exp(-\mu_n^2 Lu Fo) \left\{ \int_0^1 F_2(X) \cos \mu_n X dX + \right. \\
 & + (-1)^n Lu \int_0^{Fo} Kl_m(Fo^*) \exp(\mu_n^2 Lu Fo^*) dFo^* \Big\}. \quad (5-2-24)
 \end{aligned}$$

In the derivation of (5-2-23) and (5-2-24) it was taken into account that, when  $P_n = 0$ , conditions (5-2-13) simplify to

$$v_1^2 = \frac{1}{Lu}; \quad v_2^2 = 1.$$

5. If, in addition to the condition  $P_n = 0$ , the initial transfer-potential distributions throughout the material are constant and the Kirpichev number is also constant, then we obtain

$$\begin{aligned}
 T(X, Fo) = & (Kl_q - Ko^* Lu Kl_m) Fo - \frac{1}{6} (1 - 3X^2) Kl_q - \\
 & - \sum_{n=1}^{\infty} (-1)^n \frac{2}{\mu_n^2} \cos \mu_n X \left\{ Kl_q \exp(-\mu_n^2 Fo) + \right. \\
 & + \frac{Ko^* Lu Kl_m}{Lu - 1} [\exp(-\mu_n^2 Fo) - \exp(-\mu_n^2 Lu Fo)] \Big\}; \\
 \Theta(X, Fo) = & Kl_m \left[ Lu Fo - \frac{1}{6} (1 - 3X^2) - \right. \\
 & - \sum_{n=1}^{\infty} (-1)^n \frac{2}{\mu_n^2} \cos \mu_n X \exp(-\mu_n^2 Lu Fo) \Big].
 \end{aligned}$$

6. If, in addition to the conditions of problem 5, we assume that no phase transitions take place in the material, then we obtain the classical solutions of the heat-conduction and diffusion equations, for boundary

conditions of the second kind with sources absent:

$$T = \frac{t - t_0}{t_0} = K l_q \left[ F_0 - \frac{1}{6} (1 - 3X^2) - \sum_{n=1}^{\infty} (-1)^n \frac{2}{\mu_n^2} \cos \mu_n X \exp(-\mu_n^2 F_0) \right]; \quad (5-2-25)$$

$$\Theta = \frac{\theta_0 - \theta}{\theta_0} = K l_m \left[ Lu F_0 - \frac{1}{6} (1 - 3X^2) - \sum_{n=1}^{\infty} (-1)^n \frac{2}{\mu_n^2} \cos \mu_n X \exp(-\mu_n^2 Lu F_0) \right]. \quad (5-2-26)$$

Solutions (5-2-25) and (5-2-26) become identical if we set  $F_0 = F_0 m / Lu$ .

For low values of  $F_0$  or  $Lu F_0$  solutions (5-2-25) and (5-2-26) may be written as

$$T = 2 K l_q \sqrt{F_0} \sum_{n=1}^{\infty} \left[ i \operatorname{erfc} \frac{(2n-1) - X}{2 \sqrt{F_0}} + i \operatorname{erfc} \frac{(2n-1) + X}{2 \sqrt{F_0}} \right];$$

$$\Theta = 2 K l_m \sqrt{Lu F_0} \sum_{n=1}^{\infty} \left[ i \operatorname{erfc} \frac{(2n-1) - X}{2 \sqrt{Lu F_0}} + i \operatorname{erfc} \frac{(2n-1) + X}{2 \sqrt{Lu F_0}} \right],$$

where

$$i \operatorname{erfc} z = \frac{1}{\sqrt{\pi}} e^{-z^2} - z \operatorname{erfc} z.$$

The general nature of the transfer-potential distributions is evident from Figure 5-8.

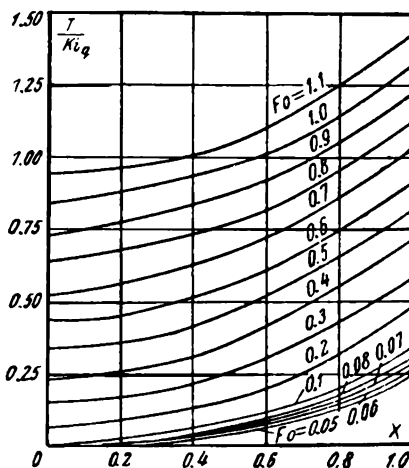


FIGURE 5-8. The temperature field of an infinite plate, for constant thermal-flux density at the surface

#### b) Solution for an infinite cylinder ( $\Gamma=1$ )

The method of solving system of equations (4-1-2) and (4-1-3) for an infinite cylinder or sphere, with boundary conditions (5-1-1) and (5-1-2) and initial conditions (5-1-3), is similar to the solution method for an infinite plate.

A finite Hankel integral transformation with respect to variable  $X$

has the general form

$$\{\varphi(\mu, F_0)\}_H = \int_0^1 \varphi(X, F_0) X J_0(\mu X) dX, \quad (5-2-27)$$

where  $\mu$  represents the roots of the characteristic equation  $J_1(\mu) = 0$ . Some values of these characteristic-equation roots are given in Table 4-9, row 2.

The inverse Hankel transformation for the problem at hand is given by the formula

$$\varphi(X, F_0) = 2\{\varphi(0, F_0)\}_H + 2 \sum_{n=1}^{\infty} \frac{J_0(\mu_n X)}{J_0^2(\mu_n)} \{\varphi(\mu_n, F_0)\}_H. \quad (5-2-28)$$

Let us now apply transformation (5-2-27) to the groups of terms in differential equations (4-1-2) and (4-1-3) which have the form

$$\frac{\partial^2 \varphi(X, F_0)}{\partial X^2} + \frac{1}{X} \frac{\partial \varphi(X, F_0)}{\partial X}.$$

If boundary conditions (5-1-1) and (5-1-2) are assumed valid, then the following relations are obtained:

$$\left\{ \begin{aligned} \int_0^1 \left( \frac{\partial^2 T}{\partial X^2} + \frac{1}{X} \frac{\partial T}{\partial X} \right) X J_0(\mu X) dX &= J_0(\mu) \frac{\partial T(1, F_0)}{\partial X} - \mu^2 T_H = J_0(\mu) \text{Ki}_q(F_0) - \mu^2 T_H; \\ \int_0^1 \left( \frac{\partial^2 \Theta}{\partial X^2} + \frac{1}{X} \frac{\partial \Theta}{\partial X} \right) X J_0(\mu X) dX &= J_0(\mu) [\text{Pn Ki}_q(F_0) + \text{Ki}_m(F_0)] - \mu^2 \Theta_H. \end{aligned} \right. \quad (5-2-29)$$

Next we multiply all the terms in equations (4-1-2) and (4-1-3) by  $X J_0(\mu X)$  and integrate with respect to  $X$  from 0 to 1 to obtain, using (5-2-29), a system of ordinary differential equations similar to (5-2-6), but in which  $(-1)^n$  is replaced by  $J_0(\mu)$ . After being transformed by a Hankel transformation, initial conditions (5-1-3) become

$$\begin{aligned} T_{0H} &= T_H(\mu, 0) = \int_0^1 X F_1(X) J_0(\mu X) dX, \\ \Theta_{0H} &= \Theta_H(\mu, 0) = \int_0^1 X F_2(X) J_0(\mu X) dX. \end{aligned} \quad (5-2-30)$$

The application of Laplace integral transformation (5-2-8) to the obtained system of differential equations transforms it into a system of algebraic equations in  $T_{HL}$  and  $\Theta_{HL}$ . The solutions of the latter equations are

$$\begin{aligned} T_{HL} &= J_0(\mu) \frac{s + \text{Lu} \mu^2}{R(s)} \{\text{Ki}_q\}_L - J_0(\rho) \frac{\text{Ko}^* \text{Lu} s}{R(s)} \{\text{Ki}_m\}_L + \\ &+ \frac{s + \text{Lu} \mu^2}{R(s)} T_{0H} + \frac{\text{Ko}^* \text{Lu} \mu^2}{R(s)} \Theta_{0H}; \end{aligned} \quad (5-2-31)$$

$$\begin{aligned} \Theta_{HL} &= J_0(\mu) \frac{\text{Lu Pn} \mu^2}{R(s)} \{\text{Ki}_q\}_L + J_0(\rho) \frac{\text{Lu}(s + \mu^2)}{R(s)} \{\text{Ki}_m\}_L + \\ &+ \frac{\text{Lu Pn} \mu^2}{R(s)} T_{0H} + \frac{s + (1 + \text{Ko}^* \text{Pn Lu}) \mu^2}{R(s)} \Theta_{0H}, \end{aligned} \quad (5-2-32)$$

where

$$R(s) = s^2 + \left(1 + \text{Ko}^* \text{Pn} + \frac{1}{\text{Lu}}\right) \text{Lu} \mu^2 s + \text{Lu} \mu^4 = 0; \quad (5-2-33)$$

$$\{Ki_k\}_L = \int_0^\infty Ki_k(Fo) \exp(-sFo) dFo, \quad k = \begin{cases} 1 & \text{or } q \\ 2 & \text{or } m \end{cases}. \quad (5-2-34)$$

The inverse transforms of expressions (5-2-31) and (5-2-32), with respect to parameter  $s$ , are

$$T_H(\mu, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 [A_{ki}^q \{P_{ki}\}_H + B_{ki}^q \{Q_{ki}\}_H]; \quad (5-2-36)$$

$$\Theta_H(\mu, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 [A_{ki}^m \{P_{ki}\}_H + B_{ki}^m \{Q_{ki}\}_H]. \quad (5-2-37)$$

The constant coefficients  $A_{ki}$  and  $B_{ki}$  in equations (5-2-36) and (5-2-37) are defined by relations (5-2-11). Values of these constants for various Ko\*Pn and Lu are listed in Tables 5-1 and 5-2 and plotted in Figures 5-1 through 5-6. Quantities  $\{P_{ki}\}_H$  and  $\{Q_{ki}\}_H$  are defined by the corresponding expressions in (5-2-12), but with the symbol  $c$  for the Fourier cosine transformation replaced by the symbol  $H$  for the Hankel transformation and with  $(-1)^n$  replaced by  $J_0(\mu)$ .

Now, we use formula (5-2-28) to obtain from (5-2-36) and (5-2-37) the inverse transforms  $T(X, Fo)$  and  $\Theta(X, Fo)$ :

$$T(X, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 (A_{ki}^q P_{ki} + B_{ki}^q Q_{ki}); \quad (5-2-38)$$

$$\Theta(X, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 (A_{ki}^m P_{ki} + B_{ki}^m Q_{ki}), \quad (5-2-39)$$

where

$$P_{ki} = 2 \left[ \int_0^1 XF_k(X) dX + \sum_{n=1}^{\infty} \frac{J_0(\mu_n X)}{J_0^2(\mu_n)} \exp(-\mu_n^2 \nu_i^2 Lu Fo) \times \right. \\ \left. \times \int_0^1 XF_k(X) J_0(\mu_n X) dX \right]; \quad (5-2-40)$$

$$Q_{ki} = 2 \left[ \int_0^{Fo} Ki_k(Fo^*) dFo^* + \sum_{n=1}^{\infty} \frac{J_0(\mu_n X)}{J_0^2(\mu_n)} \exp(-\mu_n^2 \nu_i^2 Lu Fo) \times \right. \\ \left. \times \int_0^{Fo} Ki_k(Fo^*) \exp(\mu_n^2 \nu_i^2 Lu Fo^*) dFo^* \right],$$

$$(i=1, 2), \quad \left( k = \begin{cases} 1 & \text{for } F, \text{ and } q \text{ for } Ki \\ 2 & \text{for } F, \text{ and } m \text{ for } Ki \end{cases} \right), \quad (5-2-41)$$

$\mu_n$  being the roots of the characteristic equation  $J_1(\mu) = 0$ .

Let us consider some particular instances of solutions (5-2-38) and (5-2-39).

1. Zero initial transfer-potential distributions:  $F_i(X) = F_s(X) = 0$

$$T(X, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 B_{ki}^q Q_{ki};$$

$$\Theta(X, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 B_{ki}^m Q_{ki}.$$



2.  $Kl_q$  and  $Kl_m$  constant

$$T(X, Fo) = 2(Kl_q - Ko^* Lu Kl_m) Fo - \frac{1}{4}(1 - 2X^2) Kl_q + \sum_{k=1}^2 \sum_{i=1}^2 A_{ki}^q P_{ki} -$$

$$- \frac{1}{Lu} \sum_{n=1}^{\infty} \left\{ \frac{2J_0(\mu_n X)}{\mu_n^2 J_0(\mu_n)} \sum_{k=1}^2 \sum_{i=1}^2 Kl_k \frac{B_{ki}^q}{v_i^2} \exp(-\mu_n^2 v_i^2 Lu Fo) \right\}. \quad (5-2-42)$$

$$\Theta(X, Fo) = 2Kl_m Lu Fo - \frac{1}{4}(1 - 2X^2)(Pn Kl_q + Kl_m) + \sum_{k=1}^2 \sum_{i=1}^2 A_{ki}^m P_{ki} -$$

$$- \frac{1}{Lu} \sum_{n=1}^{\infty} \left\{ \frac{2}{\mu_n^2 J_0(\mu_n)} J_0(\mu_n X) \sum_{k=1}^2 \sum_{i=1}^2 Kl_k \frac{B_{ki}^m}{v_i^2} \exp(-\mu_n^2 v_i^2 Lu Fo) \right\}. \quad (5-2-43)$$

In the derivation of (5-2-42) and (5-2-43) we used the identity

$$\sum_{n=1}^{\infty} \frac{2J_0(\mu_n X)}{\mu_n^2 J_0(\mu_n)} = -\frac{1}{4}(1 - 2X^2).$$

3.  $F_1(X) = F_2(X) = 0$ ,  $Kl_q$  and  $Kl_m$  constant

$$T(X, Fo) = 2(Kl_q - Ko^* Lu Kl_m) Fo - \frac{1}{4}(1 - 2X^2) Kl_q -$$

$$- \sum_{n=1}^{\infty} \sum_{i=1}^2 \frac{2}{\mu_n^2 J_0(\mu_n)} C_i^q J_0(\mu_n X) \exp(-\mu_n^2 v_i^2 Lu Fo);$$

$$\Theta(X, Fo) = 2Kl_m Lu Fo - \frac{1}{4}(1 - 2X^2)(Pn Kl_q + Kl_m) -$$

$$- \sum_{n=1}^{\infty} \sum_{i=1}^2 \frac{2}{\mu_n^2 J_0(\mu_n)} C_i^m J_0(\mu_n X) \exp(\mu_n^2 v_i^2 Lu Fo),$$

where  $C_i^m$  and  $C_i^q$  are defined by (5-2-22) and their values are listed in Tables 5-3 through 5-5.

4. No thermal-gradient mass transfer:  $Pn = 0$

$$T(X, Fo) = 2 \left[ \int_0^1 X F_1(X) dX + \int_0^{Fo} [Kl_q(Fo^*) - Ko^* Lu Kl_m(Fo^*)] dFo^* + \right.$$

$$+ \sum_{n=1}^{\infty} \frac{J_0(\mu_n X)}{J_0^2(\mu_n)} \left\{ \exp(-\mu_n^2 Fo) \int_0^1 X F_1(X) J_0(\mu_n X) dX + \right.$$

$$+ \frac{Ko^* Lu}{Lu - 1} [\exp(-\mu_n^2 Fo) - \exp(-\mu_n^2 Lu Fo)] \int_0^1 X F_2(X) J_0(\mu_n X) dX \Big\} +$$

$$+ \sum_{n=1}^{\infty} \frac{J_0(\mu_n X)}{J_0^2(\mu_n)} \left\{ \exp(-\mu_n^2 Fo) \int_0^{Fo} Kl_q(Fo^*) \exp(\mu_n^2 Fo^*) dFo^* + \right.$$

$$+ \frac{Ko^* Lu}{Lu - 1} \left[ \exp(-\mu_n^2 Fo) \int_0^{Fo} Kl_m(Fo^*) \exp(\mu_n^2 Fo^*) dFo^* - \right.$$

$$\left. \left. - Lu \exp(-\mu_n^2 Lu Fo) \int_0^{Fo} Kl_m(Fo^*) \exp(\mu_n^2 Lu Fo^*) dFo^* \right] \right\} \Bigg];$$

$$\Theta(X, Fo) = 2 \left\{ \int_0^1 X F_2(X) dX + Lu \int_0^{Fo} Kl_m(Fo^*) dFo^* + \right.$$

$$+ \sum_{n=1}^{\infty} \frac{J_0(\mu_n X)}{J_0^2(\mu_n)} \exp(-\mu_n^2 Lu Fo) \left[ \int_0^1 X F_1(X) J_0(\mu_n X) dX + \right.$$

$$\left. \left. + Lu J_0(\mu_n) \int_0^{Fo} Kl_m(Fo^*) \exp(\mu_n^2 Lu Fo^*) dFo^* \right] \right\}.$$

5.  $P_n = 0$ ;  $F_1(X) = F_2(X) = 0$ ;  $Kl_q$  and  $Kl_m$  constant

$$T(X, Fo) = 2(Kl_q - Ko^* Lu Kl_m) Fo - \frac{1}{4}(1 - 2X^2) Kl_q -$$

$$- \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 J_0(\mu_n)} J_0(\mu_n X) \left\{ Kl_q \exp(-\mu_n^2 Fo) + \right.$$

$$\left. + \frac{Ko^* Lu Kl_m}{Lu - 1} [\exp(-\mu_n^2 Fo) - \exp(-\mu_n^2 Lu Fo)] \right\};$$

$$\Theta(X, Fo) = Kl_m \left[ 2 Lu Fo - \frac{1}{4}(1 - 2X^2) - \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 J_0(\mu_n)} J_0(\mu_n X) \exp(-\mu_n^2 Lu Fo) \right].$$

6.  $P_n = 0$ ;  $Ko^* = 0$ ;  $F_1(X) = F_2(X) = 0$ ;  $Kl_q$  and  $Kl_m$  constant

$$\frac{t - t_s}{t_s} = Kl_q \left[ 2 Fo - \frac{1}{4}(1 - 2X^2) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 J_0(\mu_n)} J_0(\mu_n X) \exp(-\mu_n^2 Fo) \right],$$

$$\frac{\theta_s - \theta}{\theta_s} = Kl_m \left[ 2 Lu Fo - \frac{1}{4}(1 - 2X^2) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 J_0(\mu_n)} J_0(\mu_n X) \exp(-\mu_n^2 Lu Fo) \right].$$

For low values of the Fourier number it is more convenient to use the following approximate solutions:

$$T \approx Kl_q \left[ 2 \sqrt{\frac{Fo}{X}} i \operatorname{erfc} \frac{1-X}{2\sqrt{Fo}} + \frac{(1+3X)Fo}{2X\sqrt{X}} i^3 \operatorname{erfc} \frac{1-X}{2\sqrt{Fo}} + \dots \right];$$

$$\Theta \approx Kl_m \left[ 2 \sqrt{\frac{Lu Fo}{X}} i \operatorname{erfc} \frac{1-X}{2\sqrt{Lu Fo}} + \right.$$

$$\left. + \frac{(1+3X)Lu Fo}{2X\sqrt{X}} i^3 \operatorname{erfc} \frac{1-X}{2\sqrt{Lu Fo}} + \dots \right],$$

where

$$i^3 \operatorname{erfc} z = \frac{1}{4} [\operatorname{erfc} z - 2zi \operatorname{erfc} z].$$

The nature of the transfer-potential distributions for this problem is evident from Figure 5-9.

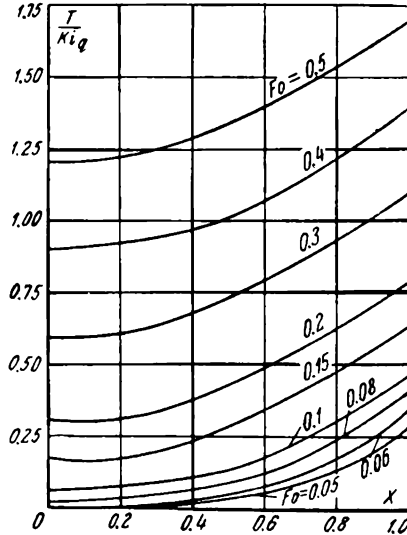


FIGURE 5-9. Temperature field for an infinite cylinder, with constant thermal-flux density at the surface

c) Solution for a sphere ( $\Gamma=2$ )

For a sphere, we apply both an integral Laplace transformation and a Fourier sine transformation. The latter has the form

$$\{\varphi(\mu, F_0)\}_S = \int_0^1 \varphi(X, F_0) X \frac{\sin \mu X}{\mu} dX, \quad (5-2-44)$$

where  $\mu$  represents the roots of the characteristic equation

$$\tan \mu = \mu.$$

Some values of these roots are given in Table 4-11, row 2.

Just as in the preceding subsections, we apply to system of equations (4-1-2) and (4-1-3) and to limiting conditions (5-1-1) and (5-1-2) first a Fourier transformation (in this case (5-2-44)) and then Laplace transformation (5-2-8) to obtain a system of algebraic equations. The latter may be solved for transforms  $T_{SL}$  and  $\theta_{SL}$  to give

$$T_{SL} = \frac{\sin \mu}{\mu} \cdot \frac{s + Lu \mu^2}{R(s)} \{Ki_q\}_L - \frac{\sin \mu}{\mu} \cdot \frac{Ko^* Lu s}{R(s)} \{Ki_m\}_L + \\ + \frac{s + Lu \mu^2}{R(s)} T_{0S} + \frac{Ko^* Lu \mu^2}{R(s)} \theta_{0S}; \quad (5-2-45)$$

$$\theta_{SL} = \frac{\sin \mu}{\mu} \cdot \frac{Lu Pn \mu^2}{R(s)} \{Ki_q\}_L + \frac{\sin \mu}{\mu} \cdot \frac{Lu (s + \mu^2)}{R(s)} \{Ki_m\}_L + \\ + \frac{Lu Pn \mu^2}{R(s)} T_{0S} + \frac{s + (1 + Ko^* Pn Lu) \mu^2}{R(s)} \theta_{0S}, \quad (5-2-46)$$

where  $R(s)$  and  $\{Ki_k\}_L$  ( $k = \begin{cases} 1 & \text{or } q \\ 2 & \text{or } m \end{cases}$ ) are defined by equations (5-2-33) and (5-2-34), and where

$$T_{0S} = T_S(\mu, 0) = \int_0^1 X F_1(X) \frac{\sin \mu X}{\mu} dX,$$

$$\theta_{0S} = \theta_S(\mu, 0) = \int_0^1 X F_2(X) \frac{\sin \mu X}{\mu} dX.$$

If we now find the inverse transforms of (5-2-45) and (5-2-46) with respect to the Laplace parameter  $s$ , then we have

$$T_S(\mu, F_0) = \sum_{k=1}^2 \sum_{i=1}^2 [A_{ki}^q \{P_{hi}\}_S + B_{ki}^q \{Q_{hi}\}_S]; \quad (5-2-47)$$

$$\theta_S(\mu, F_0) = \sum_{k=1}^2 \sum_{i=1}^2 [A_{ki}^m \{P_{hi}\}_S + B_{ki}^m \{Q_{hi}\}_S]. \quad (5-2-48)$$

Constant coefficients  $A_{hi}$  and  $B_{hi}$  in equations (5-2-47) and (5-2-48) are defined by (5-2-11) and their values are listed in Tables 5-1 and 5-2 and plotted in Figures 5-1 through 5-6. Quantities  $\{P_{hi}\}_S$  and  $\{Q_{hi}\}_S$  are defined by the corresponding expressions in (5-2-12), but with  $(-1)^n$  replaced by  $\frac{\sin \mu}{\mu}$ .

The inverse Fourier sine transformation is given by the formula

$$\varphi(X, \text{Fo}) = 3\{\varphi(0, \text{Fo})_S + 2 \sum_{n=1}^{\infty} \frac{\mu_n}{\sin^2 \mu_n} \cdot \frac{\sin \mu_n X}{X} \{\varphi(\mu_n, \text{Fo})\}_S. \quad (5-2-49)$$

When formula (5-2-49) is applied to (5-2-47) and (5-2-48), we finally obtain heat-transfer and mass-transfer potentials of

$$T(X, \text{Fo}) = \sum_{k=1}^2 \sum_{i=1}^2 (A_{ki}^q P_{ki} + B_{ki}^q Q_{ki}); \quad (5-2-50)$$

$$\Theta(X, \text{Fo}) = \sum_{k=1}^2 \sum_{i=1}^2 (A_{ki}^m P_{ki} + B_{ki}^m Q_{ki}), \quad (5-2-51)$$

where

$$P_{ki} = 3 \int_0^1 X^2 F_k(X) dX + 2 \sum_{n=1}^{\infty} \frac{\mu_n}{\sin^2 \mu_n} \frac{\sin \mu_n X}{X} \times \\ \times \exp(-\mu_n^2 \nu_i^2 \text{Lu Fo}) \int_0^1 X F_k(X) \frac{\sin \mu_n X}{\mu_n} dX; \quad (5-2-52)$$

$$Q_{ki} = 3 \int_0^{\text{Fo}} K i_k(\text{Fo}^*) d\text{Fo}^* + 2 \sum_{n=1}^{\infty} \frac{\sin \mu_n X}{X \sin \mu_n} \exp(-\mu_n^2 \nu_i^2 \text{Lu Fo}) \times \\ \times \int_0^{\text{Fo}} K i_k(\text{Fo}^*) \exp(\mu_n^2 \nu_i^2 \text{Lu Fo}^*) d\text{Fo}^*. \quad (5-2-53)$$

Let us consider some particular instances of solutions (5-2-50) and (5-2-51).

1.  $F_1(X) = F_2(X) = 0$

$$T(X, \text{Fo}) = \sum_{k=1}^2 \sum_{i=1}^2 B_{ki}^q Q_{ki};$$

$$\Theta(X, \text{Fo}) = \sum_{k=1}^2 \sum_{i=1}^2 B_{ki}^m Q_{ki}.$$

2.  $Kl_q$  and  $Kl_m$  constant

$$T(X, \text{Fo}) = 3(Kl_q - Ko^* \text{Lu } Kl_m) \text{Fo} - \frac{1}{10}(3 - 5X^2) Kl_q + \sum_{k=1}^2 \sum_{i=1}^2 A_{ki}^q P_{ki} - \\ - \frac{1}{\text{Lu}} \sum_{n=1}^{\infty} \left\{ \frac{2}{\mu_n^2 \sin \mu_n} \frac{\sin \mu_n X}{X} \sum_{k=1}^2 \sum_{i=1}^2 Kl_k \frac{B_{ki}^q}{\nu_i^2} \exp(-\mu_n^2 \nu_i^2 \text{Lu Fo}) \right\}. \quad (5-2-54)$$

$$\Theta(X, \text{Fo}) = 3Kl_m \text{Lu Fo} - \frac{1}{10}(3 - 5X^2)(Pn Kl_q + Kl_m) + \sum_{k=1}^2 \sum_{i=1}^2 A_{ki}^m P_{ki} - \\ - \frac{1}{\text{Lu}} \sum_{n=1}^{\infty} \left\{ \frac{2}{\mu_n^2 \sin \mu_n} \frac{\sin \mu_n X}{X} \sum_{k=1}^2 \sum_{i=1}^2 Kl_k \frac{B_{ki}^m}{\nu_i^2} \exp(-\mu_n^2 \nu_i^2 \text{Lu Fo}) \right\}. \quad (5-2-55)$$

To derive (5-2-54) and (5-2-55), we used the identity

$$\sum_{n=1}^{\infty} \frac{2}{\mu_n \sin \mu_n} \cdot \frac{\sin \mu_n X}{\mu_n X} = -\frac{1}{10}(3 - 5X^2).$$

3.  $F_1(X) = F_2(X) = 0$ ,  $Kl_q$  and  $Kl_m$  constant

$$\begin{aligned} T(X, Fo) &= 3(Kl_q - Ko^* Lu Kl_m) Fo - \frac{1}{10} (3 - 5X^2) Kl_q - \\ &- \sum_{n=1}^{\infty} \sum_{i=1}^2 \frac{2}{\mu_n^2 \sin \mu_n} C_i^q \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 \nu_i^2 Lu Fo), \\ \Theta(X, Fo) &= 3Kl_m Lu Fo - \frac{1}{10} (3 - 5X^2) (Pn Kl_q + Kl_m) - \\ &- \sum_{n=1}^{\infty} \sum_{i=1}^2 \frac{2}{\mu_n^2 \sin \mu_n} C_i^m \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 \nu_i^2 Lu Fo), \end{aligned}$$

where  $C_i^q$  and  $C_i^m$  are defined by relations (5-2-22) and their values are listed in Tables 5-3 through 5-5.

4.  $Pn = 0$

$$\begin{aligned} T(X, Fo) &= 3 \left\{ \int_0^1 X^2 F_1(X) dX + \int_0^{Fo} [Kl_q (Fo^*) - Ko^* Lu Kl_m (Fo^*)] dFo^* \right\} + \\ &+ \sum_{n=1}^{\infty} \frac{2}{\sin^2 \mu_n} \frac{\sin \mu_n X}{X} \left\{ \exp(-\mu_n^2 Fo) \int_0^1 X F_1(X) \sin \mu_n X dX + \right. \\ &+ \frac{Ko^* Lu}{Lu - 1} [\exp(-\mu_n^2 Fo) - \exp(-\mu_n^2 Lu Fo)] \int_0^1 X F_2(X) \sin \mu_n X dX \Big\} + \\ &+ \sum_{n=1}^{\infty} \frac{2}{\sin \mu_n} \frac{\sin \mu_n X}{X} \left\{ \exp(-\mu_n^2 Fo) \int_0^{Fo} [Kl_q (Fo^*) + \right. \\ &\quad \left. + \frac{Ko^* Lu}{Lu - 1} Kl_m (Fo^*)] \exp(\mu_n^2 Fo^*) dFo^* - \right. \\ &\quad \left. - \frac{Ko^* Lu^2}{Lu - 1} \exp(-\mu_n^2 Lu Fo) \int_0^{Fo} Kl_m (Fo^*) \exp(\mu_n^2 Lu Fo^*) dFo^* \right\}; \\ \Theta(X, Fo) &= 3 \left[ \int_0^1 X^2 F_2(X) dX + Lu \int_0^{Fo} Kl_m (Fo^*) dFo^* \right] + \\ &+ \sum_{n=1}^{\infty} \frac{2}{\sin^2 \mu_n} \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 Lu Fo) \left[ \int_0^1 X F_2(X) \sin \mu_n X dX + \right. \\ &\quad \left. + Lu \sin \mu_n \int_0^{Fo} Kl_m (Fo^*) \exp(\mu_n^2 Lu Fo^*) dFo^* \right]. \end{aligned}$$

5.  $Pn = 0$ ;  $F_1(X) = F_2(X) = 0$ ;  $Kl_q$  and  $Kl_m$  constant

$$\begin{aligned} T(X, Fo) &= 3(Kl_q - Ko^* Lu Kl_m) Fo - \frac{1}{10} (3 - 5X^2) Kl_q - \\ &- \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \sin \mu_n} \frac{\sin \mu_n X}{X} \left\{ Kl_q \exp(-\mu_n^2 Fo) + \right. \\ &\quad \left. + \frac{Ko^* Lu Kl_m}{Lu - 1} [\exp(-\mu_n^2 Fo) - \exp(-\mu_n^2 Lu Fo)] \right\}; \\ \Theta(X, Fo) &= Kl_m \left[ 3 Lu Fo - \frac{1}{10} (3 - 5X^2) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \sin \mu_n} \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 Lu Fo) \right]. \end{aligned}$$

6.  $Pn = 0$ ;  $Ko^* = 0$ ;  $F_1(X) = F_2(X) = 0$ ;  $Kl_q$  and  $Kl_m$  constant

$$T(X, Fo) = Kl_q \left[ 3 Fo - \frac{1}{10} (3 - 5X^2) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \sin \mu_n} \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 Fo) \right];$$

$$\Theta(X, Fo) = Kl_m \left[ 3LuFo - \frac{1}{10}(3 - 5X^2) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \sin \mu_n} \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 LuFo) \right].$$

For low values of the Fourier number it is more convenient to use the following solution for ( $X \neq 0$ ):

$$T(X, Fo) = \frac{Kl_q}{X} \left[ \exp(Fo - 1 + X) \operatorname{erfc} \left( \frac{1-X}{2\sqrt{Fo}} - \sqrt{Fo} \right) - \operatorname{erfc} \frac{1-X}{2\sqrt{Fo}} \right];$$

$$\Theta(X, Fo) = \frac{Kl_m}{X} \left[ \exp(LuFo - 1 + X) \operatorname{erfc} \left( \frac{1-X}{2\sqrt{LuFo}} - \sqrt{LuFo} \right) - \operatorname{erfc} \frac{1-X}{2\sqrt{LuFo}} \right]$$

and the following solution for  $X = 0$ :

$$T(X, Fo) \approx Kl_q \left\{ [\exp(Fo - 1)] \operatorname{erfc} \left( \frac{1}{2\sqrt{Fo}} - \sqrt{Fo} \right) \right\};$$

$$\Theta(X, Fo) \approx Kl_m \left\{ \exp(LuFo - 1) \operatorname{erfc} \left( \frac{1}{2\sqrt{LuFo}} - \sqrt{LuFo} \right) \right\}.$$

The transfer-potential distributions as functions of the dimensionless coordinate, for various values of the Fourier numbers ( $Fo_q$  and  $Fo_m$ ), are shown in Figure 5-10.

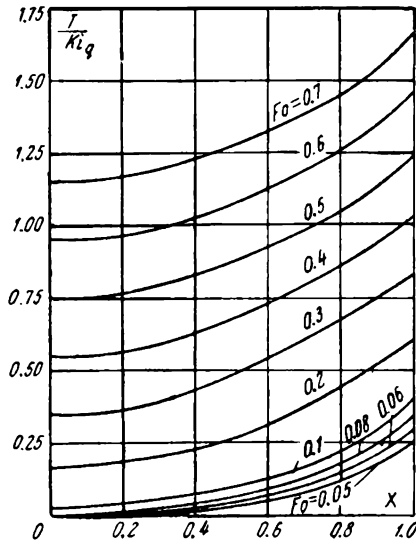


FIGURE 5-10. Temperature field for a sphere with constant thermal-flux density at the surface

#### d) Analysis of the solutions

Let us now analyze the solutions obtained in this section. In general, as shown by equations (5-2-14) and (5-2-15), (5-2-38) and (5-2-39), (5-2-50) and (5-2-51), the dimensionless transfer potentials depend on two groups of terms. The first group,  $A_{hi}P_{hi}$ , determines the effect of the initial distributions of  $T$  and  $\Theta$  on their unsteady fields; in this case the initial distributions of temperature and mass-transfer potential affect each type of field. The second group of terms,  $B_{hi}Q_{hi}$ , takes into account the effect of the specific heat

flux ( $Ki_q$ ) and mass flux ( $Ki_m$ ) at the surface of the body. The special properties of the material itself are accounted for by the dimensionless numbers  $Lu$ ,  $Ko^*Pn$ , and  $Pn$  (which enter into the expressions for  $A_{hi}$  and  $B_{hi}$ ) and also by  $v_i$  (which is contained in coefficients  $P_{hi}$  and  $Q_{hi}$ ).

When no thermal-gradient mass transfer exists in the material, the solutions are simplified considerably, especially for constant initial potential distributions and constant Kirpichev numbers. In the latter case the temperature in the body is determined by the superposition of three temperature fields ( $T=T_1+T_2+T_3$ ). The field ( $T_1$ ) corresponds to heating of the material without taking into account the effects of mass exchange and phase transitions. The second field ( $T_2$ ) takes into account the effect of heat sinks (or sources) which are at rest. Finally, the third field ( $T_3$ ) characterizes the effect of moving heat sinks, coupled to the mass transfer. For an infinite plate, for example,

$$T_1 = Ki_q \left[ Fo - \frac{1}{6} (1 - 3X^2) - \sum_{n=1}^{\infty} (-1)^n \frac{2}{\mu_n^2} \cos \mu_n X \exp(-\mu_n^2 Fo) \right];$$

$$T_2 = -Ko^*LuKi_m \left[ Fo + \frac{1}{Lu-1} \sum_{n=1}^{\infty} (-1)^n \frac{2}{\mu_n^2} \cos \mu_n X \exp(-\mu_n^2 Fo) \right];$$

$$T_3 = \frac{Ko^*LuKi_m}{Lu-1} \sum_{n=1}^{\infty} (-1)^n \frac{2}{\mu_n^2} \cos \mu_n X \exp(-\mu_n^2 Lu Fo).$$

If  $Ko^* = 0$  (no phase transitions in the material), then we obtain the familiar classical solution for the heat-conduction or diffusion equation. The difference between the first and second fields is formally only a difference in

Fourier numbers. If we set  $Fo_q = Fo = \frac{Fom}{Lu}$  in the corresponding expressions, then these fields become completely the same.

The infinite series entering into the solutions converge rapidly. Thus, if appropriate tables and graphs of the characteristic-equation roots and coefficients are available, the solutions are quite convenient for engineering calculations.

### 5-3. Generalized Boundary Conditions of the Second Kind

Let us now consider several solutions of problems for which the boundary conditions are more general than those in the previous section. System of equations (4-1-2) and (4-1-3) will be solved with boundary conditions (5-1-4) and (5-1-2), the rest of the limiting conditions remaining as before. Since the method for solving such problems does not differ from the solution methods in the previous section, only the final results in a generalized form will be given.

The generalized solutions of system of equations (4-1-2) and (4-1-3) for the above limiting conditions are

$$T(X, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 (A_{hi}^k P_{hi} + B_{hi}^k Q_{hi}); \quad (5-3-1)$$

$$\theta(X, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 (A_{ki}^m P_{ki} + B_{ki}^m Q_{ki}), \quad (5-3-2)$$

where  $A_{ki}^q, B_{ki}^q, A_{ki}^m, B_{ki}^m$  are defined by the respective expressions of (5-2-11);

$$\begin{aligned} B_{2i}^q &= (-1)^i \frac{\epsilon K_o Lu v_i^2 + (v_i^2 - 1) N}{v_1^2 - v_2^2} = (-1)^i \frac{(v_i^2 - 1 + \epsilon) K_o Lu}{v_1^2 - v_2^2}; \\ B_{2i}^m &= (-1)^i \frac{Lu(1/Lu - v_i^2) - P_n N}{v_1^2 - v_2^2} = (-1)^i \frac{Lu(1/Lu - v_i^2) - (1 - \epsilon) K_o P_n Lu}{v_1^2 - v_2^2}; \\ N &= (1 - \epsilon) Lu K_o; \\ P_{hi} &= (\Gamma + 1) \int_0^1 X^\Gamma F_h(X) dX + 2 \sum_{n=1}^{\infty} C_n^P \Phi_n(\mu_n X) \exp(-\mu_n^2 v_i^2 Lu Fo) \times \\ &\quad \times \int_0^1 X^\Gamma F_h(X) \Phi_n(\mu_n X) dX; \\ Q_{hi} &= (\Gamma + 1) \int_0^{Fo} Ki_h(Fo^*) dFo^* + 2 \sum_{n=1}^{\infty} C_n^Q \Phi_n(\mu_n X) \exp(-\mu_n^2 v_i^2 Lu Fo) \times \\ &\quad \times \int_0^{Fo} Ki_h(Fo^*) \exp(\mu_n^2 v_i^2 Lu Fo^*) dFo^*. \end{aligned}$$

For an infinite plate ( $\Gamma=0$ )

$$C_0^P = 1; C_0^Q = (-1)^n; \Phi_0(\mu_n X) = \cos \mu_n X; \mu_n = n\pi (n=1, 2, \dots).$$

For an infinite cylinder ( $\Gamma=1$ )

$$C_1^P = \frac{1}{J_0^2(\mu_n)}; C_1^Q = \frac{1}{J_0(\mu_n)}; \Phi_1(\mu_n X) = J_0(\mu_n X);$$

where  $\mu_n$  are the roots of characteristic equation  $J_1(\mu)=0$ .

For a sphere ( $\Gamma=2$ )

$$C_2^P = \frac{\mu_n^2}{\sin^2 \mu_n}; C_2^Q = \frac{\mu_n}{\sin \mu_n}; \Phi_2(\mu_n X) = \frac{\sin \mu_n X}{\mu_n X};$$

where  $\mu_n$  are the roots of characteristic equation  $\tan \mu = \mu$ .

Let us now consider some particular solutions of (5-3-1) and (5-3-2).

1.  $F_1(X) = F_2(X) = 0$

$$T(X, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 B_{ki}^q Q_{ki};$$

$$\theta(X, Fo) = \sum_{k=1}^2 \sum_{i=1}^2 B_{ki}^m Q_{ki}.$$

2.  $Ki_q$  and  $Ki_m$  constant,  $F_1(X) = F_2(X) = 0$

$$\begin{aligned} T(X, Fo) &= (\Gamma + 1)(Ki_q - Ko Lu Ki_m) Fo - \frac{1}{2} (X_r - X^2)(Ki_q - NKi_m) - \\ &\quad - \sum_{n=1}^{\infty} \sum_{i=1}^2 \Lambda_n C_n^Q \Phi_n(\mu_n X) \exp(-\mu_n^2 v_i^2 Lu Fo); \\ \theta(X, Fo) &= (\Gamma + 1) Ki_m Lu Fo - \frac{1}{2} (X_r - X^2) [P_n Ki_q + (1 - P_n N) Ki_m] - \\ &\quad - \sum_{n=1}^{\infty} \sum_{i=1}^2 \Lambda_n C_n^m \Phi_n(\mu_n X) \exp(-\mu_n^2 v_i^2 Lu Fo), \end{aligned}$$



where

$$C_1^q = \frac{(Kl_q - N Kl_m) v_2^2 - \frac{1}{Lu} (Kl_q - Ko Lu Kl_m)}{v_2^2 - v_1^2};$$

$$C_2^q = - \frac{(Kl_q - N Kl_m) v_1^2 - \frac{1}{Lu} (Kl_q - Ko Lu Kl_m)}{v_2^2 - v_1^2};$$

$$C_1^m = \frac{[Pn Kl_q + (1 - N Pn) Kl_m] v_2^2 - Kl_m}{v_2^2 - v_1^2};$$

$$C_2^m = - \frac{[Pn Kl_q + (1 - N Pn) Kl_m] v_1^2 - Kl_m}{v_2^2 - v_1^2}.$$

For an infinite plate

$$\Lambda_0 = (-1)^n \frac{2}{\mu_n^2} \text{ and } \chi_0 = 1/4.$$

For an infinite cylinder

$$\Lambda_1 = \frac{2}{\mu_n^2 J_0(\mu_n)} \text{ and } \chi_1 = 1/4.$$

For a sphere

$$\Lambda_2 = \frac{2}{\mu_n \sin \mu_n} \text{ and } \chi_2 = 1/4.$$

3. No thermal mass transfer ( $Pn=0$ )

$$T(X, Fo) = (\Gamma + 1) \left\{ \int_0^1 X^r F_1(X) dX + \int_0^{Fo} [Ki_q(Fo^*) - Ko Lu Ki_m(Fo^*)] dFo^* \right\} +$$

$$+ 2 \sum_{n=1}^{\infty} C_r^p \Phi_r(\mu_n X) \left\{ \exp(-\mu_n^2 Fo) \int_0^1 X^r F_1(X) \Phi_r(\mu_n X) dX + \right.$$

$$+ \frac{Ko Lu}{Lu - 1} [\exp(-\mu_n^2 Fo) - \exp(-\mu_n^2 Lu Fo)] \int_0^1 X^r F_2(X) \Phi_r(\mu_n X) dX \Big\} +$$

$$+ 2 \sum_{n=1}^{\infty} C_r^q \Phi_r(\mu_n X) \left\{ \exp(-\mu_n^2 Fo) \int_0^{Fo} [Ki_q(Fo^*) - \right.$$

$$- Ko Lu Ki_m(Fo^*)] \exp(\mu_n^2 Fo^*) dFo^* + \frac{Ko Lu^2}{Lu - 1} \left[ \exp(-\mu_n^2 Fo) \int_0^{Fo} Ki_m(Fo^*) \times \right.$$

$$\times \exp(\mu_n^2 Fo^*) dFo^* - \exp(-\mu_n^2 Lu Fo) \int_0^{Fo} Ki_m(Fo^*) \exp(\mu_n^2 Lu Fo^*) dFo^* \Big\};$$

$$\theta(X, Fo) = (\Gamma + 1) \left[ \int_0^1 X^r F_2(X) dX + Lu \int_0^{Fo} Ki_m(Fo^*) dFo^* \right] +$$

$$+ 2 \sum_{n=1}^{\infty} C_r^p \Phi_r(\mu_n X) \exp(-\mu_n^2 Lu Fo) \left\{ \int_0^1 X^r F_2(X) \Phi_r(\mu_n X) dX + \right.$$

$$+ \frac{Lu}{C_r^q} \int_0^{Fo} Ki_m(Fo^*) \exp(\mu_n^2 Lu Fo^*) dFo^* \Big\}.$$

4. No thermal mass transfer, constant fluxes ( $Pn = 0$ ;  $Ki_q$  and  $Ki_m$  constant,  $F_1(X) = F_2(X) = 0$ )

$$\begin{aligned}
T(X, Fo) &= (\Gamma + 1)(Ki_q - Ko Lu Ki_m) Fo - \frac{1}{2}(\chi_r - X^2)[Ki_q - \\
&\quad - (1 - \varepsilon) Ko Lu Ki_m] - \sum_{n=1}^{\infty} \Lambda_r \Phi_r(\mu_n X) \left\{ Ki_q \exp(-\mu_n^2 Fo) + \right. \\
&\quad \left. + \frac{Ko Lu Ki_m}{Lu - 1} \left[ 1 - (1 - \varepsilon) Lu \right] \exp(-\mu_n^2 Fo) - \varepsilon \exp(-\mu_n^2 Lu Fo) \right\}; \\
\theta(X, Fo) &= Ki_m \left[ (\Gamma + 1) Lu Fo - \frac{1}{2}(\chi_r - X^2) - \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \Lambda_r \Phi_r(\mu_n X) \exp(-\mu_n^2 Lu Fo) \right].
\end{aligned}$$

5. No phase transitions or thermal mass transfer in the material, constant fluxes at the surface ( $Pn = 0$ ,  $\varepsilon = 0$ ,  $Ki_q$  and  $Ki_m$  constant,  $F_1(X) = F_2(X) = 0$ )

$$\begin{aligned}
T(X, Fo) &= (Ki_q - Ko Lu Ki_m) \left[ (\Gamma + 1) Fo - \frac{1}{2}(\chi_r - X^2) - \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \Lambda_r \Phi_r(\mu_n X) \exp(-\mu_n^2 Fo) \right]; \\
\theta(X, Fo) &= Ki_m \left[ (\Gamma + 1) Lu Fo - \frac{1}{2}(\chi_r - X^2) - \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \Lambda_r \Phi_r(\mu_n X) \exp(-\mu_n^2 Lu Fo) \right].
\end{aligned}$$

The dimensionless ratios

$$\frac{T(X, Fo)}{Ki_q - Ko Lu Ki_m} \quad \text{and} \quad \frac{\theta}{Ki_m}$$

may be determined from Figures 5-8 through 5-10 as functions of the dimensionless spatial coordinate, for various values of the Fourier number ( $Fo_q = Fo$  or  $Fo_m = Fo Lu$ ), for the above body shapes. In the figures the required quantity must be read off along the ordinate axis rather than  $T/Ki$ .

#### 5-4. A Method for Solving Systems of Heat-Transfer and Mass-Transfer Equations

The method known as d'Alembert's method is often used to solve the equations of heat conduction and diffusion, that is, the differential equations describing uncoupled transfer. For a linear system of ordinary differential equations with constant coefficients d'Alembert's method is very effective for finding the first integrals. P.S. Henry, and later Crank and Smirnov, have shown that a similar substitution can be used to simplify considerably the system of equations describing heat and mass transfer. System (4-1-2) and (4-1-3), for example, can be reduced to a system of two uncoupled equations like the heat-conduction equation, in which the compound variables  $Z_i$  take the place of the "potentials." These variables are linear combinations of  $T$  and  $\theta$ , so that  $Z_i = p_i T + q_i \theta$  (where  $p_i$  and  $q_i$  are constants). Similar transformations for solving the complicated systems of equations required to describe the kinetics of irreversible processes make it possible to use the large group of known solutions for "pure" heat or mass conduction to study diverse coupled transfer phenomena.

Let us now demonstrate the method using the transformation of system of equations (4-1-2) and (4-1-3) and boundary conditions (5-1-2) and (5-1-4) as an example. The system of equations describing molecular and macroscopic-molecular heat and mass transfer for a two-dimensional or three-dimensional body, with boundary conditions of the first and second kinds, may be transformed by similar means.

Given the system of equations

$$\frac{\partial T}{\partial Fo} = \frac{\partial^2 T}{\partial X^2} - \varepsilon Ko \frac{\partial \theta}{\partial Fo}, \quad (5-4-1)$$

$$\frac{\partial \theta}{\partial Fo} = Lu \frac{\partial^2 \theta}{\partial X^2} - Lu Pn \frac{\partial^2 T}{\partial X^2}, \quad (5-4-2)$$

we may determine  $\frac{\partial^2 T}{\partial X^2}$  from equation (5-4-1) and then substitute it into (5-4-2) to obtain

$$\begin{aligned} \frac{\partial^2 T}{\partial X^2} &= \frac{\partial T}{\partial Fo} + \varepsilon Ko \frac{\partial \theta}{\partial Fo}; \\ \frac{\partial^2 \theta}{\partial X^2} &= Pn \frac{\partial T}{\partial Fo} + \left( \frac{1}{Lu} + \varepsilon Ko Pn \right) \frac{\partial \theta}{\partial Fo}. \end{aligned}$$

Next, we multiply the first and second equations by the constants  $p$  and  $q$  respectively and then add them, to obtain

$$\frac{\partial^2}{\partial X^2} (pT + q\theta) = \frac{\partial}{\partial Fo} \left\{ (p + q Pn) T + \left[ p\varepsilon Ko + q \left( \frac{1}{Lu} + \varepsilon Ko Pn \right) \right] \theta \right\}. \quad (5-4-3)$$

Equation (5-4-3) has the form of a heat-conduction equation in  $Z = pT + q\theta$ , provided that

$$\frac{p + q Pn}{p} = \frac{p\varepsilon Ko + q \left( \frac{1}{Lu} + \varepsilon Ko Pn \right)}{q} = v^2. \quad (5-4-4)$$

It follows from (5-4-4) that the quantity  $v$  obeys the following biquadratic equation:

$$v^4 - \left( 1 + \varepsilon Ko Pn + \frac{1}{Lu} \right) v^2 + \frac{1}{Lu} = 0,$$

so that

$$v_i^2 = \frac{1}{2} \left[ \left( 1 + \varepsilon Ko Pn + \frac{1}{Lu} \right) + (-1)^i \sqrt{\left( 1 + \varepsilon Ko Pn + \frac{1}{Lu} \right)^2 - \frac{4}{Lu}} \right] \quad (i=1, 2). \quad (5-4-5)$$

Therefore, each value of  $v_i^2$  corresponds to certain values of  $p_i$  and  $q_i$ . Consequently, system of equations (5-4-1) and (5-4-2) can be rewritten as follows, taking into account (5-4-3) and (5-4-5):

$$\left. \begin{aligned} \frac{\partial Z_i}{\partial Fo} &= \frac{1}{v_i^2} \frac{\partial^2 Z_i}{\partial X^2}; \\ \frac{\partial Z_i}{\partial Fo} &= \frac{1}{v_i^2} \frac{\partial^2 Z_i}{\partial X^2}; \\ Z_i &= p_i T(X, Fo) + q_i \theta(X, Fo) \quad (i=1, 2), \end{aligned} \right\} \quad (5-4-6)$$

where  $1/v_i^2$  represents, from the physical aspect, a coefficient of potential conductivity (namely, the thermal diffusivity or the diffusion coefficient).

It is still necessary to determine the constant coefficients  $p_i$  and  $q_i$ . Since  $v_1^2$  and  $v_2^2$  are related by the formulas  $v_1^2 + v_2^2 = 1 + \epsilon \text{Ko Pn} + \frac{1}{\text{Lu}}$  and  $v_1^2 v_2^2 = \frac{1}{\text{Lu}}$ , two of the four equations given by (5-4-4) must be independent (two equations are obtained for  $v_1^2$  and two equations are obtained for  $v_2^2$ ). Consequently, two coefficients may be selected arbitrarily, for example, we may choose  $p_1 = q_2 = 1$ . Then, from (5-4-4) we obtain

$$\left. \begin{aligned} p_1 &= 1; \\ q_1 &= \frac{v_1^2 - 1}{\text{Pn}}; \\ p_2 &= \frac{\text{Pn}}{v_2^2 - 1}; \\ q_2 &= 1. \end{aligned} \right\} \quad (5-4-7)$$

The limiting conditions for this problem are

$$-\frac{\partial T(1, \text{Fo})}{\partial X} + \text{Ki}_q(\text{Fo}) - N \text{Ki}_m(\text{Fo}) = 0; \quad (5-4-8)$$

$$-\frac{\partial \theta(1, \text{Fo})}{\partial X} + \text{Pn} \frac{\partial T(1, \text{Fo})}{\partial X} + \text{Ki}_m(\text{Fo}) = 0; \quad (5-4-9)$$

$$T(X, 0) = F_1(X); \quad \theta(X, 0) = F_2(X); \quad (5-4-10)$$

$$\frac{\partial T(0, \text{Fo})}{\partial X} = \frac{\partial \theta(0, \text{Fo})}{\partial X} = 0, \quad (5-4-11)$$

where  $N = (1 - \epsilon) \text{Lu Ko}$ .

These conditions may be transformed in the same way as system (5-4-1) and (5-4-2) was transformed previously. From (5-4-8) we determine  $\frac{\partial T(1, \text{Fo})}{\partial X}$  and substitute it into (5-4-9). Then, by multiplying the heat-exchange boundary condition by the constant  $p$  and the mass-exchange boundary condition by the constant  $q$  and adding equations, we obtain, after regrouping terms,

$$\frac{\partial}{\partial X} Z(1, \text{Fo}) = (p + q \text{Pn}) \text{Ki}_q(\text{Fo}) - [pN - q(1 - N \text{Pn})] \text{Ki}_m(\text{Fo}). \quad (5-4-12)$$

In the derivation of (5-4-12) constants  $p$  and  $q$  remained arbitrary. Therefore, they may be chosen to match the values  $p_i$  and  $q_i$  of the system, that is,  $i=1$  corresponds to the heat-exchange boundary condition and  $i=2$  corresponds to the mass-exchange boundary condition.

It is easy to show that after transformation initial conditions (5-4-10) become

$$Z_i(X, 0) = p_i F_1(X) + q_i F_2(X) = \Phi_i(X), \quad (5-4-13)$$

while symmetry conditions (5-4-11) become

$$\frac{\partial Z_i(0, \text{Fo})}{\partial X} = 0. \quad (5-4-14)$$

Thus, as a result of transforming the system of coupled equations (5-4-1) and (5-4-2) and limiting conditions (5-4-8) and (5-4-11), by means of the new variable  $Z_i(X, \text{Fo}) = p_i T(X, \text{Fo}) + q_i \theta(X, \text{Fo})$ , we obtained two uncoupled equations of the type of the heat-conduction equation. The first ( $i=1$ ) and second ( $i=2$ ) equations together with their corresponding limiting

conditions may be written finally as

$$\begin{aligned}\frac{\partial Z_i}{\partial Fo} &= \frac{1}{v_i^2} \frac{\partial^2 Z_i}{\partial X^2}; \\ Z_i(X, 0) &= \Phi_i(X); \\ \frac{\partial Z_i(0, Fo)}{\partial X} &= 0; \\ \frac{\partial Z_i(1, Fo)}{\partial X} &= (p_i + q_i Pn) Ki_q(Fo) - [p_i N - q_i(1 - N Pn)] Ki_m(Fo),\end{aligned}$$

where  $p_i$  and  $q_i$  ( $i=1, 2$ ) are defined by (5-4-7).

Using the solutions of the transformed differential equations, it is easy to find final solutions for the transfer potentials  $T(X, Fo)$  and  $\Theta(X, Fo)$ . Let the solution of the first differential equation be  $\varphi_1(X, Fo)$  and that of the second be  $\varphi_2(X, Fo)$ :

$$\begin{aligned}T(X, Fo) + q_1 \Theta(X, Fo) &= \varphi_1(X, Fo); \\ \Theta(X, Fo) + p_2 T(X, Fo) &= \varphi_2(X, Fo).\end{aligned}$$

Then, if we take into account relations (5-4-7), this system of equations gives the answer to the problem at hand as

$$\left. \begin{aligned}T(X, Fo) &= -\frac{v_2^2 - 1}{v_1^2 - 1} \left[ \varphi_1(X, Fo) - \frac{v_1^2 - 1}{Pn} \varphi_2(X, Fo) \right]; \\ \Theta(X, Fo) &= \frac{v_2^2 - 1}{v_1^2 - 1} \left[ \frac{Pn}{v_2^2 - 1} \varphi_1(X, Fo) - \varphi_2(X, Fo) \right].\end{aligned} \right\} \quad (5-4-15)$$

When the initial differential equations (5-4-1) and (5-4-2) are transformed, they become somewhat similar to the equations describing two coupled oscillations. Therefore, Henry suggested that solutions (5-4-15) for these equations might be interpreted physically as describing a temperature "wave" accompanied by a diffusion (mass) "wave", the latter traveling at the same velocity as and having an amplitude proportional to the temperature wave. The relation between the two waves is determined only by the properties of the medium. Similarly, the diffusion "wave" is accompanied by an additional temperature "wave". Even though one of the external conditions (for example the mass-transfer potential) varies, still all the properties of the two mass waves and two temperature waves will be observed, although some of these properties may be very slight if the interaction is weak.

## 5-5. Unsteady Potential Fields for Uncoupled Transfer with Continuously Acting Sources

It was shown in the previous section that the system of equations describing heat and mass transfer may be reduced to a system of uncoupled equations for the combined potential  $Z$ . In relation to this, it is interesting to consider solutions of the following type of equations:

$$\frac{\partial Z(X, Fo)}{\partial Fo} = \frac{\partial^2 Z(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \cdot \frac{\partial Z(X, Fo)}{\partial X} + P_0(X, Fo), \quad (5-5-1)$$

for the initial condition

$$Z(X, 0) = F(X), \quad (5-5-2)$$

the boundary condition

$$-\frac{\partial Z(1, F_0)}{\partial X} + \text{Ki}(F_0) = 0 \quad (5-5-3)$$

and the symmetry condition

$$\frac{\partial Z(0, F_0)}{\partial X} = 0, \quad (5-5-4)$$

where  $Z = \frac{z - z_*}{z_*}$  is the dimensionless potential ( $z_*$  being a fixed value of the potential); and  $Po = \frac{R^2}{\lambda z_*} w$  is the Pomerantsev number, which represents the ratio of the amount of heat or mass released by the source  $w$  per unit time in the volume  $R$  (the volume of a parallelepiped with a base  $1 \text{ cm}^2$  and a height  $R$ ) to the maximum amount of heat or mass that can be transferred by heat or mass conduction through a unit of its surface per unit time.

The solution of equation (5-5-1) is also of independent interest, since it describes the numerous processes of pure heat conduction or mass conduction (diffusion) which can occur in the presence of heat or mass sources (or sinks). Such problems are encountered, to cite only a few examples, during the heating of a body by an electrical current (as well as during dielectric and induction heating), during the decay of a radioactive substance or the absorption of radiation, during various chemical reactions, and during the condensation or vaporization of liquid.

All problems involving sources may be divided into two types: 1) problems with constant or varying sources which are active throughout the whole process (continuously-acting heat and mass sources); and 2) problems with transient sources which are active only during an infinitesimal time interval. Examples of the latter are problems of the heat exchange in a conductor in which a short circuit has occurred, in which case the internal heat source acts practically instantaneously. In this section only the most representative problems with continuously-acting sources will be considered.

The strength of a heat or mass source, or the Pomerantsev number which characterizes this source strength, may in general depend not only on the spatial coordinates and the Fourier number but also on the dimensionless transfer potentials. The most characteristic problem of this type is diffusion in the presence of chemical transformations. In the analysis of this problem a Pomerantsev number which is independent of the potential  $Z$  will be used.

#### a) Solution for an infinite plate ( $\Gamma = 0$ )

Let us first apply Fourier cosine transformation (5-2-1) to equation (5-5-1); then, taking into account conditions (5-5-3) and (5-5-4), we obtain

$$\text{where} \quad \frac{dZ_0}{dF_0} + \mu^2 Z_0 = (-1)^n \text{Ki}(F_0) + Po(\mu, F_0),$$

$$Z_0 = Z(\mu, F_0) = \int_0^1 Z(X, F_0) \cos \mu X dX;$$

$$Po(\mu, F_0) = \int_0^1 Po(X, F_0) \cos \mu X dX.$$

$$\mu = n\pi \quad (n = 0, 1, 2, \dots).$$

The solution of this equation is

$$Z_0 = \exp(-\mu^2 Fo) \left\{ C(\mu) + \int_0^{Fo} [(-1)^n Ki(Fo^*) + Po(\mu, Fo^*)] \exp(\mu^2 Fo^*) dFo^* \right\}. \quad (5-5-5)$$

The constant  $C(\mu)$  may be determined from initial condition (5-5-2):

$$Z(\mu, 0) = C(\mu) = \int_0^1 F(X) \cos \mu X dX.$$

The inverse transform of (5-5-5) is found using formula (5-2-3). After the necessary transformations the final solution is

$$\begin{aligned} Z(X, Fo) = & \int_0^1 F(X) dX + \int_0^{Fo} [Ki(Fo^*) + \int_0^1 Po(X, Fo^*) dX] dFo^* + \\ & + 2 \sum_{n=1}^{\infty} \cos \mu_n X \exp(-\mu_n^2 Fo) \int_0^1 F(X) \cos \mu_n X dX + \\ & + 2 \sum_{n=1}^{\infty} \cos \mu_n X \int_0^{Fo} [(-1)^n Ki(Fo^*) + \int_0^1 Po(X, Fo^*) \cos \mu_n X dX] \exp[-\mu_n^2 (Fo - Fo^*)] dFo^*. \end{aligned} \quad (5-5-6)$$

In the case of zero initial transfer-potential distribution and constant flux at the surface of the material, that is,  $F(X) = 0$  and  $Ki(Fo)$  constant, solution (5-5-6) simplifies to

$$\begin{aligned} Z(X, Fo) = & Ki \left[ Fo - \frac{1}{6} (1 - 3X^2) - \sum_{n=1}^{\infty} (-1)^n \frac{2}{\mu_n^2} \cos \mu_n X \exp(-\mu_n^2 Fo) \right] + \\ & + \int_0^{Fo} dFo^* \int_0^1 Po(X, Fo^*) dX + 2 \sum_{n=1}^{\infty} \cos \mu_n X \exp(-\mu_n^2 Fo) \times \\ & \times \int_0^{Fo} \exp(\mu_n^2 Fo^*) dFo^* \int_0^1 Po(X, Fo^*) \cos \mu_n X dX. \end{aligned} \quad (5-5-7)$$

The first series in (5-5-7) represents the solution of the problem in the absence of sources ( $Po = 0$ ), namely solution (5-2-25). The second series indicates the effect on the dimensionless-potential distribution of a heat or mass source which varies in space and time.

From solutions (5-5-6) or (5-5-7) it is possible to obtain a number of particular solutions. Let us consider some of these solutions with the conditions that  $F(X) = 0$  and  $Ki$  is constant.

1. Constant source strength:  $Po(X, Fo) = Po_1$  is constant

where

$$Z(X, Fo) = Po_1 Fo + \Phi_0(X, Fo), \quad (5-5-8)$$

$$\Phi_0(X, Fo) = Ki \left[ Fo - \frac{1}{6} (1 - 3X^2) - \sum_{n=1}^{\infty} (-1)^n \frac{2}{\mu_n^2} \cos \mu_n X \exp(-\mu_n^2 Fo) \right],$$

represents the solution of the problem in the absence of a source.

2. The source strength is a linear function of the spatial coordinate:  $Po(X, Fo) = Po_1(1 - X)$

$$Z(X, Fo) = Po_1 \left\{ \frac{1}{2} Fo + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi X}{(2n-1)^2} \cos \mu_n X [1 - \exp(-\mu_n^2 Fo)] \right\} + \Phi_0(X, Fo). \quad (5-5-9)$$

3. The source strength is a quadratic function of the spatial coordinate  $Po(X, Fo) = Po_1(1 - X^2)$

$$Z(X, Fo) = Po_1 \left\{ \frac{2}{3} Fo - \sum_{n=1}^{\infty} (-1)^n \frac{4}{\mu_n^2} \cos \mu_n X [1 - \exp(-\mu_n^2 Fo)] \right\} + \Phi_0(X, Fo) \quad (5-5-10)$$

4. The source strength is an exponential function of the coordinate  $Po(X, Fo) = Po_1 \exp(-bX)$ .

$$Z(X, Fo) = Po_1 \left\{ \frac{1 - e^{-b}}{b} Fo + \sum_{n=1}^{\infty} [1 - (-1)^n e^{-b}] \right\} \times \\ \times \frac{2b}{\mu_n^2(b^2 + \mu_n^2)} \cos \mu_n X [1 - \exp(-\mu_n^2 Fo)] + \Phi_0(X, Fo) \quad (5-5-11)$$

The general nature of the distribution of the potential  $Z(X, Fo)$  in a symmetrical infinite plate when the source strength is an exponential function of the spatial coordinate is evident from Figure 5-11.

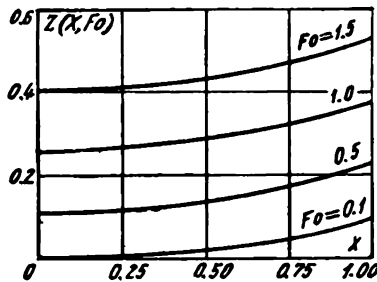


FIGURE 5-11. The distribution of potential  $Z(X, Fo)$  in an infinite plate, when the source strength is an exponential function of the coordinate (for  $Po_1 = 0.1$  and  $b = 0.1$ )

It should be noted that, qualitatively, the potential distribution does not change its form for the various types of relations between source strength and coordinate. This is due to the very rapid convergence of the infinite sums entering into expressions (5-5-9) and (5-5-11). Even at  $Fo = 0.1$  these sums have essentially no effect on the value of  $Z(X, Fo)$ . The differences which are observed in the potential are determined by the first term, which shifts the initial reference point of the curve but does not change its shape. If, for example, we assume that  $b = 0.1$ , then for the above problems one through four the initial reference point is given by the following relations:

for  $Po_1 = 0.1$

1)  $0.1 Fo$ ; 2)  $0.05 Fo$ ; 3)  $0.07 Fo$ ; 4)  $0.1 Fo$ ;

for  $Po_1 = 1.0$

1)  $1.0 Fo$ ; 2)  $0.5 Fo$ ; 3)  $0.7 Fo$ ; 4)  $1.0 Fo$ .

5. The source strength is a linear function of time:  $Po(X, Fo) = Po_1(1 + Pd'Fo)$

$$Z(X, Fo) = Po_1 Fo \left( 1 + \frac{1}{2} Pd' Fo \right) + \Phi_0(X, Fo) \quad (5-5-12)$$

Here  $Pd' = \frac{k}{a} R^2$  is the Predvoditelev number, which may be defined as the maximum variation rate of the relative specific source strength, with respect to the Fourier number:

$$-\left[ \frac{d(w/w_0)}{dFo} \right]_{\max} = \frac{k}{a} R^2 = Pd',$$

where  $k$  is a constant which is numerically equal to the maximum relative variation rate of the specific source strength.



6. The source strength is an exponential function of time:  $Po(X, Fo) = Po_1 \exp(-Pd' Fo)$

$$Z(X, Fo) = \frac{Po_1}{Pd'} [1 - \exp(-Pd' Fo)] + \Phi_0(X, Fo). \quad (5-5-13)$$

7. The source strength is a periodical function of time:  $Po(X, Fo) = Po_1 \cos Pd' Fo$

$$Z(X, Fo) = \frac{Po_1}{Pd'} \sin Pd' Fo + \Phi_0(X, Fo). \quad (5-5-14)$$

8. The sources strength varies with the  $n$ th power of the time:  $Po(X, Fo) = Po_1 (Pd' Fo)^n$

$$Z(X, Fo) = \frac{(Pd' Fo)^n}{n+1} Po_1 Fo + \Phi_0(X, Fo). \quad (5-5-15)$$

The effect which the source strengths, depending on time and on the Predvoditelev number in various ways, have on the distribution of the dimensionless potential  $Z(X, Fo)$  is shown in Figure 5-12.

#### b) Solution for an infinite cylinder ( $\Gamma=1$ )

If we apply Hankel transformation (5-2-27) to equation (5-5-1), then taking into account conditions (5-5-3) and (5-5-4) we obtain

$$\frac{dZ_H}{dFo} - \mu^2 Z_H = J_0(\mu) Ki(Fo) + Po(\mu, Fo), \quad (5-5-16)$$

where

$$Z_H = \int_0^1 X Z(X, Fo) J_0(\mu X) dX \quad \text{and} \quad Po(\mu, Fo) = \int_0^1 X Po(X, Fo) J_0(\mu X) dX,$$

$\mu$  being the roots of the characteristic equation  $J_1(\mu_n) = 0$ . Equation (5-5-16) may be solved using formula (5-2-28). After evaluating the integration constant, we have

$$\begin{aligned} Z(X, Fo) = & 2 \left\{ \int_0^1 X F(X) dX + \int_0^{Fo} \left[ Ki(Fo^*) + \int_0^1 X Po(X, Fo) dX \right] dFo^* \right\} + \\ & + 2 \sum_{n=1}^{\infty} \frac{J_0(\mu_n X)}{J_0^2(\mu_n)} \exp(-\mu_n^2 Fo) \int_0^1 X F(X) J_0(\mu_n X) dX + 2 \sum_{n=1}^{\infty} \frac{J_0(\mu_n X)}{J_0^2(\mu_n)} \int_0^{Fo} \left[ J_0(\mu) Ki(Fo^*) + \right. \\ & \left. + \int_0^1 X Po(X, Fo^*) J_0(\mu X) dX \right] \exp[-\mu_n^2 (Fo - Fo^*)] dFo^*. \end{aligned} \quad (5-5-17)$$

For zero initial potential distribution and for a constant Kirpichev number solution (5-5-17) simplifies to

$$\begin{aligned} Z(X, Fo) = & Ki \left[ 2Fo - \frac{1}{4} (1 - 2X^2) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 J_0(\mu_n)} J_0(\mu_n X) \exp(-\mu_n^2 Fo) \right] + \\ & + 2 \int_0^{Fo} dFo^* \int_0^1 X Po(X, Fo^*) dX + \sum_{n=1}^{\infty} \frac{2}{J_0^2(\mu_n)} \exp(-\mu_n^2 Fo) \times \\ & \times \int_0^{Fo} \exp(\mu_n^2 Fo^*) dFo^* \int_0^1 X Po(X, Fo^*) J_0(\mu X) dX. \end{aligned} \quad (5-5-18)$$

When the Pomerantsev number is zero, we obtain the previously derived solution for the potential distribution in the absence of sources. If we denote this distribution by  $\Phi_1(X, Fo)$ , then we have

$$\Phi_1(X, Fo) = Ki \left[ 2Fo - \frac{1}{4} (1 - 2X^2) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 J_0(\mu_n)} J_0(\mu_n X) \exp(-\mu_n^2 Fo) \right].$$

Let us find the solutions for some particular cases, for zero initial conditions and constant  $Ki$ .

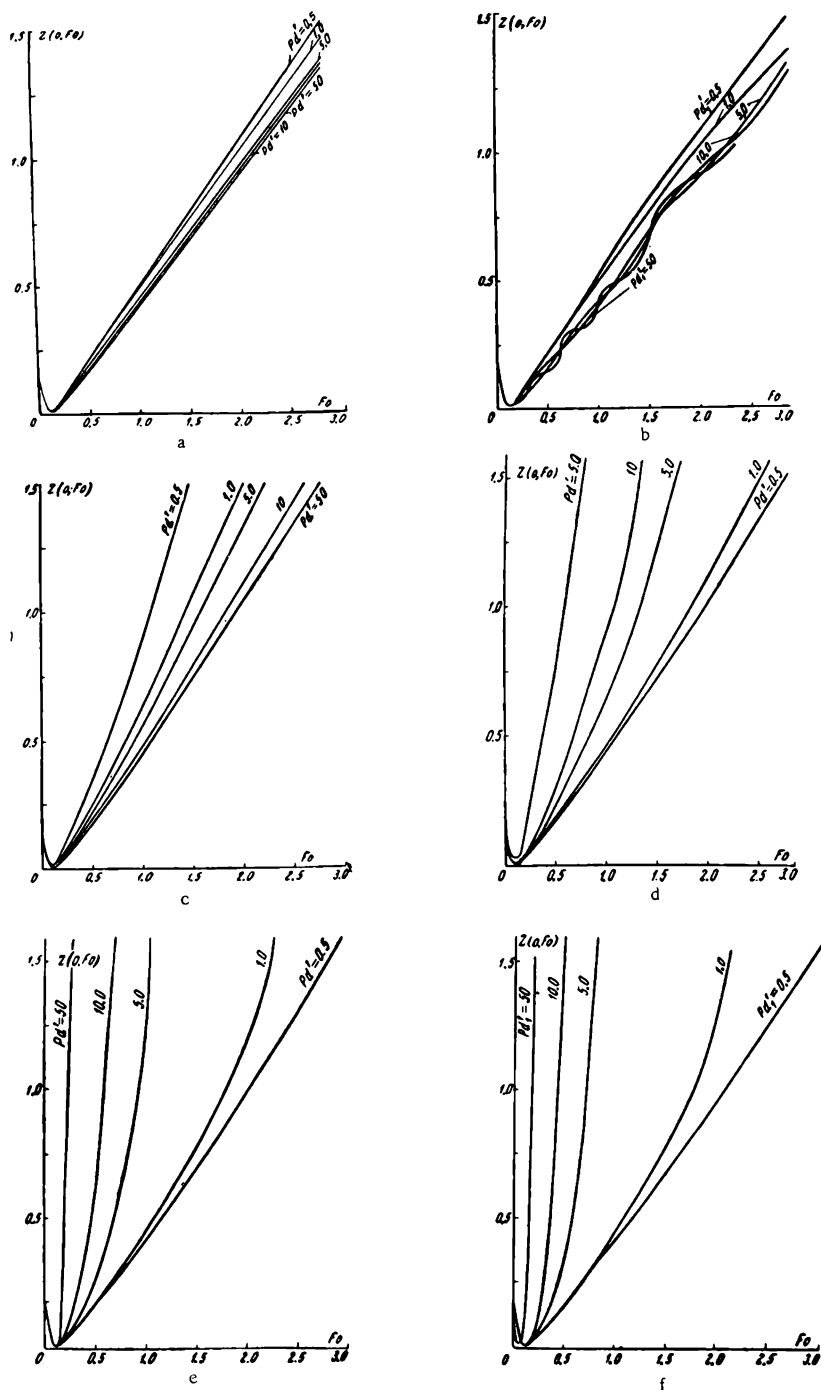


FIGURE 5-12. Distributions of the potential  $Z(X, Fo)$  in an infinite plate, for different laws of source-strength variation with time ( $Kl = 0.5$  and  $Po_1 = 0.1$ )

a)  $Po = Po_1 e^{-Pd'Fo}$ ; b)  $Po = Po_1 \cos Pd'Fo$ ; c)  $Po = Po_1 \sqrt{Pd'Fo}$ ; d)  $Po = Po_1 \cdot Pd'Fo$ ;  
e)  $Po = Po_1 (Pd'Fo)^2$ ; f)  $Po = Po_1 (Pd'Fo)^3$ .

1. Constant source:  $Po(X, Fo) = Po_1$  is constant

$$Z(X, Fo) = Po_1 Fo + \Phi_1(X Fo). \quad (5-5-19)$$

The unsteady-state transfer-potential field for a constant source distribution in an infinite cylinder is shown in Figure 5-13, a.

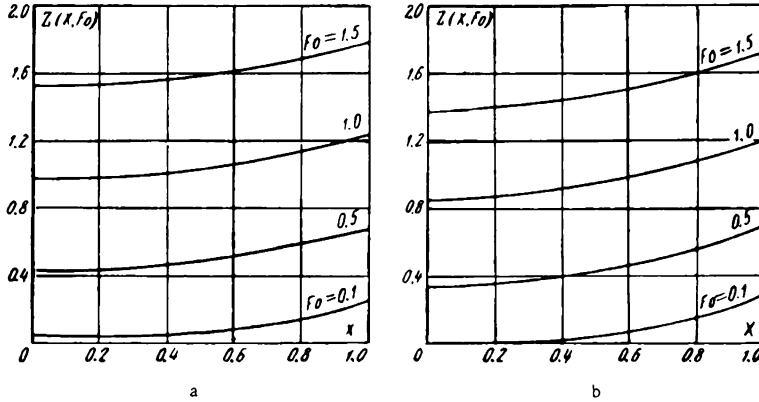


FIGURE 5-13. Distribution of the potential  $Z(X, Fo)$  in an infinite cylinder  
a)  $Po$  constant; b)  $Po = Po_1(1-X^2)$ .

2. The source is a quadratic function of the spatial coordinate:  $Po(X, Fo) = Po_1(1-X^2)$

$$Z(X, Fo) = Po_1 \left\{ \frac{1}{3} Fo + \sum_{n=1}^{\infty} \frac{4[2J_1(\mu_n) - J_0(\mu_n)]}{\mu_n^3 J_0^2(\mu_n)} J_0(\mu_n X) [1 - \exp(-\mu_n^2 Fo)] \right\} + \Phi_1(X, Fo). \quad (5-5-20)$$

Curves of the potential  $Z$  as function of  $X$ , for various values of  $Fo$ , are shown in Figure 5-13, b.

3. The source is a linear function of time:  $Po(X, Fo) = Po_1(1 + Pd' Fo)$

$$Z(X, Fo) = Po_1 Fo \left( 1 + \frac{1}{2} Pd' Fo \right) + \Phi_1(X, Fo). \quad (5-5-21)$$

4. The source is an exponential function of time:  $Po(X, Fo) = Po_1 \exp(-Pd' Fo)$

$$Z(X, Fo) = \frac{Po_1}{Pd'} [1 - \exp(-Pd' Fo)] + \Phi_1(X, Fo). \quad (5-5-22)$$

5. The source is a periodic function of time:  $Po(X, Fo) = Po_1 \cos Pd' Fo$

$$Z(X, Fo) = \frac{Po_1}{Pd'} \sin Pd' Fo + \Phi_1(X, Fo). \quad (5-5-23)$$

6. The source varies with the  $n$ th power of the time:  $Po(X, Fo) = Po_1 (Pd' Fo)^n$

$$Z(X, Fo) = \frac{(Pd' Fo)^n}{n+1} Po_1 Fo + \Phi_1(X, Fo). \quad (5-5-24)$$

The distributions of the potential  $Z(X, Fo)$  at the center of an infinite cylinder, for  $Po = Po_1 (Pd' Fo)^2$ , are shown in Figure 5-14.

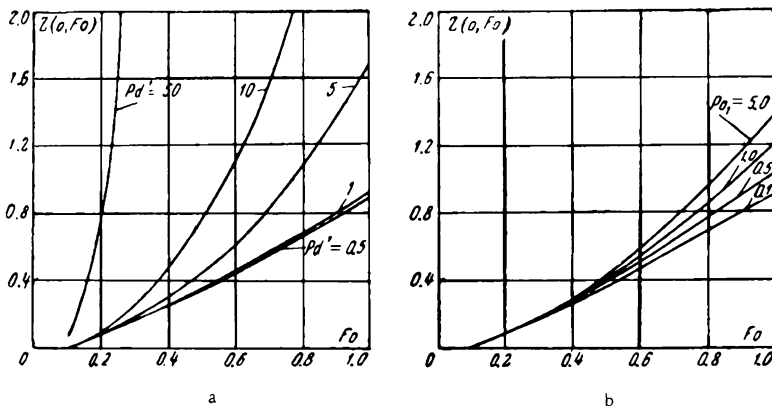


FIGURE 5-14. Distribution of the potential  $Z(X, Fo)$  along the cylinder axis, for  $Po = Po_1(Pd'Fo)^2$

a)  $Z = f(Pd')$ , for  $Po_1 = 0.1$ ; b)  $Z = f(Po_1)$ , for  $Pd' = 1.0$

### c) Solution for a sphere ( $\Gamma = 2$ )

If we apply Fourier sine transformation (5-2-44) to equation (5-1-5), then taking into account conditions (5-5-3) and (5-5-4) we obtain

$$\frac{dZ_S}{dFo} + \mu^2 Z_S = \frac{\sin \mu}{\mu} Kl(Fo) + Po(\mu, Fo), \quad (5-5-25)$$

where

$$Z_S = \int_0^1 X Z(X, Fo) \frac{\sin \mu X}{\mu} dX, \quad Po(\mu, Fo) = \int_0^1 X Po(X, Fo) \frac{\sin \mu X}{\mu} dX,$$

$\mu_n$  being the roots of the characteristic equation  $\tan \mu = \mu$ .

Equation (5-5-25) may be solved using formula (5-2-49). After evaluating the integration constant and forming the necessary transformations, we obtain

$$\begin{aligned} Z(X, Fo) = & 3 \left\{ \int_0^1 X^2 F(X) dX + \int_0^{Fo} [Kl(Fo^*) + \int_0^1 X^2 Po(X, Fo^*) dX] dFo^* \right\} + \\ & + 2 \sum_{n=1}^{\infty} \frac{\mu_n}{\sin^2 \mu_n} \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 Fo) \int_0^1 X F(X) \frac{\sin \mu_n X}{X} dX + \\ & + \sum_{n=1}^{\infty} \frac{2}{\sin^2 \mu_n} \frac{\sin \mu_n X}{X} \int_0^{Fo} \left[ \sin \mu_n Kl(Fo^*) + \int_0^1 X Po(X, Fo^*) \sin \mu_n X dX \right] \times \\ & \times \exp[-\mu_n^2 (Fo - Fo^*)] dFo^*. \end{aligned} \quad (5-5-26)$$

For zero initial distribution of the potential  $Z$  and a constant Kirpichev number, solution (5-5-26) simplifies to

$$\begin{aligned} Z(X, Fo) = & Kl \left[ 3Fo - \frac{1}{10} (3 - 5X^2) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \sin \mu_n} \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 Fo) \right] + \\ & + 3 \int_0^{Fo} dFo^* \int_0^1 X^2 Po(X, Fo^*) dX + \sum_{n=1}^{\infty} \frac{2}{\sin^2 \mu_n} \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 Fo) \times \\ & \times \int_0^{Fo} \exp(\mu_n^2 Fo^*) dFo^* \int_0^1 X Po(X, Fo^*) \sin \mu_n X dX. \end{aligned} \quad (5-5-27)$$

If we set  $P_0=0$  in (5-5-27), then we obtain the solution of the problem in the absence of sources (see § 5-2, c, problem 6).

Let us find some particular solutions of (5-5-27). For convenience of notation, we write (as in the previous problems in this section) the known solution of the problem without sources as  $\Phi_1(X, Fo)$ , where in this case ( $\Gamma=2$ ). Thus

$$\Phi_2(X, Fo) = Kt \left[ 3Fo - \frac{1}{10} (3 - 5X^2) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \sin \mu_n} \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 Fo) \right].$$

1. Constant source strength:  $P_0(X, Fo) = P_0$  is constant

$$Z(X, Fo) = P_0 Fo + \Phi_2(X, Fo). \quad (5-5-28)$$

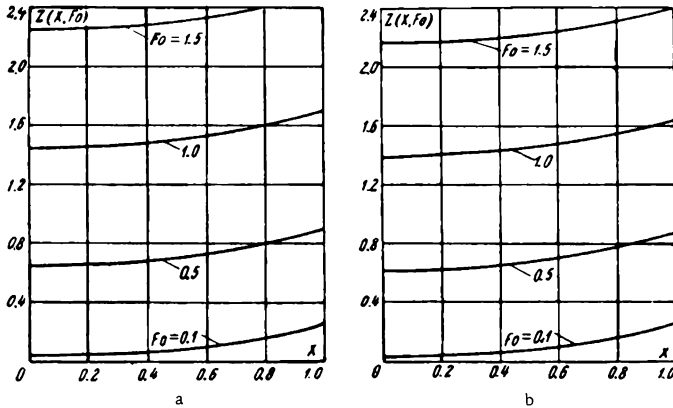


FIGURE 5-15. Distributions of the potential  $Z(X, Fo)$  in a sphere

a)  $Z=f(Pd')$ , for  $P_0=0.1$ ; b)  $Z=f(P_0)$ , for  $Pd'=1.0$

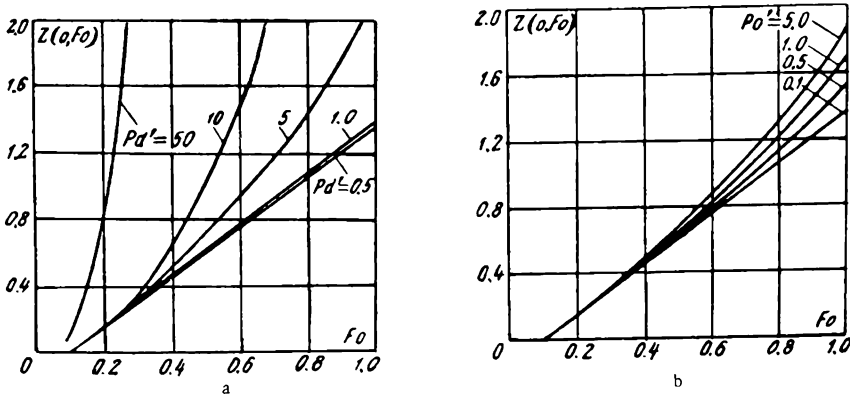


FIGURE 5-16. Variation in potential at the center of a sphere, for  $P_0=P_0(Pd'Fo)^2$

a)  $Z=f(Pd')$ , for  $P_0=0.1$ ; b)  $Z=f(P_0)$ , for  $Pd'=1.0$

2. The source strength is a linear function of the spatial coordinate:  $P_0(X, Fo) = P_0(1 - X)$

$$Z(X, Fo) = P_0 \left\{ \frac{1}{4} Fo + \sum_{n=1}^{\infty} 2 \frac{2 - (2 + \mu_n^2) \cos \mu_n}{\mu_n^5 \sin^2 \mu_n} \cdot \frac{\sin \mu_n X}{X} [1 - \exp(-\mu_n^2 Fo)] \right\} + \Phi_2(X, Fo). \quad (5-5-29)$$

3. The source strength is a quadratic function of the coordinate:  $P_0(X, Fo) = P_0(1 - X^2)$

$$Z(X, Fo) = P_0 \left\{ \frac{2}{5} Fo - \sum_{n=1}^{\infty} \frac{2}{\mu_n^4 \sin \mu_n} \cdot \frac{\sin \mu_n X}{X} [1 - \exp(-\mu_n^2 Fo)] \right\} + \Phi_2(X, Fo). \quad (5-5-30)$$

4. The source strength is an exponential function of the coordinate:  $P_0(X, Fo) = P_0 \exp(-bX)$

$$Z(X, Fo) = P_0 \left\{ -3Fo \frac{2+b}{b^2} + \sum_{n=1}^{\infty} \frac{2b[2\mu_n - e^{-b}(b^2 + 2b + 2 + \mu_n^2) \sin \mu_n]}{\mu_n^2 \sin^2 \mu_n (\mu_n^2 + b^2)^2} \times \right. \\ \left. \times \frac{\sin \mu_n X}{X} [1 - e^{-\mu_n^2 Fo}] \right\} + \Phi_2(X, Fo). \quad (5-5-31)$$

Distributions of the dimensionless potential  $Z$  in a sphere, with source strengths which depend in various ways on the coordinate, are shown in Figure 5-15.

5. The source is a linear function of time:  $P_0(X, Fo) = P_0(1 + Pd' Fo)$

$$Z(X, Fo) = P_0 Fo \left( 1 + \frac{1}{2} Pd' Fo \right) + \Phi_2(X, Fo). \quad (5-5-32)$$

6. The source is an exponential function of time:  $P_0(X, Fo) = P_0 \exp(-Pd' Fo)$

$$Z(X, Fo) = \frac{P_0}{Pd'} [1 - \exp(-Pd' Fo)] + \Phi_2(X, Fo). \quad (5-5-33)$$

7. The source is a periodic function of time:  $P_0(X, Fo) = P_0 \cos Pd' Fo$

$$Z(X, Fo) = \frac{P_0}{Pd'} \sin Pd' Fo + \Phi_2(X, Fo). \quad (5-5-34)$$

8. The source varies with the  $n$ th power of the time:  $P_0(X, Fo) = P_0(Pd' Fo)^n$

$$Z(X, Fo) = \frac{(Pd' Fo)^n}{n+1} P_0 Fo + \Phi_2(X, Fo). \quad (5-5-35)$$

Distributions of the dimensionless potential  $Z(X, Fo)$  at the center of a sphere when the source strength is a quadratic function of time are plotted in Figure 5-16 for various values of  $Pd'$  (graph a) and  $P_0$  (graph b). It should be noted that functions  $Z=f(Fo)$  are very similar for different sources and are located in the immediate vicinity of one another on the graphs.

#### d) Solution for an infinite hollow cylinder, No heat sources ( $P_0=0$ )

Differential equation (5-5-1) and conditions (5-5-3) and (5-5-4) may be rewritten as follows for the given problem:

$$\frac{\partial Z(X, Fo)}{\partial Fo} = \frac{1}{X} \frac{\partial}{\partial X} \left[ X \frac{\partial Z(X, Fo)}{\partial X} \right], \quad (5-5-36)$$

$$0 \leq Fo < \infty, \quad 1 \leq X \leq \infty;$$

$$-\frac{\partial Z(1, Fo)}{\partial X} + Kl_1(Fo) = 0; \quad (5-5-37)$$

$$-\frac{\partial Z(\infty, Fo)}{\partial X} + Kl_2(Fo) = 0, \quad (5-5-38)$$

where  $\infty = \frac{R_2}{R_1}$ . All the dimensionless numbers are referred to the inner radius  $R_1$ . Following Trofimov [13], we solve the problem by means of the finite integral Hankel transformation

$$Z_H = Z(p, Fo) = \int_1^{\infty} X Z(X, Fo) M(pX) dX, \quad (5-5-39)$$

where

$$M(pX) = I_0(pX) \frac{K_1(p)}{I_1(p)} + K_0(pX). \quad (5-5-40)$$

$\mu$  being the roots of the characteristic equation

$$I_1(p)K_1(p\pi) - K_1(p)I_1(p\pi) = 0. \quad (5-5-41)$$

When transformation (5-5-39) is applied to equation (5-5-36) and boundary conditions (5-5-37) and (5-5-38) are taken into account, we obtain

$$\frac{dZ_H}{dFo} = [M(p\pi)K_1(Fo) - M(p)K_1(Fo)] + p^2 Z_H, \quad (5-5-42)$$

where in this case initial condition (5-5-2) is written as

$$Z(p, 0) = \int_0^{\pi} XF(X) M(pX) dX.$$

The solution of equation (5-5-42) is

$$Z(p, Fo) = \left\{ \int_0^{Fo} [M(p\pi)K_1(Fo^*) - M(p)K_1(Fo^*)] \exp(-p^2 Fo^*) dFo^* + \int_0^{\pi} XF(X) M(pX) dX \right\} \exp(p^2 Fo). \quad (5-5-43)$$

The inverse transform of (5-5-43) is obtained using the inversion formula

$$Z(X, Fo) = \sum_{p=0}^{\infty} \frac{Z(p, Fo)}{\int_0^{\pi} XM(pX) dX} M(pX).$$

After the inversion and certain necessary simplifications, we obtain

$$\begin{aligned} Z(X, Fo) = & \frac{2}{\pi - 1} \left\{ \int_0^{\pi} XF(X) dX + \int_0^{Fo} [K_1(Fo^*) - K_1(Fo^*)] dFo^* \right\} + \\ & + \sum_{n=1}^{\infty} 2p_n^2 \frac{I_1^2(p_n) I_1^2(p_n\pi)}{I_1^2(p_n) - I_1^2(p_n\pi)} \left[ I_0(p_n X) \frac{K_1(p_n)}{I_1(p_n)} + K_0(p_n X) \right] \left\{ \int_0^{\pi} XF(X) \times \right. \\ & \times \left[ I_0(p_n X) \frac{K_1(p_n)}{I_1(p_n)} + K_0(p_n X) \right] dX + \\ & \left. + \frac{1}{p} \int_0^{Fo} \left[ \frac{K_1(Fo^*)}{I_1(p\pi)} - \frac{K_1(Fo^*)}{I_1(p)} \right] \exp(-p^2 Fo^*) dFo^* \right\} e^{p^2 Fo}. \end{aligned}$$

The modified Bessel functions  $I_1$  and  $K_1$  diverge for real values of their arguments. Thus, characteristic equation (5-5-41) will have purely imaginary roots  $p = i\mu$ . As a result, it is more convenient to introduce ordinary Bessel functions. In this case characteristic equation (5-5-41) and transformation kernel (5-5-40) become

$$\begin{aligned} J_1(\mu) Y_1(\mu\pi) - Y_1(\mu) J_1(\mu\pi) &= 0; \\ M(\mu X) &= \frac{\pi}{2} [J_0(\mu X) \frac{Y_1(\mu)}{J_1(\mu)} - Y_0(\mu X)]. \end{aligned} \quad (5-5-44)$$

Values of the roots  $\mu_n$  of characteristic equation (5-5-44) are listed in Table 5-6.

TABLE 5-6

The first six roots  $\mu_n$  of characteristic equation  $J_1(\mu) Y_1(\mu\pi) - Y_1(\mu) J_1(\mu\pi) = 0$ .

$\pi$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_5$	$\mu_6$
1.2	15.728	31.426	47.130	62.837	78.544	94.251
1.5	6.322	12.586	18.863	25.143	31.424	37.706
2.0	3.197	6.312	9.444	12.581	15.720	18.859

Consequently, the final solution of the problem is

$$\begin{aligned}
 Z(X, Fo) = & \frac{2}{x-1} \left\{ \int_1^x XF(X) dX + \int_0^{Fo} [Kl_2(Fo^*) - Kl_1(Fo^*)] dFo^* \right\} + \\
 & + \pi \sum_{n=1}^{\infty} \mu_n^2 \frac{J_1^2(\mu_n) J_1^2(\mu_n x)}{J_1^2(\mu_n) - J_1^2(x\mu_n)} [J_0(\mu_n X) \frac{Y_1(\mu_n)}{J_1(\mu_n)} - Y_0(\mu_n X)] \left\{ \frac{\pi}{2} \int_1^x XF(X) \times \right. \\
 & \times [J_0(\mu_n X) \frac{Y_1(\mu_n)}{J_1(\mu_n)} - Y_0(\mu_n X)] dX - \int_0^{Fo} \frac{1}{\mu_n} \left[ \frac{Kl_2(Fo^*)}{J_1(x\mu_n)} - \frac{Kl_1(Fo^*)}{J_1(\mu_n)} \right] \times \\
 & \times \exp(\mu_n^2 Fo^*) dFo^* \Big\} \exp(-\mu_n^2 Fo). \quad (5-5-45)
 \end{aligned}$$

In the particular case of a constant Kirpichev number at the surface of the hollow cylinder (that is, for  $Kl_1 = 0$  and  $Kl_2 = Kl$  constant) and of zero initial distribution of the dimensionless potential  $F(X) = 0$ , (5-5-45) becomes

$$\begin{aligned}
 T(X, Fo) = & \frac{t-t_0}{t_0} = Kl \left\{ \frac{x^2}{x^2-1} \left[ 2 \frac{Fo}{x^2} - \frac{1}{4} \left( 1 - \frac{2}{x^2} X^2 \right) - \right. \right. \\
 & \left. \left. - \frac{1}{x^2} \left( \ln X + \frac{x^2}{x^2-1} \ln \frac{1}{x} + \frac{3}{4} \right) \right] + \right. \\
 & \left. + \pi \sum_{n=1}^{\infty} \frac{1}{\mu_n} \frac{J_1(\mu_n) J_1(x\mu_n)}{J_1^2(\mu_n) - J_1^2(x\mu_n)} [J_0(\mu_n X) Y_1(\mu_n) - Y_0(\mu_n X) J_1(\mu_n)] \right\} \exp(-\mu_n^2 Fo).
 \end{aligned}$$

As  $R_1 \rightarrow 0$ , (5-5-45) becomes the solution for a solid cylinder, while as  $\left(1 - \frac{1}{x}\right) \rightarrow 0$  it becomes the solution for a plane infinite plate with boundary conditions which are identical to those of the problem at hand.



## Chapter VI

### UNSTEADY FIELDS OF HEAT-TRANSFER AND MASS-TRANSFER POTENTIAL WITH BOUNDARY CONDITIONS OF THE THIRD KIND

#### 6-1. The Dimensionless Boundary Conditions

The system of differential heat-transfer and mass-transfer equations with boundary conditions of the third kind may be used to represent a very wide range of phenomena, including nonisothermal dissolution, heterogeneous reactions which take place according to diffusion kinetics, convective drying, and electrodiffusion. In this case the boundary conditions relate the transfer potentials at the surface of the body to the corresponding potentials of the surroundings, by means of specified values of the heat-exchange and mass-exchange coefficients or, which is the same thing, by means of laws describing the convective heat exchange and mass exchange at the surface. Newton's law is selected to describe convective heat exchange, while Dalton's law or some other experimentally established law (for example, Nernst's law or Shchukarev's law) is selected to represent mass-transfer phenomena at the surface of the body.

The heat-exchange coefficient  $\alpha_q$  and mass-exchange coefficient  $\alpha_m$  entering into Newton's law or Dalton's law (or a similar mass-exchange law) are in general functions of the regime parameters and the state of the surface of the material. To simplify the solutions, we will assume that coefficients  $\alpha_q$  and  $\alpha_m$ , as well as the temperature  $t_c$  of the surroundings, are constant and equal over the whole surface of the body. Heat and mass transfer with surroundings in which the temperature and the exchange coefficients vary will be considered in the next chapter.

The dimensionless boundary conditions of the third kind, as applied to the system of equations describing heat and mass transfer accompanied by phase or chemical transformations (system of equations (4-1-2) and (4-1-3)), may be written as

$$\frac{\partial T(1, Fo)}{\partial X} - Bi_q [1 - T(1, Fo)] + (1 - \epsilon) Ko Lu Ki_m = 0; \quad (6-1-1)$$

$$-\frac{\partial \theta(1, Fo)}{\partial X} + Pn \frac{\partial T(1, Fo)}{\partial X} + Ki_m = 0. \quad (6-1-2)$$

The main problems to be considered in this chapter will correspond to the following three conditions for the dimensionless mass flux  $Ki_m$ :

- a)  $Ki_m$  constant; b)  $Ki_m = Bi_m [1 - \theta(1, Fo)]$ ; c)  $Ki_m = Ki_m(Fo)$ ;

the most typical condition for the dimensionless mass flux during convective heat and mass transfer being the second one.

The two new similarity criteria which appear in boundary conditions (6-1-1) and (6-1-2) are  $Bi_q = \frac{\alpha_q R}{\lambda_q}$  and  $Bi_m = \frac{\alpha_m R}{\lambda_m}$ , the Biot numbers for heat exchange and mass exchange.

When they are not specified, the dimensionless transfer potentials will be assumed to be

$$T = \frac{t - t_*}{t_c - t_*} \text{ and } \theta = \frac{\theta_* - \theta}{\theta_* - \theta_{eq}}$$

[here, subscript "c" again indicates the surroundings and subscript "eq" indicates equilibrium]. When the initial distributions of transfer potential are constant:

$$T(X, 0) = \theta(X, 0) = 0, \quad (6-1-3)$$

the quantities  $t_*$  and  $\theta_*$  have a subscript 0. When the initial distributions are parabolic:

$$T(X, 0) = -(1 - X^2)W \text{ and } \theta(X, 0) = -(1 - X^2)V, \quad (6-1-4)$$

the quantities  $t_*$  and  $\theta_*$  have a subscript s (surface). Here,  $W = \frac{t_s - t_{ce}}{t_c - t_s}$  and  $V = \frac{\theta_{ce} - \theta_s}{\theta_s - \theta_{eq}}$  are simplexes [dimensionless parameters] for the nonuniformity of the initial distribution of the heat-transfer and mass-transfer potentials. At the same time, the similarity criteria for these problems will be referred to  $\Delta t = t_c - t_*$  and  $\Delta \theta = \theta_* - \theta_{eq}$ . In this chapter the expanded form  $Ko^*$  of the Kossovich number will be used, instead of  $Ko^*$ .

Analytical solutions of the one-dimensional system of heat-transfer and mass-transfer equations (4-1-2) and (4-1-3), with boundary conditions of the third kind, have been obtained for the complete system of equations by Lykov and Mikhailov /1 through 5/, Gamayunov /6/, and Prudnikov /7, 8/. Particular solutions for the incomplete system of equations have been considered by Lykov /9/, Mikhailov /10, 11/, Polonskaya /12/, and Smirnov /13, 14/. The effects of individual criteria for the similarity of heat and mass [transfer] on the heat and mass exchange have been investigated by Lykov /10, 15/, Mikhailov /16 through 19/, and Lebedev /20/.

## 6-2. Unsteady Fields of Heat-Transfer and Mass-Transfer Potential. Constant Mass Flux at Surface of Body

Let us now consider solutions of system of equations (4-1-2) and (4-1-3) for bodies of regular shape. Boundary conditions (6-1-1) and (6-1-2), symmetry conditions (4-1-5), and initial conditions (6-1-3) are assumed. The quantity  $Ki_m$  entering into the boundary conditions will be taken as constant. Laplace integral transformations will be used to solve the problems in this section.

### a) Solution for an infinite plate ( $\Gamma = 0$ )

The transform solution of system of equations (4-1-2) and (4-1-3), assuming initial conditions (6-1-3) and symmetry conditions (4-1-5), has the form (see § 4-2, a):

$$T_L(X, s) = \sum_{i=1}^2 B_i \operatorname{ch} \sqrt{s} v_i X; \quad (6-2-1)$$

$$\Theta_L(X, s) = -\frac{1}{s \text{Ko}} \sum_{i=1}^2 B_i (1 - v_i^2) \text{ch} \sqrt{s} v_i X, \quad (6-2-2)$$

where

$$v_i^2 = \frac{1}{2} \left[ \left( 1 + s \text{Ko} \text{Pn} + \frac{1}{\text{Lu}} \right) + (-1)^i \sqrt{\left( 1 + s \text{Ko} \text{Pn} + \frac{1}{\text{Lu}} \right)^2 - \frac{4}{\text{Lu}}} \right].$$

To determine the constants  $B_i$ , boundary conditions (6-1-1) and (6-1-2) are used. When the Laplace transformation

$$F_L(X, s) = \int_0^{\infty} F(X, s) \exp(-s F_0) d F_0$$

is applied to these conditions, we obtain

$$T'_L(1, s) - \text{Bi}_q \left[ \frac{1}{s} - T_L(1, s) \right] + \frac{1}{s} (1 - s) \text{Ko Lu Kl}_m = 0; \quad (6-2-3)$$

$$-\Theta'_L(1, s) + \text{Pn } T'_L(1, s) + \frac{1}{s} \text{Kl}_m = 0. \quad (6-2-4)$$

Then, equations (6-2-1) and (6-2-2) combine with conditions (6-2-3) and (6-2-4) to give a system of algebraic equations in  $B_i$ :

$$\sum_{i=1}^2 B_i Q_i + \frac{s \text{Ko } \bar{K}_1 - 1}{s} = 0;$$

$$\sum_{i=1}^2 B_i P_i + \frac{s \text{Ko Kl}_m}{s} = 0,$$

the solutions of which are

$$B_1 = \frac{(s \text{Ko } \bar{K}_1 - 1) P_2 - s \text{Ko Kl}_m Q_2}{s(P_1 Q_2 - P_2 Q_1)}; \quad (6-2-5)$$

$$B_2 = -\frac{(s \text{Ko } \bar{K}_1 - 1) P_1 - s \text{Ko Kl}_m Q_1}{s(P_1 Q_2 - P_2 Q_1)}. \quad (6-2-6)$$

Here

$$Q_i = \text{ch} \sqrt{s} v_i + \frac{1}{\text{Bi}_q} \sqrt{s} v_i \text{sh} \sqrt{s} v_i;$$

$$P_i = [s \text{Ko Pn} + (1 - v_i^2)] \sqrt{s} v_i \text{sh} \sqrt{s} v_i;$$

$$\bar{K}_1 = \frac{1 - s}{s} \text{Lu } \frac{\text{Kl}_m}{\text{Bi}_q}.$$

To determine the inverse transform, let us find the roots of the denominator of  $B_i$ :

$$\psi(s) = s(P_1 Q_2 - P_2 Q_1) = s \{ [s \text{Ko Pn} + (1 - v_1^2)] \sqrt{s} v_1 \text{sh} \sqrt{s} v_1 Q_2 -$$

$$- [s \text{Ko Pn} + (1 - v_2^2)] \sqrt{s} v_2 \text{sh} \sqrt{s} v_2 Q_1 \} = s^4 \{ [s \text{Ko Pn} + (1 - v_1^2)] (v_1^2 + \frac{v_1^4}{3!} s + \dots) \times$$

$$\times Q_2 - [s \text{Ko Pn} + (1 - v_2^2)] (v_2^2 + \frac{v_2^4}{3!} s + \dots) Q_1 \} = s^4 \varphi(s) = 0,$$

where  $\varphi(s)$  is the expression in braces. It follows from this expression that:

- 1)  $s = s_0 = 0$  is a double zero root;
- 2)  $s = s_m$  are the roots of the simultaneous solution of the system of equations:

$$\left. \begin{aligned} \sin v_{1\mu} &= 0; \\ \sin v_{2\mu} &= 0, \end{aligned} \right\} \quad (6-2-7)$$

where  $s_m = -\mu_m^2$ ;

- 3)  $s = s_n$  are the roots of the equation

$$x \cot v_{1\mu} - \cot v_{2\mu} + \frac{v_2 - v_1}{\text{Bi}_q} \mu = 0, \quad (6-2-8)$$

where

$$s_n = -\mu_n^2 \text{ and } x = \frac{\varepsilon K_0 P_n + (1 - v_2^2)}{\varepsilon K_0 P_n + (1 - v_1^2)} \cdot \frac{v_2}{v_1}.$$

The inverse transforms of (6-2-1) and (6-2-2) are found using the generalized expansion theorem for multiple roots:

$$L^{-1} \left[ \frac{\Phi_h(s)}{\psi(s)} \right] = \lim_{s \rightarrow 0} \left\{ \left[ F_0 \frac{\Phi_h(s)}{\psi(s)} + \frac{\Phi'_h(s)}{\psi(s)} - \frac{\Phi_h(s) \psi'(s)}{[\psi(s)]^2} \right] \exp(s F_0) \right\} + \\ + \sum_{n=1}^{\infty} \frac{\Phi_h(s_n)}{\psi'(s_n)} \exp(s_n F_0) + \sum_{m=1}^{\infty} \frac{\Phi_h(s_m)}{\psi'(s_m)} \exp(s_m F_0) \quad (k=1,2),$$

where

$$\Phi_1(s) = [(\varepsilon K_0 \bar{K}_1 - 1) P_2 - \varepsilon K_0 K_{1m} Q_2] \operatorname{ch} \sqrt{s} v_1 X - \\ - [(\varepsilon K_0 \bar{K}_1 - 1) P_1 - \varepsilon K_0 K_{1m} Q_1] \operatorname{ch} \sqrt{s} v_2 X; \\ \Phi_2(s) = -\frac{1}{\varepsilon K_0} \{ (\varepsilon K_0 \bar{K}_1 - 1) P_2 - \varepsilon K_0 K_{1m} Q_2 \} (1 - v_1^2) \operatorname{ch} \sqrt{s} v_1 X - \\ - [(\varepsilon K_0 \bar{K}_1 - 1) P_1 - \varepsilon K_0 K_{1m} Q_1] (1 - v_2^2) \operatorname{ch} \sqrt{s} v_2 X \}.$$

After the necessary calculations have been made, the unsteady potential distributions are found to be

$$T(X, F_0) = 1 - \varepsilon K_0 \bar{K}_1 - \frac{1}{2} \varepsilon K_0 \operatorname{Lu} K_{1m} \left( 1 - X^2 + \frac{2}{\operatorname{Bi}_q} \right) - \\ - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \cos v_i \mu_n X \exp(-\mu_n^2 F_0) - \\ - \frac{\varepsilon K_0 K_{1m}}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 C_{mi} \cos v_i \mu_m X \exp(-\mu_m^2 F_0); \quad (6-2-9)$$

$$\theta(X, F_0) = K_{1m} \left[ \operatorname{Lu} F_0 - \frac{1}{2} (1 + \varepsilon K_0 P_n \operatorname{Lu}) \left( \frac{1}{3} - X^2 \right) \right] + \\ + \frac{1}{\varepsilon K_0} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2) \cos v_i \mu_n X \exp(-\mu_n^2 F_0) + \\ + \frac{K_{1m}}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 C_{mi} (1 - v_i^2) \cos v_i \mu_m X \exp(-\mu_m^2 F_0). \quad (6-2-10)$$

The following coefficients enter into (6-2-9) and (6-2-10):

$$C_{n1} = \frac{2}{\mu_n \psi_n} [(1 - \varepsilon K_0 \bar{K}_1) P_{n2} + \varepsilon K_0 K_{1m} Q_{n2}]; \quad (6-2-11)$$

$$C_{n2} = -\frac{2}{\mu_n \psi_n} [(1 - \varepsilon K_0 \bar{K}_1) P_{n1} + \varepsilon K_0 K_{1m} Q_{n1}]; \quad (6-2-12)$$

$$\psi_n = v_1 A_{n1} P_{n2} + v_2 B_{n2} Q_{n1} - v_2 A_{n2} P_{n1} - v_1 B_{n1} Q_{n2}; \quad (6-2-13)$$

$$Q_{ni} = \cos v_i \mu_n - \frac{1}{\operatorname{Bi}_q} v_i \mu_n \sin v_i \mu_n; \quad (6-2-14)$$

$$P_{ni} = -[\varepsilon K_0 P_n + (1 - v_i^2)] v_i \mu_n \sin v_i \mu_n; \quad (6-2-15)$$

$$A_{ni} = \left( 1 + \frac{1}{\operatorname{Bi}_q} \right) \sin v_i \mu_n + \frac{1}{\operatorname{Bi}_q} v_i \mu_n \cos v_i \mu_n; \quad (6-2-16)$$

$$B_{ni} = \sin v_i \mu_n + v_i \mu_n \cos v_i \mu_n; \quad (6-2-17)$$

$$C_{m1} = \frac{2}{\mu_m^2 \cos v_1 \mu_m}, \quad C_{m2} = -\frac{2}{\mu_m^2 \cos v_2 \mu_m}, \quad (6-2-18)$$

where  $\mu_n$  are the roots of characteristic equation (6-2-8) and  $\mu_m$  are the roots of equations (6-2-7).

An analysis of these solutions shows that the infinite sums in (6-2-9) and (6-2-10) converge quite rapidly. Thus, for Fourier numbers above 0.2 to 0.3, we need only retain, for practical calculations, two characteristic-equation roots and two terms in each series. With decreasing Fourier numbers, the number of series terms which must be kept increases. Thus, for  $Fo = 0.1$  four terms, and for  $Fo = 0.05$  six terms, must be used. The infinite sum over  $m$  converges the most rapidly, and starting with  $Fo = 0.05$  it can generally be disregarded during calculation. For the above reasons, for  $Fo \geq 0.3$  it is convenient to calculate unsteady fields of transfer potential using the simplified formulas:

$$\begin{aligned} T(X, Fo) \approx & 1 - \epsilon Ko \bar{K}_1 - \frac{1}{2} \epsilon Ko Lu Ki_m \left( 1 - X^2 + \frac{2}{Bi_q} \right) - \\ & - \sum_{n=1}^2 \sum_{i=1}^2 C_{ni} \cos v_i \mu_n X \exp(-\mu_n^2 Fo); \\ \theta(X, Fo) \approx & Ki_m \left[ Lu Fo - \frac{1}{2} (1 + \epsilon Ko Pn Lu) \left( \frac{1}{3} - X^2 \right) \right] + \\ & + \frac{1}{\epsilon Ko} \sum_{n=1}^2 \sum_{i=1}^2 C_{ni} (1 - v_i^2) \cos v_i \mu_n X \exp(-\mu_n^2 Fo). \end{aligned}$$

Values of the first two roots  $\mu_n$  of characteristic equation (6-2-8) are listed in Tables 6-1 and 6-2 for various values of  $Lu$ ,  $\epsilon Ko Pn$ , and  $Bi_q$ , and the curves are plotted in Figures 6-1 through 6-3. Values of constant coefficients  $C_{ni}$  for various combinations of dimensionless numbers are listed in Tables 6-3 through 6-8 and the curves are plotted in Figures 6-4 through 6-9.

In terms of average dimensionless transfer potentials solutions (6-2-9) and (6-2-10) may be written as

$$\begin{aligned} \bar{T}(Fo) = & 1 - \epsilon Ko \bar{K}_1 - \frac{1}{2} \epsilon Ko Lu Ki_m \left( \frac{2}{3} + \frac{2}{Bi_q} \right) - \\ & - \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} \exp(-\mu_n^2 Fo) - \frac{\epsilon Ko Ki_m}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 D_{mi} \exp(-\mu_m^2 Fo); \\ \bar{\theta}(Fo) = & Ki_m Lu Fo + \frac{1}{\epsilon Ko} \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} (1 - v_i^2) \exp(-\mu_n^2 Fo) + \\ & + \frac{Ki_m}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 D_{mi} (1 - v_i^2) \exp(-\mu_m^2 Fo), \end{aligned}$$

where

$$D_{ni} = C_{ni} \frac{\sin v_i \mu_n}{v_i \mu_n}; \quad D_{mi} = C_{mi} \frac{\sin v_i \mu_m}{v_i \mu_m}.$$

Values of coefficients  $D_{ni}$ , as well as of  $D_{rn} = \sum_{i=1}^2 D_{ni}$  and  $D_{\theta n} = \frac{1}{\epsilon Ko} \times \sum_{i=1}^2 D_{ni} (1 - v_i^2)$ , are listed in Tables 6-3 through 6-8 and plotted in Figures 6-4 through 6-9.

Steady-state distributions of temperature and mass-transfer potential are established for Fourier numbers from 1.0 to 2.5. The steady state is first established with respect to temperature (for  $X = 1$  and  $Fo = 0.2$  to

0.7) and then with respect to mass-transfer potential (for  $X = 0$  and  $Fo \approx 2.5$ ). For the steady state the dimensionless potentials correspond to those obtained by Polonskaya /11/ [12?]:

$$T = 1 - \varepsilon Ko \bar{K}_1 - \frac{1}{2} \varepsilon Ko Lu Ki_m \left( 1 - X^* + \frac{2}{Bi_q} \right); \quad (6-2-19)$$

$$\theta = Ki_m \left[ Lu Fo - \frac{1}{2} (1 + \varepsilon Ko Pn Lu) (1/3 - X^*) \right]. \quad (6-2-20)$$

Thus, in the steady state parabolic distributions of the dimensionless transfer potentials are established in the material. Equation (6-2-19) shows that the phase-transition ratio  $\varepsilon$  is

$$\varepsilon = \frac{2\lambda_q}{\rho q_m R} (t_s - t_{ce}).$$

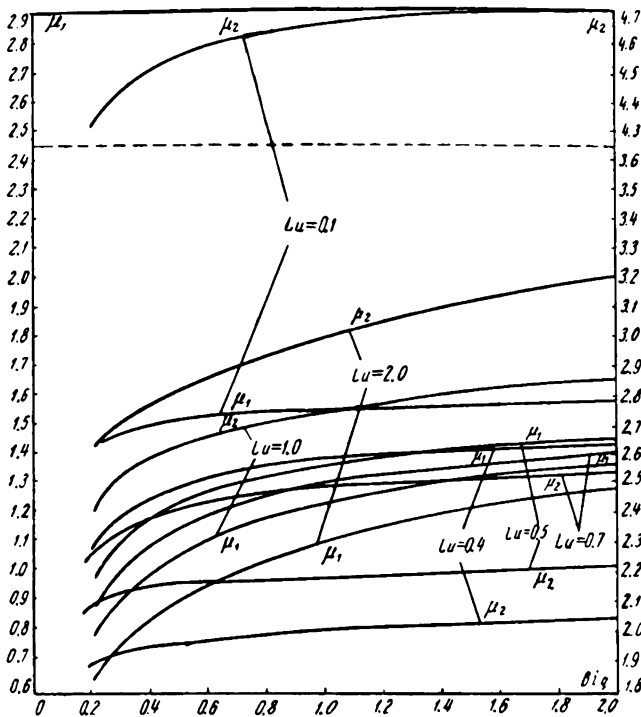


FIGURE 6-1. Characteristic-equation roots as functions of  $Bi_q$  (from 0 to 2) and  $Lu$ , for an infinite plate

This means that, when the temperature drop and the thermophysical properties of the material are known, it is possible to determine  $\varepsilon$  and consequently to evaluate the flux of matter in the vapor state. According to the experiments of Polonskaya and Lebedev: for plaster plates  $\varepsilon = 0.045$ ; for clay plates  $\varepsilon = 0.75$  to  $1.0$ ; for sand plates  $\varepsilon = 0.2$  to  $0.4$ , and for wooden plates  $\varepsilon = 0.09$  to  $0.2$ .

If we differentiate the various equations obtained above with respect to time, then we will get the rates of variation of the local and average transfer potentials.

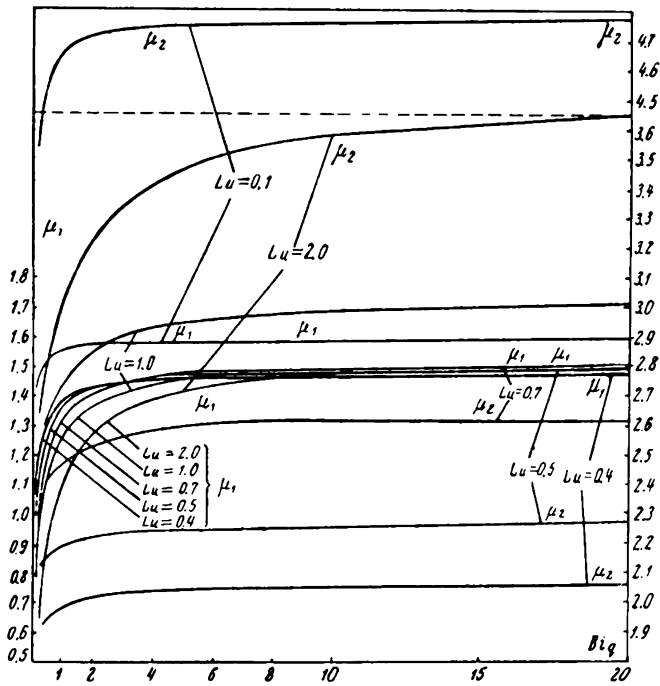


FIGURE 6.2. Characteristic-equation roots as functions of  $Bi_q$  (from 1 to 20) and  $Lu$ , for an infinite plate

TABLE 6-1

Infinite Plate. Roots of characteristic equation (6-2-8) as functions of  $Lu$  and  $Bi_q$  (for  $\epsilon$  Ko Pn = 0.3)

$Bi_q$	$Lu = 0.1$		$Lu = 0.4$		$Lu = 0.5$		$Lu = 0.7$		$Lu = 1.0$		$Lu = 2.0$	
	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$
0.20	1.420	4.309	1.066	1.886	0.996	2.083	0.882	2.257	0.771	2.393	0.625	2.630
0.25	1.450	4.375	1.131	1.911	1.065	2.116	0.953	2.300	0.843	2.464	0.692	2.674
0.30	1.472	4.427	1.178	1.925	1.114	2.130	1.008	2.331	0.899	2.512	0.744	2.712
0.35	1.488	4.471	1.214	1.934	1.156	2.140	1.056	2.356	0.945	2.550	0.788	2.745
0.40	1.500	4.505	1.242	1.940	1.189	2.148	1.094	2.378	0.986	2.579	0.828	2.775
0.45	1.511	4.532	1.265	1.948	1.218	2.154	1.126	2.395	1.023	2.602	0.863	2.806
0.50	1.519	4.555	1.284	1.950	1.240	2.159	1.156	2.408	1.055	2.621	0.898	2.827
0.55	1.526	4.574	1.300	1.951	1.260	2.162	1.179	2.417	1.081	2.636	0.926	2.850
0.60	1.531	4.590	1.314	1.954	1.277	2.166	1.200	2.425	1.107	2.650	0.955	2.871
0.70	1.540	4.614	1.336	1.964	1.304	2.171	1.236	2.439	1.148	2.675	1.000	2.905
0.80	1.547	4.633	1.356	1.974	1.330	2.175	1.270	2.449	1.188	2.699	1.033	2.942
0.90	1.551	4.649	1.366	1.984	1.344	2.178	1.288	2.460	1.212	2.720	1.073	2.972
1.00	1.555	4.659	1.378	1.991	1.358	2.180	1.308	2.470	1.237	2.740	1.105	3.000
1.50	1.566	4.694	1.412	2.017	1.400	2.203	1.363	2.511	1.318	2.810	1.216	3.124
2.00	1.573	4.711	1.428	2.030	1.423	2.218	1.400	2.534	1.361	2.854	1.280	3.216
3.00	1.579	4.730	1.443	2.040	1.446	2.234	1.437	2.561	1.408	2.902	1.348	3.336
4.00	1.582	4.739	1.452	2.045	1.458	2.243	1.458	2.576	1.433	2.930	1.389	3.411
5.00	1.584	4.744	1.457	2.048	1.465	2.248	1.470	2.585	1.448	2.949	1.412	3.464
6.00	1.585	4.748	1.461	2.050	1.470	2.251	1.478	2.591	1.458	2.959	1.430	3.500
7.00	1.586	4.750	1.463	2.052	1.475	2.253	1.482	2.594	1.466	2.968	1.442	3.520
8.00	1.587	4.752	1.465	2.053	1.477	2.255	1.485	2.599	1.471	2.976	1.451	3.541
9.00	1.587	4.754	1.467	2.055	1.479	2.256	1.487	2.601	1.476	2.983	1.459	3.560
10.00	1.588	4.755	1.468	2.056	1.482	2.257	1.488	2.604	1.479	2.988	1.465	3.576
15.00	1.589	4.759	1.471	2.058	1.487	2.260	1.494	2.609	1.492	3.000	1.484	3.622
20.00	1.590	4.761	1.472	2.060	1.489	2.262	1.497	2.612	1.498	3.009	1.493	3.644
30.00	1.590	4.764	1.474	2.061	1.490	2.264	1.499	2.616	1.503	3.016	1.502	3.670
40.00	1.591	4.765	1.475	2.062	1.491	2.265	1.501	2.617	1.506	3.018	1.507	3.682
50.00	1.591	4.766	1.475	2.062	1.492	2.266	1.501	2.618	1.508	3.022	1.510	3.690
$\infty$	1.592	4.767	1.478	2.065	1.494	2.268	1.503	2.622	1.515	3.030	1.520	3.715

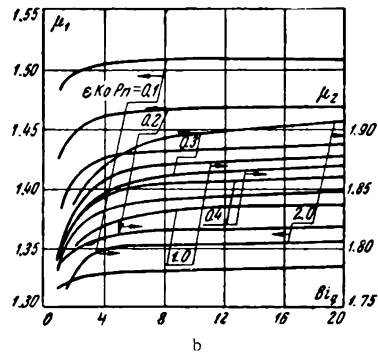
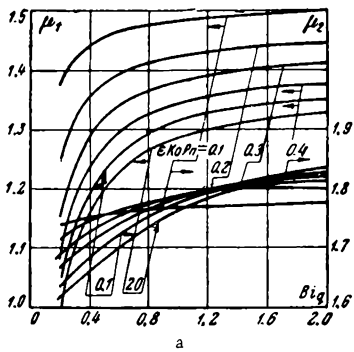


FIGURE 6-3. Characteristic-equation roots as functions of  $Bi_q$  and  $\varepsilon Ko Pn$ , for an infinite plate

a)  $Bi_q$  from 0 to 2; b)  $Bi_q$  from 1 to 20

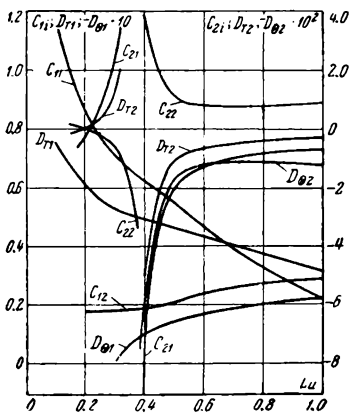


FIGURE 6-4. Constant coefficients  $C_{n1}$ ,  $D_{Tn}$  and  $D_{\Theta n}$  as functions of  $Lu$ , for an infinite plate

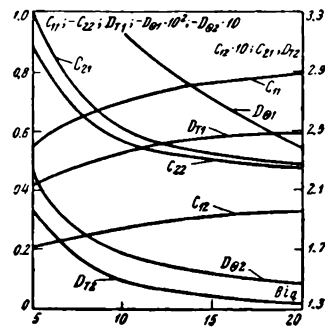


FIGURE 6-5. Constant coefficients  $C_{n1}$ ,  $D_{Tn}$ , and  $D_{\Theta n}$  as functions of  $Bi_q$ , for an infinite plate

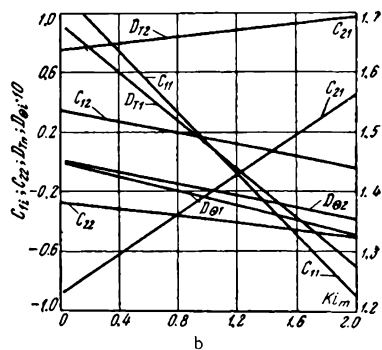
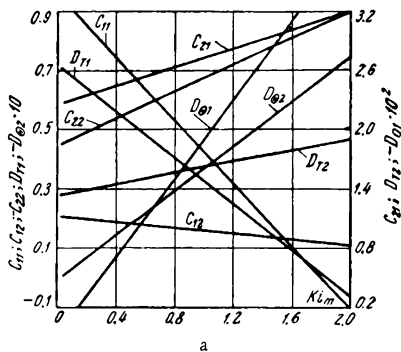


FIGURE 6-6. Constant coefficients  $C_{n1}$ ,  $D_{Tn}$ , and  $D_{\Theta n}$  as functions of  $Ki_m$ , for an infinite plate

a)  $Ko = 3$ ; b)  $Ko = 5$



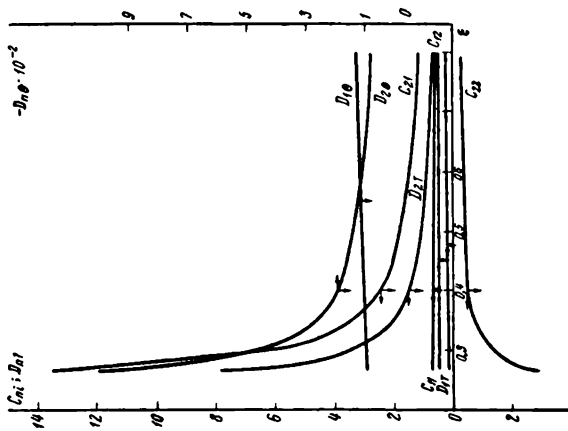


FIGURE 6-7. Constant coefficients  $C_{n,i}$ ,  $D_{Tn}$ , and  $D_{\theta n}$  as functions of  $\epsilon$ , for an infinite plate

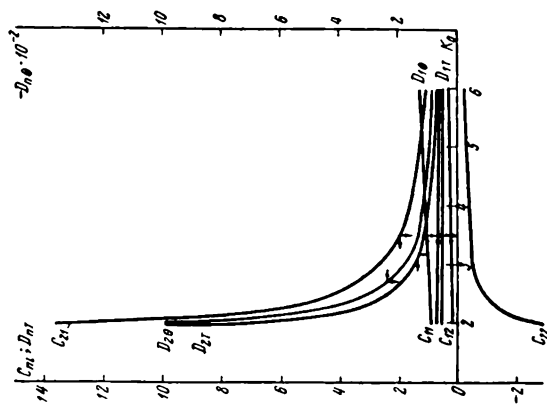


FIGURE 6-8. Constant coefficients  $C_{n,i}$ ,  $D_{Tn}$ , and  $D_{\theta n}$  as functions of  $K_0$ , for an infinite plate

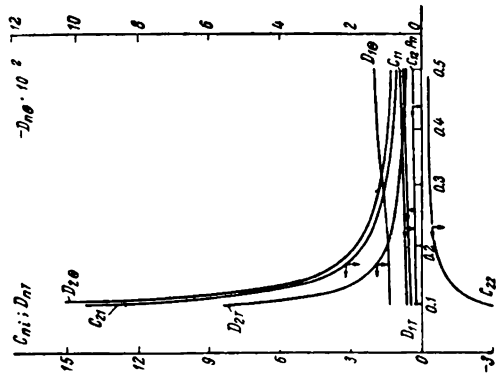


FIGURE 6-9. Constant coefficients  $C_{n,i}$ ,  $D_{Tn}$ , and  $D_{\theta n}$  as functions of  $Pn$ , for an infinite plate

TABLE 6-2

Infinite Plate. Roots of characteristic equation (6-2-8) as functions of  $\varepsilon$  Ko Pn and  $Bi_q$  (for  $Lu = 0.3$ )

$Bi_q$	$\varepsilon$ Ko Pn = 0.1		$\varepsilon$ Ko Pn = 0.2		$\varepsilon$ Ko Pn = 0.3		$\varepsilon$ Ko Pn = 0.4		$\varepsilon$ Ko Pn = 0.5		$\varepsilon$ Ko Pn = 0.6	
	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$
0.20	1.373	1.734	1.246	1.714	1.158	1.687	1.080	1.665	1.050	1.640	0.975	1.615
0.25	1.397	1.740	1.286	1.723	1.202	1.696	1.137	1.678	1.085	1.651	1.040	1.629
0.30	1.415	1.744	1.316	1.729	1.240	1.708	1.179	1.690	1.130	1.665	1.090	1.640
0.35	1.430	1.747	1.338	1.737	1.268	1.719	1.212	1.697	1.162	1.675	1.130	1.652
0.40	1.440	1.750	1.355	1.744	1.289	1.727	1.236	1.707	1.192	1.686	1.159	1.661
0.45	1.448	1.752	1.368	1.750	1.307	1.735	1.256	1.717	1.210	1.697	1.180	1.673
0.50	1.454	1.754	1.378	1.754	1.320	1.742	1.272	1.727	1.225	1.706	1.197	1.683
0.55	1.459	1.755	1.387	1.758	1.331	1.749	1.285	1.735	1.245	1.712	1.212	1.692
0.60	1.464	1.757	1.394	1.761	1.340	1.754	1.296	1.742	1.263	1.723	1.226	1.701
0.70	1.470	1.759	1.405	1.767	1.355	1.762	1.313	1.752	1.278	1.737	1.249	1.718
0.80	1.475	1.761	1.413	1.775	1.366	1.772	1.326	1.764	1.293	1.754	1.265	1.735
0.90	1.479	1.763	1.419	1.779	1.374	1.779	1.336	1.774	1.306	1.766	1.277	1.748
1.00	1.482	1.765	1.424	1.783	1.378	1.784	1.344	1.782	1.314	1.776	1.287	1.760
1.50	1.492	1.770	1.439	1.794	1.402	1.802	1.367	1.806	1.339	1.808	1.314	1.811
2.00	1.497	1.773	1.447	1.800	1.410	1.813	1.378	1.821	1.351	1.826	1.327	1.836
3.00	1.501	1.775	1.455	1.806	1.419	1.825	1.390	1.838	1.363	1.847	1.343	1.859
4.00	1.503	1.776	1.459	1.809	1.424	1.830	1.395	1.847	1.368	1.859	1.348	1.870
5.00	1.505	1.777	1.461	1.811	1.427	1.833	1.398	1.852	1.372	1.864	1.350	1.879
6.00	1.506	1.777	1.463	1.812	1.429	1.837	1.401	1.856	1.375	1.867	1.350	1.884
7.00	1.506	1.777	1.464	1.812	1.430	1.838	1.402	1.858	1.377	1.869	1.351	1.888
8.00	1.507	1.778	1.465	1.813	1.431	1.839	1.403	1.860	1.378	1.870	1.351	1.891
9.00	1.507	1.778	1.465	1.813	1.432	1.840	1.404	1.861	1.380	1.871	1.351	1.894
10.00	1.507	1.778	1.465	1.814	1.432	1.841	1.404	1.862	1.381	1.872	1.352	1.895
15.00	1.508	1.783	1.466	1.815	1.434	1.843	1.406	1.866	1.383	1.874	1.352	1.901
20.00	1.508	1.783	1.467	1.816	1.435	1.845	1.407	1.868	1.384	1.875	1.352	1.905
30.00	1.509	1.783	1.468	1.817	1.435	1.846	1.408	1.870	1.385	1.876	1.352	1.907
40.00	1.509	1.784	1.468	1.818	1.436	1.847	1.408	1.871	1.385	1.877	1.352	1.909
50.00	1.509	1.784	1.469	1.819	1.436	1.848	1.409	1.871	1.385	1.877	1.353	1.910
$\infty$	1.510	1.784	1.469	1.820	1.437	1.849	1.409	1.873	1.386	1.880	1.353	1.911

It is clear from the foregoing that the solution methods used here for an infinite plate do not in principle differ from those employed for the corresponding problem with boundary conditions of the first kind (see § 4-2, a). Therefore, for the cylinder and sphere only the final results will be given.

b) Solution for an infinite cylinder ( $\Gamma = 1$ )

$$\begin{aligned}
 T(X, Fo) = & 1 - \varepsilon Ko \bar{K}_1 - \frac{1}{2} \varepsilon Ko Lu Ki_m \left( 1 - X^2 + \frac{2}{Bi_q} \right) - \\
 & - \sum_{n=1}^{\infty} \sum_{l=1}^2 C_{nl} J_0(\nu_l \mu_n X) \exp(-\mu_n^2 Fo) - \frac{\varepsilon Ko Kl_m}{\nu_2^2 - \nu_1^2} \times \\
 & \times \sum_{m=1}^{\infty} \sum_{l=1}^2 C_{ml} J_0(\nu_l \mu_m X) \exp(-\mu_m^2 Fo); \quad (6-2-21)
 \end{aligned}$$

$$\begin{aligned}
 \theta(X, Fo) = & Ki_m \left[ 2Lu Fo - \frac{1}{2} (1 + \varepsilon Ko Pn Lu) \left( \frac{1}{2} - X^2 \right) \right] + \\
 & + \frac{1}{\varepsilon Ko} \sum_{n=1}^{\infty} \sum_{l=1}^2 C_{nl} (1 - \nu_l^2) J_0(\nu_l \mu_n X) \exp(-\mu_n^2 Fo) + \\
 & + \frac{Kl_m}{\nu_2^2 - \nu_1^2} \sum_{m=1}^{\infty} \sum_{l=1}^2 C_{ml} (1 - \nu_l^2) J_0(\nu_l \mu_m X) \exp(-\mu_m^2 Fo). \quad (6-2-22)
 \end{aligned}$$

TABLE 6 - 3

The constant coefficients as functions of  $Lu$ , for an infinite plate ( $\epsilon = 0.4$ ;  $Ko = 3$ ;  $Pn = 0.25$ ;  $Bl_g = 10$ ;  $Kl_m = 0.5$ )

$Lu$	$C_{n,t}$				$D_{n,t}$				$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22	11	12	21	22	$n = 1$	$n = 2$	$n = 1$	$n = 2$
0.1	1.176	-0.0170	-0.2991	0.00955	0.753	0.00328	0.0639	0.000261	0.756	0.0642	-0.00537	-0.000323
0.3	0.692	0.184	2.501	-0.565	0.500	0.0240	1.420	0.0647	0.524	1.485	-0.00935	-0.0186
0.4	0.629	0.188	-9.059	3.737	0.454	0.0432	-4.545	-0.404	0.498	-4.949	-0.0101	0.0753
0.5	0.551	0.213	-1.872	1.076	0.404	0.0653	-0.835	-0.125	0.469	-0.960	-0.0144	0.0173
0.7	0.393	0.266	-0.935	0.779	0.298	0.108	-0.346	-0.113	0.406	-0.459	-0.0187	0.0117
1.0	0.223	0.291	-0.715	0.889	0.178	0.140	-0.238	-0.159	0.318	-0.397	-0.0214	0.0122
2.0	0.000652	0.134	0.429	-1.557	0.000583	0.0743	0.177	0.333	0.0748	0.510	-0.0279	-0.0298

TABLE 6 - 4

The constant coefficients as functions of  $Bl_g$ , for an infinite plate ( $Lu = 0.3$ ;  $\epsilon = 0.4$ ;  $Ko = 3$ ;  $Pn = 0.25$ ;  $Kl_m = 0.5$ )

$Bl_g$	$C_{n,t}$				$D_{n,t}$				$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22	11	12	21	22	$n = 1$	$n = 2$	$n = 1$	$n = 2$
5	0.542	0.171	3.284	-0.873	0.392	0.0230	1.875	0.0971	0.415	1.972	-0.0168	-0.0510
10	0.692	0.184	2.501	-0.565	0.500	0.0240	1.420	0.0647	0.524	1.485	-0.00935	-0.0186
15	0.755	0.192	2.344	-0.502	0.545	0.0248	1.329	0.0579	0.570	1.387	-0.00706	-0.0112
20	0.791	0.196	2.268	-0.471	0.571	0.0251	1.284	0.0548	0.596	1.339	-0.00544	-0.00817



TABLE 6 - 7

The constant coefficients as functions of  $K_0$ , for an infinite plate ( $Lu = 0.3$ ;  $\epsilon = 0.4$ ;  $Pn = 0.25$ ;  $KI_m = 0.5$ ;  $BI_q = 1.0$ )

$K_0$	$C_{n1}$					$D_{n1}$					$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22		11	12	21	22		$n = 1$	$n = 2$	$n = 1$	$n = 2$
1.0	0.685	0.0754	-1.439	0.290		0.462	0.00876	-0.814	-0.0150		0.471	-0.829	-0.00775	0.0107
1.5	0.693	0.107	-4.993	1.045		0.477	0.0130	-2.820	-0.0741		0.490	-2.894	-0.00837	0.0372
2.0	0.698	0.136	13.589	-2.935		0.489	0.0171	7.681	0.256		0.506	7.937	-0.00880	-0.0989
2.5	0.696	0.162	3.904	-0.865		0.496	0.0208	2.208	0.0880		0.517	2.236	-0.00950	-0.0295
3.0	0.692	0.184	2.501	-0.565		0.500	0.0240	1.420	0.0647		0.524	1.485	-0.00935	-0.0186
4.0	0.671	0.226	1.653	-0.387		0.496	0.0303	0.945	0.0527		0.527	0.998	-0.0112	-0.0125
5.0	0.637	0.258	1.314	-0.319		0.482	0.0349	0.760	0.0540		0.516	0.814	-0.0177	-0.0177
6.0	0.592	0.283	1.109	-0.273		0.450	0.0383	0.645	0.0462		0.494	0.691	-0.0134	-0.0086

TABLE 6 - 8

The constant coefficients as functions of  $Pn$ , for an infinite plate ( $Lu = 0.3$ ;  $\epsilon = 0.4$ ;  $Ko = 3$ ;  $BI_q = 1.0$ ;  $KI_m = 0.5$ )

$Pn$	$C_{n1}$					$D_{n1}$					$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22		11	12	21	22		$n = 1$	$n = 2$	$n = 1$	$n = 2$
0.08333	0.525	0.0814	-1.767	0.415		0.354	0.00946	-1.000	-0.0215		0.364	-1.021	-0.00758	0.0107
0.1250	0.570	0.107	-5.543	1.270		0.392	0.0130	-3.131	-0.0901		0.405	-3.221	-0.00832	0.0372
0.1667	0.615	0.133	14.304	-3.249		0.431	0.0167	8.085	0.283		0.447	8.368	-0.00882	-0.0997
0.2083	0.653	0.159	3.981	-0.902		0.465	0.0204	2.252	0.0917		0.486	2.344	-0.00946	-0.0296
0.2500	0.692	0.184	2.501	-0.565		0.500	0.0240	1.420	0.0647		0.524	1.485	-0.00935	-0.0186
0.3333	0.766	0.240	1.620	-0.368		0.567	0.0321	0.926	0.0501		0.599	0.976	-0.0112	-0.0126
0.4167	0.842	0.298	1.280	-0.295		0.637	0.0402	0.741	0.0450		0.677	0.786	-0.0121	-0.00967
0.5	0.925	0.360	1.086	-0.249		0.712	0.0489	0.632	0.0421		0.761	0.674	-0.0135	-0.0086

Hence  $C_{ni}$  and  $\psi_n$  are defined by equations(6-2-11) through (6-2-13), while

$$Q_{ni} = J_0(v_i \mu_n) - \frac{1}{Bl_q} v_i \mu_n J_1(v_i \mu_n); \quad (6-2-23)$$

$$P_{ni} = -[\epsilon Ko Pn + (1 - v_i^2)] v_i \mu_n J_1(v_i \mu_n); \quad (6-2-24)$$

$$A_{ni} = J_1(v_i \mu_n) + \frac{1}{Bl_q} v_i \mu_n J_0(v_i \mu_n); \quad (6-2-25)$$

$$B_{ni} = [\epsilon Ko Pn + (1 - v_i^2)] v_i \mu_n J_0(v_i \mu_n); \quad (6-2-26)$$

$$C_{mi} = -\frac{2Lu}{\mu_m^2 J_0(v_i \mu_m)}, \quad C_{m2} = \frac{2Lu}{\mu_m^2 J_0(v_2 \mu_m)}. \quad (6-2-27)$$

TABLE 6-9

Infinite cylinder. Roots  $\mu_n$  of characteristic equation (6-2-28) as functions of  $Lu$ ,  $Bl_q$ ,  $\epsilon$ , and  $Ko$

Dimensionless number		n	Lu				
			0.1	0.3	0.5	0.7	1.0
$Bl_q$	5	1	1.9967	1.7659	1.8662	1.8903	1.9036
		2	2.1896	2.2416	2.6268	2.9274	3.2233
	10	1	2.1123	1.8071	1.9651	2.0097	2.0348
		2	2.2732	2.4208	2.7814	3.0958	3.4246
	20	1	2.1326	1.8218	2.0083	2.0666	2.1001
		2	2.3676	2.5340	2.8883	3.2121	3.5634
$\epsilon$	0.2	1	2.1424	1.9020	2.0577	2.0890	2.1044
		2	2.2495	2.3527	2.7599	3.1473	3.5806
	0.8	1	1.1430	1.6771	1.8233	1.8765	3.0020
		2	2.0584	2.4863	2.7813	1.9108	3.2154
$Ko$	3	1	1.1893	1.8800	2.0375	2.0724	2.0901
		2	2.1361	2.3696	2.7659	3.1366	3.5446
	7	1	2.0899	1.7488	1.9030	1.9547	1.9828
		2	2.2903	2.4550	2.7867	3.0569	3.3299

Note. The values of the roots for  $Bl_q = 10$ ;  $\epsilon = 0.4$ ;  $Ko = 5$ , as well as for all the values of  $Kl_m$  and the corresponding  $Lu$ , are equal. In the table they are given under  $Bl_q = 10$ .

TABLE 6-10

Infinite cylinder. Roots  $\mu_n$  of characteristic equation (6-2-28) as functions of  $Pn$  (for  $Lu = 0.3$ )

n	Pn			
	0.1	0.25	0.40	0.55
1	1.9263	1.8071	1.7232	1.6561
2	2.3334	2.4208	2.4677	2.4926
3	—	3.5429	3.4249	3.3334
4	—	4.9965	4.8265	4.6698

Characteristic-equation roots  $\mu_n$  are found by solving the equation

$$[\epsilon Ko Pn + (1 - v_1^2)] \frac{J_0(v_2 \mu_n)}{v_2 \mu_n J_1(v_2 \mu_n)} - [\epsilon Ko Pn + (1 - v_2^2)] \frac{J_0(v_1 \mu_n)}{v_1 \mu_n J_1(v_1 \mu_n)} = \frac{v_2^2 - v_1^2}{Bl_q}, \quad (6-2-28)$$

while  $\mu_m$  are roots of the system of equations

$$\left. \begin{aligned} J_1(\nu_i \mu_m) &= 0; \\ J_1(\nu_s \mu_m) &= 0. \end{aligned} \right\} \quad (6-2-29)$$

When the dimensionless potentials are averaged over the spatial coordinate, we obtain

$$\begin{aligned} \bar{T}(Fo) &= 1 - \varepsilon Ko \bar{K}_1 - \frac{1}{2} \varepsilon Ko Lu Ki_m \left( \frac{1}{2} + \frac{2}{Bl_q} \right) - \\ &- \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} \exp(-\mu_n^2 Fo) - \frac{\varepsilon Ko Ki_m}{\nu_2^2 - \nu_1^2} \sum_{m=1}^{\infty} \sum_{l=1}^2 D_{ml} \exp(-\mu_m^2 Fo); \\ \bar{\theta}(Fo) &= 2Ki_m Lu Fo + \frac{1}{\varepsilon Ko} \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} (1 - \nu_i^2) \exp(-\mu_n^2 Fo) + \\ &+ \frac{Ki_m}{\nu_2^2 - \nu_1^2} \sum_{m=1}^{\infty} \sum_{l=1}^2 D_{ml} (1 - \nu_l^2) \exp(-\mu_m^2 Fo), \end{aligned}$$

where

$$D_{ni} = 2C_{ni} \frac{J_1(\nu_i \mu_n)}{\nu_i \mu_n} \quad \text{and} \quad D_{ml} = 2C_{ml} \frac{J_1(\nu_l \mu_m)}{\nu_l \mu_m}.$$

The simplifications of the general solutions for an infinite plate may be extended to the corresponding solutions for the infinite cylinder and the sphere. Certain fluctuations observed in the values of the Fourier number thus become unimportant.

Values of the roots of characteristic equation (6-2-28), for various combinations of dimensionless numbers, are given in Tables 6-9 and 6-10. Values of constant coefficients  $C_{ni}$  and  $D_{ni}$  are given in Tables 6-11 through 6-15.

### c) Solution for a sphere ( $\Gamma=2$ )

$$\begin{aligned} T(X, Fo) &= 1 - \varepsilon Ko \bar{K}_1 - \frac{1}{2} \varepsilon Ko Lu Ki_m \left( 1 - X^* + \frac{2}{Bl_q} \right) + \\ &+ \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \frac{\sin \nu_i \mu_n X}{X} \exp(-\mu_n^2 Fo) + \frac{\varepsilon Ko Ki_m}{\nu_2^2 - \nu_1^2} \times \\ &\times \sum_{m=1}^{\infty} \sum_{l=1}^2 C_{ml} \frac{\sin \nu_l \mu_m X}{X} \exp(-\mu_m^2 Fo); \end{aligned} \quad (6-2-30)$$

$$\begin{aligned} \theta(X, Fo) &= Ki_m [3Lu Fo - \frac{1}{2} (1 + \varepsilon Ko Pn Lu) \left( \frac{3}{5} - X^* \right)] - \\ &- \frac{1}{\varepsilon Ko} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - \nu_i^2) \frac{\sin \nu_i \mu_n X}{X} \exp(-\mu_n^2 Fo) - \\ &- \frac{Ki_m}{\nu_2^2 - \nu_1^2} \sum_{m=1}^{\infty} \sum_{l=1}^2 C_{ml} (1 - \nu_l^2) \frac{\sin \nu_l \mu_m X}{X} \exp(-\mu_m^2 Fo). \end{aligned} \quad (6-2-31)$$

The quantities  $C_{ni}$  and  $\psi_n$  in (6-2-30) and (6-2-31) are defined by equations (6-2-11) through (6-2-13), while

$$Q_{ni} = \left( 1 - \frac{1}{Bl_q} \right) \sin \nu_i \mu_n + \frac{1}{Bl_q} \nu_i \mu_n \cos \nu_i \mu_n; \quad (6-2-32)$$

TABLE 6-11  
The constant coefficients as functions of  $Lu$  and  $Bl_\phi$  for an infinite cylinder

Lu	$Bl_\phi$	$C_{n1}$		$C_{n2}$		$D_{n1}$		$D_{n2}$	
		$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
0.1	5	1.1502	0.2408	-0.7728·10 <sup>-1</sup>	0.8604·10 <sup>-1</sup>	0.6861	0.1276	0.3760·10 <sup>-2</sup>	0.6994·10 <sup>-2</sup>
	10	0.4058	1.0833	-0.9124·10 <sup>-1</sup>	0.1040	0.2260	0.5425	0.1238·10 <sup>-2</sup>	0.2973·10 <sup>-2</sup>
	20	0.1534	1.3766	-0.6025·10 <sup>-1</sup>	0.7562·10 <sup>-1</sup>	0.8434·10 <sup>-1</sup>	0.6438	0.4621·10 <sup>-2</sup>	0.3528·10 <sup>-2</sup>
0.3	5	0.2343	0.8451	0.1313	-0.2604	0.1660	0.4736	0.9222·10 <sup>-2</sup>	0.2631·10 <sup>-1</sup>
	10	0.1301	1.0987	0.1016	-0.2426	0.9057·10 <sup>-1</sup>	0.5514	0.5032·10 <sup>-2</sup>	0.3063·10 <sup>-1</sup>
	20	0.9396·10 <sup>-1</sup>	1.2004	0.8520·10 <sup>-1</sup>	-0.2348	0.6501·10 <sup>-1</sup>	0.5578	0.3612·10 <sup>-2</sup>	0.3099·10 <sup>-1</sup>
0.5	5	0.3416	0.4651	0.2031	-0.4041	0.2452	0.2304	0.3866·10 <sup>-1</sup>	0.3633·10 <sup>-1</sup>
	10	0.2990	0.7109	0.2318	-0.4389	0.2065	0.3187	0.3256·10 <sup>-1</sup>	0.5025·10 <sup>-1</sup>
	20	0.2605	0.8488	0.2328	-0.4446	0.1768	0.3529	0.2787·10 <sup>-1</sup>	0.5564·10 <sup>-1</sup>
0.7	5	0.2232	0.3356	0.1816	-0.4695	0.1670	0.1573	0.4863·10 <sup>-1</sup>	0.4582·10 <sup>-1</sup>
	10	0.2598	0.5314	0.2608	-0.5473	0.1867	0.2238	0.5437·10 <sup>-1</sup>	0.6518·10 <sup>-1</sup>
	20	0.2556	0.6627	0.2892	-0.5777	0.1800	0.2535	0.5244·10 <sup>-1</sup>	0.7384·10 <sup>-1</sup>
1.0	5	0.4236·10 <sup>-1</sup>	0.2234	0.5073·10 <sup>-1</sup>	-0.4930	0.3347·10 <sup>-1</sup>	0.1066	0.1673·10 <sup>-1</sup>	0.5328·10 <sup>-1</sup>
	10	0.1418	0.3735	0.2028	-0.2294	0.1081	0.1500	0.5406·10 <sup>-1</sup>	0.7950·10 <sup>-1</sup>
	20	0.1707	0.4700	0.2710	-0.6973	0.1278	0.1836	0.6390·10 <sup>-1</sup>	0.9181·10 <sup>-1</sup>



TABLE 6-12  
The constant coefficients as functions of  $Lu$  and  $Kl_m$ , for an infinite cylinder

$Lu$	$Kl_m$	$C_{n1}$		$C_{n3}$		$D_{n1}$		$D_{n3}$	
		$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
0.1	0.1	0.4537	1.0716	-0.1020	0.1029	0.2526	0.5366	0.1384·10 <sup>-2</sup>	0.2940·10 <sup>-2</sup>
	0.5	0.4058	1.0833	-0.9124·10 <sup>-1</sup>	0.1040	0.2260	0.5425	0.1238·10 <sup>-2</sup>	0.2973·10 <sup>-2</sup>
	1.0	0.3459	1.0979	-0.7778·10 <sup>-2</sup>	0.1054	0.1926	0.5498	0.1056·10 <sup>-2</sup>	0.3013·10 <sup>-2</sup>
0.3	0.1	0.2686	1.1197	0.2098	-0.2473	0.1870	0.5620	0.1039·10 <sup>-1</sup>	0.3122·10 <sup>-1</sup>
	0.5	0.1301	1.0987	0.1016	-0.2426	0.9057·10 <sup>-1</sup>	0.5514	0.5032·10 <sup>-2</sup>	0.3063·10 <sup>-1</sup>
	1.0	-0.4306·10 <sup>-1</sup>	1.0724	-0.3363·10 <sup>-1</sup>	-0.2368	-0.2998·10 <sup>-1</sup>	0.5382	-0.1666·10 <sup>-2</sup>	0.2990·10 <sup>-1</sup>
0.5	0.1	0.5421	0.6952	0.4203	-0.4292	0.3745	0.3117	0.5904·10 <sup>-1</sup>	0.4914·10 <sup>-1</sup>
	0.5	0.2990	0.7109	0.2318	-0.4389	0.2065	0.3187	0.3256·10 <sup>-1</sup>	0.5025·10 <sup>-1</sup>
	1.0	-0.4959·10 <sup>-2</sup>	0.7305	-0.3845·10 <sup>-2</sup>	-0.4510	-0.3426·10 <sup>-2</sup>	0.3275	-0.5401·10 <sup>-2</sup>	0.5163·10 <sup>-1</sup>
0.7	0.1	0.5482	0.5221	0.5504	-0.5378	0.3939	0.2199	0.1147	0.2404·10 <sup>-1</sup>
	0.5	0.2598	0.5314	0.2608	-0.5473	-0.1867	0.2238	0.5437·10 <sup>-1</sup>	0.6518·10 <sup>-1</sup>
	1.0	-0.1008	0.5430	-0.1012	-0.5592	-0.7243·10 <sup>-1</sup>	0.2287	-0.2110·10 <sup>-1</sup>	0.6661·10 <sup>-1</sup>
1.0	0.1	0.4668	0.3868	0.6681	-0.6520	0.3550	0.1647	0.1780	0.8236·10 <sup>-1</sup>
	0.5	0.1418	0.3735	0.2028	-0.2294	0.1081	0.1500	0.5406·10 <sup>-1</sup>	0.7950·10 <sup>-1</sup>
	1.0	-0.2645	0.3657	-0.3786	-0.6010	-0.2017	0.1518	-0.1008	0.7592·10 <sup>-1</sup>

TABLE 6-13  
The constant coefficients as functions of  $Lu$  and  $s$ , for an infinite cylinder

$Lu$	$s$	$C_{n1}$		$C_{n2}$		$D_{n1}$		$D_{n2}$	
		$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
0.1	0.2	0.6234	0.8857	$-0.7474 \cdot 10^{-1}$	$0.8230 \cdot 10^{-1}$	0.3344	0.4416	$0.9715 \cdot 10^{-2}$	$0.1283 \cdot 10^{-2}$
	0.4	0.4058	1.0833	$-0.9124 \cdot 10^{-1}$	0.1040	0.2260	0.5425	$0.1238 \cdot 10^{-2}$	$0.2973 \cdot 10^{-2}$
	0.8	$-0.2653 \cdot 10^{-1}$	0.2295	$-0.4414 \cdot 10^{-1}$	$-0.9581 \cdot 10^{-1}$	$-0.2281 \cdot 10^{-1}$	$0.1361 \cdot 10^{-1}$	$-0.2237 \cdot 10^{-2}$	$0.1334 \cdot 10^{-2}$
0.3	0.2	0.2226	1.1156	0.1200	$-0.1842$	0.1430	0.5477	$0.4990 \cdot 10^{-2}$	$0.1911 \cdot 10^{-1}$
	0.4	0.1301	1.0987	0.1016	$-0.2426$	$0.9057 \cdot 10^{-1}$	0.5514	$0.5032 \cdot 10^{-2}$	$0.3063 \cdot 10^{-1}$
	0.8	$0.7333 \cdot 10^{-2}$	0.9848	$0.8170 \cdot 10^{-2}$	$-0.3124$	$0.5620 \cdot 10^{-2}$	0.5281	$0.4312 \cdot 10^{-2}$	$0.4051 \cdot 10^{-1}$
0.5	0.2	0.5841	0.5756	0.2763	$-0.3514$	0.3636	0.2276	$0.4475 \cdot 10^{-1}$	$0.2801 \cdot 10^{-1}$
	0.4	0.2990	0.7109	0.2318	$-0.4369$	0.2065	0.3187	$0.3256 \cdot 10^{-1}$	$0.5025 \cdot 10^{-1}$
	0.8	$0.2020 \cdot 10^{-1}$	0.7039	$0.2356 \cdot 10^{-1}$	$-0.4848$	$0.1566 \cdot 10^{-1}$	0.3737	$0.2687 \cdot 10^{-1}$	$0.6412 \cdot 10^{-1}$
0.7	0.2	0.5131	0.4416	0.3671	$-0.4478$	0.3331	0.1447	$0.6354 \cdot 10^{-1}$	$0.4064 \cdot 10^{-1}$
	0.4	$0.2598$	0.5314	0.2608	$-0.5473$	0.1867	0.2238	$0.5437 \cdot 10^{-1}$	$0.6518 \cdot 10^{-1}$
	0.8	$-0.4038 \cdot 10^{-1}$	0.5216	$-0.5438 \cdot 10^{-1}$	$-0.5684$	$-0.3235 \cdot 10^{-1}$	0.2639	$-0.8566 \cdot 10^{-2}$	$0.7518 \cdot 10^{-1}$
1.0	0.2	0.3283	0.3395	0.4252	$-0.5514$	0.2293	$0.8986 \cdot 10^{-1}$	0.1398	$0.6088 \cdot 10^{-1}$
	0.4	0.1418	0.3735	0.2028	$-0.2294$	0.1081	0.1500	$0.5406 \cdot 10^{-1}$	$0.7950 \cdot 10^{-1}$
	0.8	$-0.1610$	0.3553	$-0.2612$	$-0.5977$	$-0.1345$	0.2065	$-0.5136 \cdot 10^{-1}$	$0.7888 \cdot 10^{-1}$

TABLE 6-14

The constant coefficients as functions of  $Lu$  and  $Ko$ , for an infinite cylinder

$Lu$	$Ko$	$C_{n1}$		$C_{n2}$		$D_{n1}$		$D_{n2}$	
		$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
0.1	3	-0.7222·10 <sup>-3</sup>	0.5696	-0.1123·10 <sup>-1</sup>	-0.8061·10 <sup>-1</sup>	-0.6055·10 <sup>-3</sup>	0.3080	-0.2086·10 <sup>-4</sup>	0.1061·10 <sup>-2</sup>
	5	0.4058	1.0833	-0.9124·10 <sup>-1</sup>	0.1040	0.2260	0.5425	0.1238·10 <sup>-3</sup>	0.2973·10 <sup>-3</sup>
	7	0.3082	1.1479	-0.9376·10 <sup>-1</sup>	0.1142	-0.1762	0.5774	0.1292·10 <sup>-3</sup>	0.4233·10 <sup>-3</sup>
0.3	3	0.2119	1.1490	0.1258	-0.2046	0.1388	0.5658	0.5533·10 <sup>-3</sup>	0.2256·10 <sup>-1</sup>
	5	0.1301	1.0987	0.1016	-0.2426	0.9037·10 <sup>-1</sup>	0.5514	0.5032·10 <sup>-3</sup>	0.3063·10 <sup>-1</sup>
	7	0.9626·10 <sup>-1</sup>	1.0665	0.8279·10 <sup>-1</sup>	-0.2559	0.8867·10 <sup>-1</sup>	0.5418	0.4218·10 <sup>-3</sup>	0.3328·10 <sup>-1</sup>
0.5	3	0.5484	0.6480	0.2976	-0.3948	0.3501	0.2637	0.4668·10 <sup>-1</sup>	0.3516·10 <sup>-1</sup>
	5	0.2990	0.7109	0.2318	-0.4389	0.2065	0.3187	0.3256·10 <sup>-1</sup>	0.5025·10 <sup>-1</sup>
	7	0.1377	0.6959	0.1317	-0.4449	0.1006	0.3369	0.1687·10 <sup>-1</sup>	0.5652·10 <sup>-1</sup>
0.7	3	0.5037	0.5010	0.3957	-0.5092	0.3354	0.1750	0.6631·10 <sup>-1</sup>	0.5026·10 <sup>-1</sup>
	5	0.2598	0.5314	0.2608	-0.5473	0.1867	0.2238	0.5437·10 <sup>-1</sup>	0.6518·10 <sup>-1</sup>
	7	0.8322·10 <sup>-1</sup>	0.5086	0.9660·10 <sup>-1</sup>	-0.5299	0.2302·10 <sup>-1</sup>	0.2429	0.1768·10 <sup>-1</sup>	0.6813·10 <sup>-1</sup>
1.0	3	0.3494	0.3859	0.4633	-0.6356	0.2495	0.1256	0.1452	0.7313·10 <sup>-1</sup>
	5	0.1418	0.3735	0.2028	-0.2294	0.1081	0.1500	0.5406·10 <sup>-1</sup>	0.7960·10 <sup>-1</sup>
	7	-0.3788·10 <sup>-1</sup>	0.3344	-0.5746·10 <sup>-1</sup>	-0.5646	-0.3021·10 <sup>-1</sup>	0.1670	-0.1338·10 <sup>-1</sup>	0.7400·10 <sup>-1</sup>

$$P_{ni} = [\varepsilon \text{Ko Pn} + (1 - \nu_i^2)] (\nu_i \mu_n \cos \nu_i \mu_n - \sin \nu_i \mu_n); \quad (6-2-33)$$

$$A_{ni} = \cos \nu_i \mu_n - \frac{1}{\text{Bi}_q} \nu_i \mu_n \sin \nu_i \mu_n; \quad (6-2-34)$$

$$B_{ni} = -[\varepsilon \text{Ko Pn} + (1 - \nu_i^2)] \nu_i \mu_n \sin \nu_i \mu_n; \quad (6-2-35)$$

$$C_{m1} = 2\text{Lu} \frac{\text{Bi}_q - 1 + \frac{(1 - \varepsilon) \text{Ko Lu Kl}_m - \text{Bi}_q}{\varepsilon \text{Ko Kl}_m} [\varepsilon \text{Ko Pn} + (1 - \nu_2^2)]}{\mu_m^2 \sin \nu_1 \mu_m}; \quad (6-2-36)$$

$$C_{m2} = -2\text{Lu} \frac{\text{Bi}_q - 1 + \frac{(1 - \varepsilon) \text{Ko Lu Kl}_m - \text{Bi}_q}{\varepsilon \text{Ko Kl}_m} [\varepsilon \text{Ko Pn} + (1 - \nu_2^2)]}{\mu_m^2 \sin \nu_2 \mu_m}; \quad (6-2-37)$$

where  $\mu_n$  are the roots of the characteristic equation

$$\frac{E_1 \tan \nu_2 \mu_n - E_2 \tan \nu_1 \mu_n}{E_2 \nu_1 - E_1 \nu_2} = \frac{\mu_n}{\text{Bi}_q - 1}, \quad (6-2-38)$$

TABLE 6-15

The constant coefficients as functions of Pn, for an infinite cylinder

Pn	n\j	C <sub>nj</sub>	D <sub>nj</sub>
0.1	11	0.1013	0.6370 · 10 <sup>-1</sup>
	21	1.2080	0.5922
	12	0.4858 · 10 <sup>-1</sup>	0.1870 · 10 <sup>-2</sup>
	22	-0.1814	0.1739 · 10 <sup>-1</sup>
0.25	11	0.1301	0.9057 · 10 <sup>-1</sup>
	21	1.0987	0.5514
	12	0.1016	0.5032 · 10 <sup>-2</sup>
	22	-0.2426	0.3063 · 10 <sup>-1</sup>
0.40	11	0.1210	0.8982
	21	1.0284	0.5363
	12	0.1207	0.6324 · 10 <sup>-2</sup>
	22	-0.2856	0.3776 · 10 <sup>-1</sup>
0.55	11	0.1095	0.8507 · 10 <sup>-1</sup>
	21	0.9543	0.5190
	12	0.1277	0.6732 · 10 <sup>-2</sup>
	22	-0.3227	0.4106 · 10 <sup>-1</sup>

in which

$$E_i = [\varepsilon \text{Ko Pn} + (1 - \nu_i^2)] (\nu_i \mu_n - \tan \nu_i \mu_n),$$

and where  $\mu_m$  are the roots of the system of equations

$$\left. \begin{aligned} \cos \nu_1 \mu_m &= 0; \\ \cos \nu_2 \mu_m &= 0. \end{aligned} \right\} \quad (6-2-39)$$

For the average dimensionless transfer potentials solutions (6-2-30) and (6-2-31) become

$$\begin{aligned}\bar{T}(Fo) &= 1 - \epsilon Ko \bar{K}_1 - \frac{1}{2} \epsilon Ko Lu Ki_m \left( \frac{2}{5} + \frac{2}{Bi_q} \right) + \\ &+ \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} \exp(-\mu_n^2 Fo) + \frac{\epsilon Ko Ki_m}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 D_{mi} \exp(-\mu_m^2 Fo); \\ \bar{\theta}(Fo) &= 3 Ki_m Lu Fo - \frac{1}{\epsilon Ko} \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} (1 - v_i^2) \exp(-\mu_n^2 Fo) - \\ &- \frac{Ki_m}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 D_{mi} (1 - v_i^2) \exp(-\mu_m^2 Fo),\end{aligned}$$

where

$$D_{ni} = 3C_{ni} \frac{\sin v_i \mu_n - v_i \mu_n \cos v_i \mu_n}{v_i^2 \mu_n^2} \quad \text{and} \quad D_{mi} = 3C_{mi} \frac{\sin v_i \mu_m - v_i \mu_m \cos v_i \mu_m}{v_i^2 \mu_m^2}.$$

#### d) Generalized solutions

A comparison of the solutions obtained for the various body shapes shows that these solutions have much in common. Thus, it is possible to write all the previous solutions in the following generalized form:

$$\begin{aligned}T(X, Fo) &= 1 - (1 - \epsilon) Ko Lu \frac{Ki_m}{Bi_q} - \frac{1}{2} \epsilon Ko Lu Ki_m \left( 1 - X^2 + \frac{2}{Bi_q} \right) - \\ &- \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \Phi_r(v_i \mu_n X) \exp(-\mu_n^2 Fo) - \frac{\epsilon Ko Ki_m}{v_2^2 - v_1^2} \times \\ &\times \sum_{m=1}^{\infty} \sum_{i=1}^2 C_{mi} \Phi_r(v_i \mu_m X) \exp(-\mu_m^2 Fo); \quad (6-2-40)\end{aligned}$$

$$\begin{aligned}\theta(X, Fo) &= Ki_m \left[ (\Gamma + 1) Lu Fo - \frac{1}{2} (1 + \epsilon Ko Pn Lu) (\chi_r - X^2) \right] + \\ &+ \frac{1}{\epsilon Ko} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2) \Phi_r(v_i \mu_n X) \exp(-\mu_n^2 Fo) + \\ &+ \frac{Ki_m}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 C_{mi} (1 - v_i^2) \Phi_r(v_i \mu_m X) \exp(-\mu_m^2 Fo). \quad (6-2-41)\end{aligned}$$

For an infinite plate  $\Gamma=0$ , for an infinite cylinder  $\Gamma=1$ , and for a sphere  $\Gamma=2$ . Here  $C_{ni}$  and  $\psi_n$  are defined in all cases by equations (6-2-11), (6-2-12), and (6-2-13). The quantities  $Q_{ni}$ ,  $P_{ni}$ ,  $A_{ni}$ , and  $B_{ni}$  for a plate are defined by equations (6-2-14) through (6-2-17), for a cylinder by equations (6-2-23) through (6-2-27), and for a sphere by equations (6-2-32) through (6-2-35). Coefficients  $C_{mi}$  for a plate, cylinder, and sphere are defined by (6-2-18), (6-2-27), and (6-2-36) and (6-2-37). The characteristic-equation roots  $\mu_n$  for these body shapes are the roots of equations (6-2-8), (6-2-28), and (6-2-38), while  $\mu_m$  are the roots of systems of equations (6-2-7), (6-2-29), and (6-2-39). In the generalized equations, the following quantities have been introduced:

$$\Phi_r(v_i \mu_h X) = \cos v_i \mu_h X, \quad \chi_0 = 1/3;$$

$$\Phi_1(v_i \mu_h X) = J_0(v_i \mu_h X), \quad \chi_1 = 1/2;$$

$$\Phi_2(v_i \mu_h X) = -\frac{\sin v_i \mu_h X}{X}, \quad \chi_2 = 3/5.$$

If a steady state is established, solutions (6-40) and (6-41) simplify to

$$T = 1 - (1 - \epsilon) \text{Ko Lu} \frac{\text{Ki}_m}{\text{Bi}_q} - \frac{1}{2} \epsilon \text{Ko Lu Ki}_m \left(1 - X^2 + \frac{2}{\text{Bi}_q}\right); \quad (6-2-42)$$

$$\theta = \text{Ki}_m [(\Gamma + 1) \text{Lu Fo} - \frac{1}{2} (1 + \epsilon \text{Ko Pn Lu})(\chi_r - X^2)]. \quad (6-2-43)$$

### 6-3. Unsteady Fields of Heat-Transfer and Mass-Transfer Potential. Mass Flux at Surface of Body a Function of the Mass-Transfer Potential

Once again system of heat-transfer and mass-transfer equations (4-1-2) and (4-1-3) is solved with boundary conditions (6-1-1) and (6-1-2). In this case, however, the Kirpichev mass-exchange number is  $\text{Ki}_m = \text{Bi}_m [1 - \theta(1, \text{Fo})]$ . Solutions of the systems of equations will be given for constant (6-1-3) and parabolic (6-1-4) initial distributions of the transfer potentials. The methods for solving these problems do not differ from those employed, for the corresponding body shapes, in Chapter IV and in § 6-2, a. A detailed account of these solutions may also be found in /1, 3/.

a) Solution for an infinite plate ( $\Gamma = 0$ )

$$T(X, \text{Fo}) = 1 - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \cos v_i \mu_n X \exp(-\mu_n^2 \text{Fo}); \quad (6-3-1)$$

$$\theta(X, \text{Fo}) = 1 + \frac{1}{\epsilon \text{Ko}} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2) \cos v_i \mu_n X \exp(-\mu_n^2 \text{Fo}). \quad (6-3-2)$$

Here

$$C_{n1} = 2 \frac{(1 - \epsilon \text{Ko } K_1) P_{n2} + \epsilon \text{Ko } Q_{n2}}{\mu_n \psi_n}; \quad (6-3-3)$$

$$C_{n2} = -2 \frac{(1 - \epsilon \text{Ko } K_1) P_{n1} + \epsilon \text{Ko } Q_{n1}}{\mu_n \psi_n}; \quad (6-3-4)$$

$$\psi_n = v_1 A_{n1} P_{n2} + v_2 B_{n2} Q_{n1} - v_2 A_{n2} P_{n1} - v_1 B_{n1} Q_{n2}; \quad (6-3-5)$$

$$A_{ni} = \left[1 + \frac{1}{\text{Bi}_q} + (1 - v_i^2) K_1\right] \sin v_i \mu_n + \frac{1}{\text{Bi}_q} v_i \mu_n \cos v_i \mu_n; \quad (6-3-6)$$

$$B_{ni} = (1 - v_i^2) \sin v_i \mu_n + \frac{(1 - v_i^2) + \epsilon \text{Ko Pn}}{\text{Bi}_m} (\sin v_i \mu_n + v_i \mu_n \cos v_i \mu_n); \quad (6-3-7)$$

$$Q_{ni} = [1 + (1 - v_i^2) K_1] \cos v_i \mu_n - \frac{1}{\text{Bi}_q} v_i \mu_n \sin v_i \mu_n; \quad (6-3-8)$$

$$P_{ni} = (1 - v_i^2) \cos v_i \mu_n - \frac{(1 - v_i^2) + \epsilon \text{Ko Pn}}{\text{Bi}_m} v_i \mu_n \sin v_i \mu_n; \quad (6-3-9)$$

$$K_1 = \frac{1 - \epsilon}{\epsilon} \text{Lu} \frac{\text{Bi}_m}{\text{Bi}_q},$$

and  $\mu_n$  are the roots of the characteristic equation

$$P_{n1} Q_{n2} - P_{n2} Q_{n1} = 0. \quad (6-3-10)$$

As in all the previous problems the values of the characteristic-equation roots  $\mu_n$  can be obtained using graphical methods and then these can be determined more accurately using the familiar methods of numerical analysis (for example, the methods of Newton, Bairstow, or Lin). To determine the roots by graphical methods, let us rewrite equation (6-3-10) in a somewhat different form:

$$\frac{M}{N} = \mu_n / Bi_q, \tag{6-3-11}$$

where

$$\begin{aligned} M &= P_{n1} \cos v_1 \mu_n - P_{n2} \cos v_2 \mu_n; \\ N &= [(1 - v_1^2) \frac{1 - \epsilon}{\epsilon} Lu Bi_m \frac{\cos v_1 \mu_n}{\mu_n} - v_1 \sin v_1 \mu_n] P_{n2} - \\ &\quad - (1 - v_2^2) \frac{1 - \epsilon}{\epsilon} Lu Bi_m \frac{\cos v_2 \mu_n}{\mu_n} - v_2 \sin v_2 \mu_n] P_{n1}. \end{aligned}$$

If the left side of (6-3-11) is denoted by  $\Phi_\mu = \Phi(\mu)$  and the right side by  $y = \frac{1}{Bi_q} \mu$ , then the points at which the  $\Phi_\mu$  curves intersect a  $y$  line on the  $(\mu, \Phi_\mu)$  graph indicate the values of the roots  $\mu_n$  for the specified combination of  $Lu$ ,  $\epsilon$ ,  $Ko$ ,  $Pn$ ,  $Bi_m$ , and  $Bi_q$  (Figure 6-10). Figures 6-11 through 6-18 show

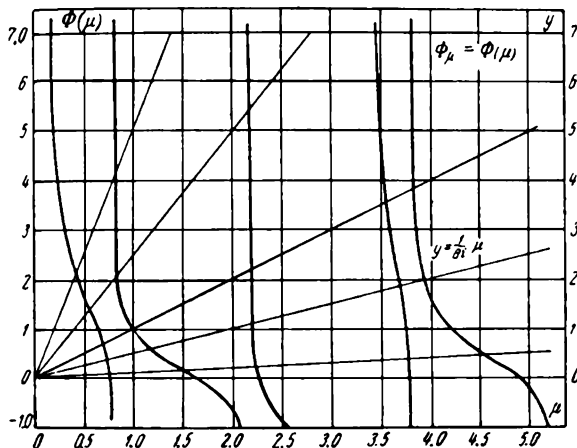


FIGURE 6-10. Method for determining the roots of characteristic equation (6-3-10), for an infinite plate

the characteristic-equation roots  $\mu_n$  as functions of  $Lu$ ,  $Bi_m$ ,  $\epsilon Ko Pn$ , or  $\epsilon$  for values of  $Bi_q$  from 0.1 to 2.0 or 20. The corresponding tables of the characteristic-equation roots are given in /1, 3/.

Both equation (6-3-11) and Figure 6-10 indicate that an infinite series of roots  $\mu_n$  exists, whose absolute values increase as follows:

$$\mu_1 < \mu_2 < \dots \mu_n < \dots$$

Because of this inequality of the roots, the infinite series in equations (6-3-1) and (6-3-2) converge quite rapidly. Calculations show that, as the Fourier number increases, the error introduced by neglecting series terms after the first two decreases rapidly. Whereas, for the center of the plate,

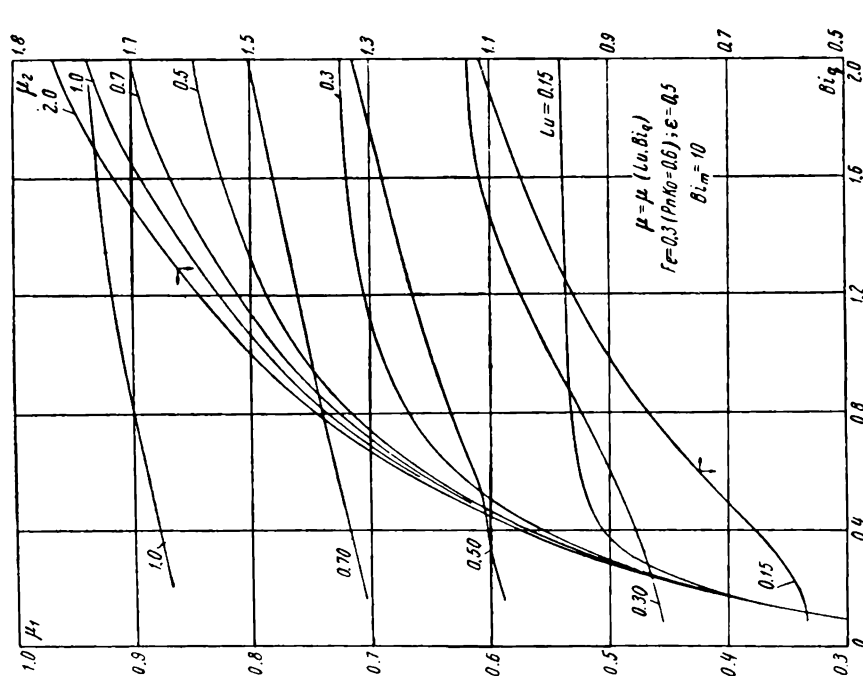


FIGURE 6-11. Characteristic-equation roots as functions of  $Bi_q$  and  $Lu$  (for  $Bi_q$  from 0.1 to 2.0), for an infinite plate

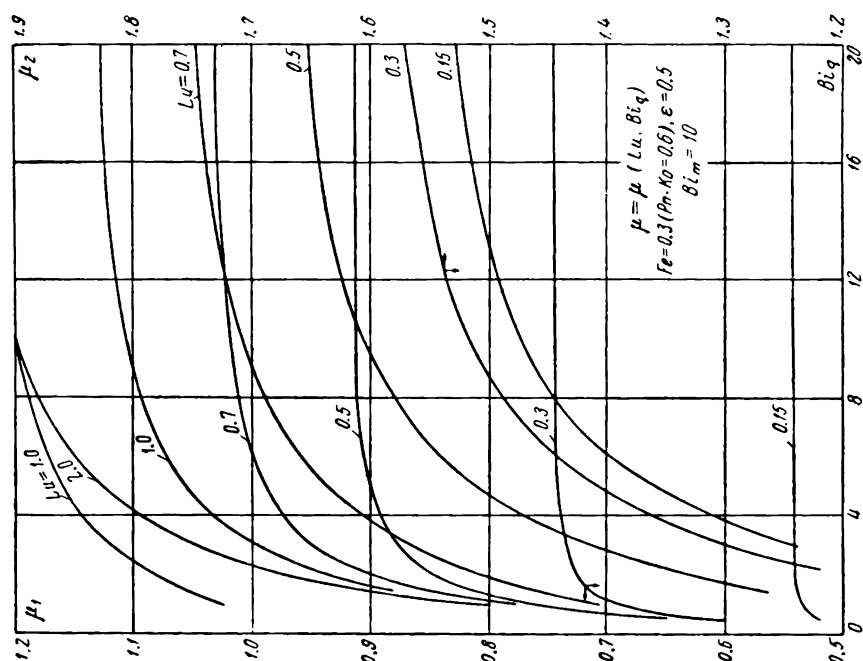


FIGURE 6-12. Characteristic-equation roots as functions of  $Bi_q$  and  $Lu$  (for  $Bi_q$  from 1 to 20), for an infinite plate



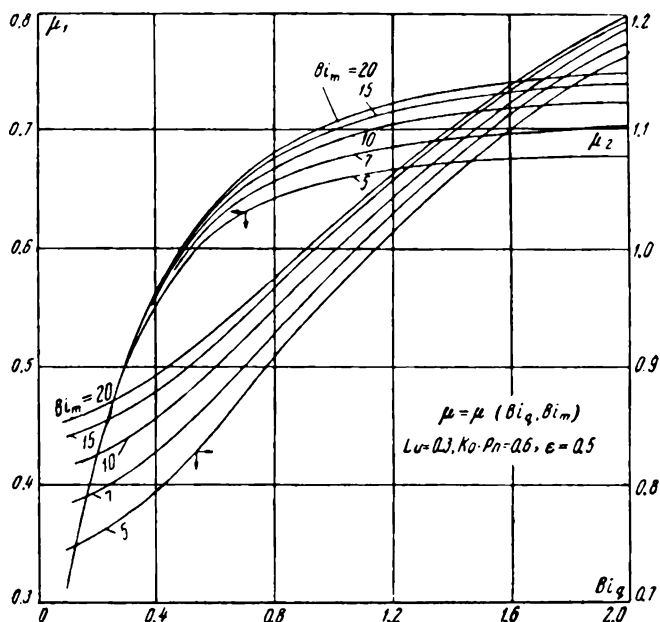


FIGURE 6-13. Characteristic-equation roots as functions of  $Bi_q$  and  $Bi_m$  (for  $Bi_q$  from 0.1 to 2.0), for an infinite plate

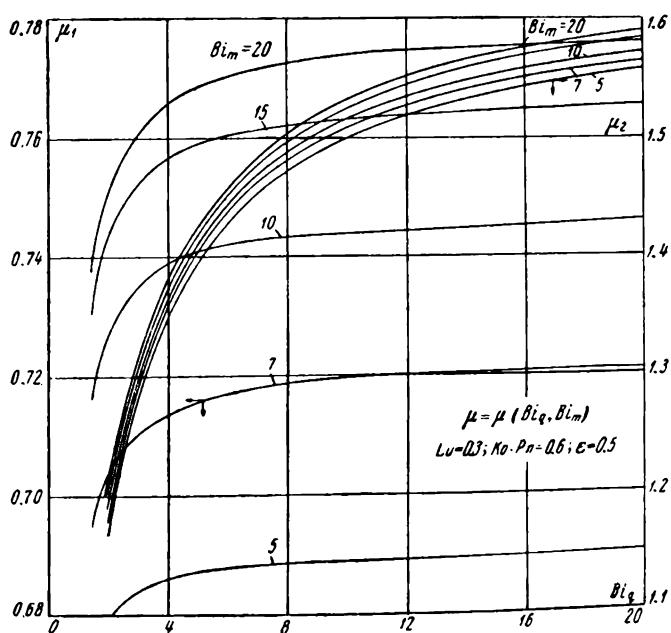


FIGURE 6-14. Characteristic-equation roots as functions of  $Bi_m$  and  $Bi_q$  (for  $Bi_q$  from 2 to 20), for an infinite plate

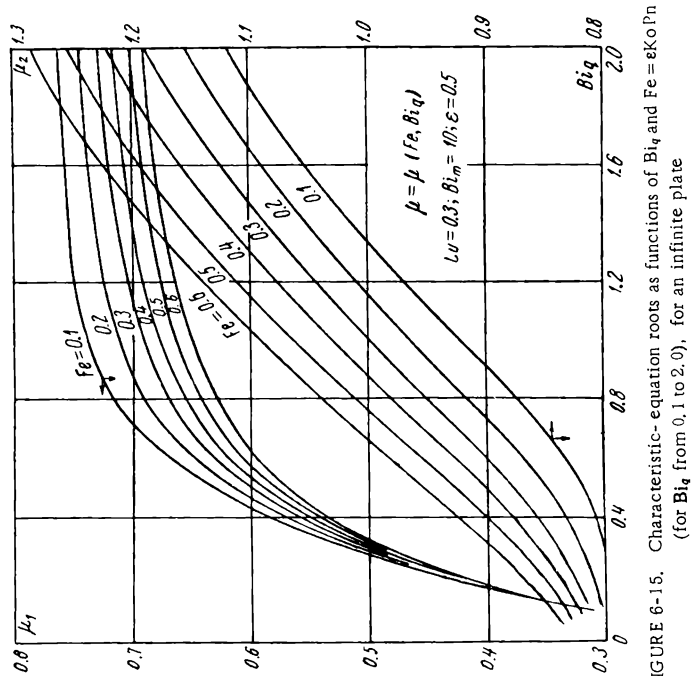


FIGURE 6-15. Characteristic- equation roots as functions of  $Bi_q$  and  $Fe = \epsilon Ko Pn$  (for  $Bi_q$  from 0.1 to 2.0), for an infinite plate

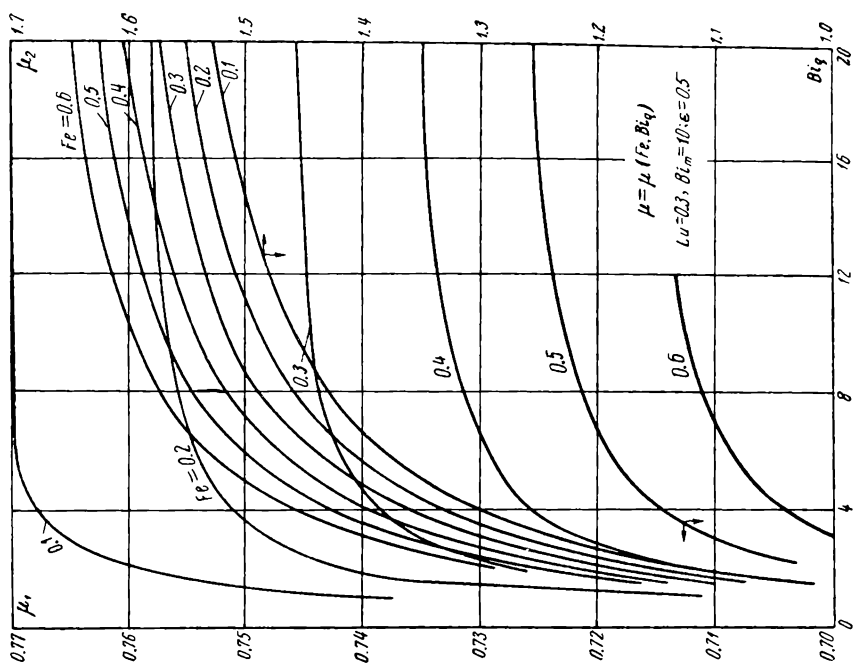


FIGURE 6-16. Characteristic-equation roots as functions of  $Bi_q$  and  $Fe = \epsilon Ko Pn$  (for  $Bi_q$  from 1.0 to 20), for an infinite plate

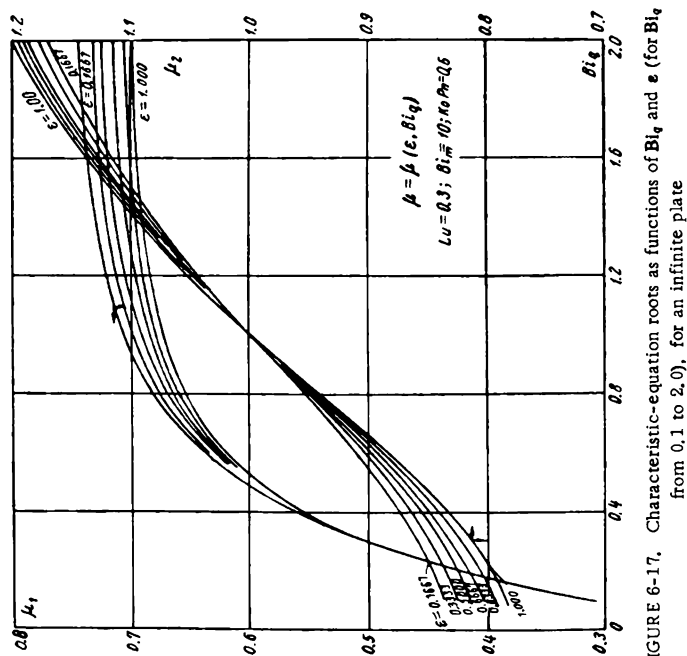


FIGURE 6-17. Characteristic-equation roots as functions of  $Bi_q$  and  $\epsilon$  (for  $Bi_q$  from 0.1 to 2.0), for an infinite plate

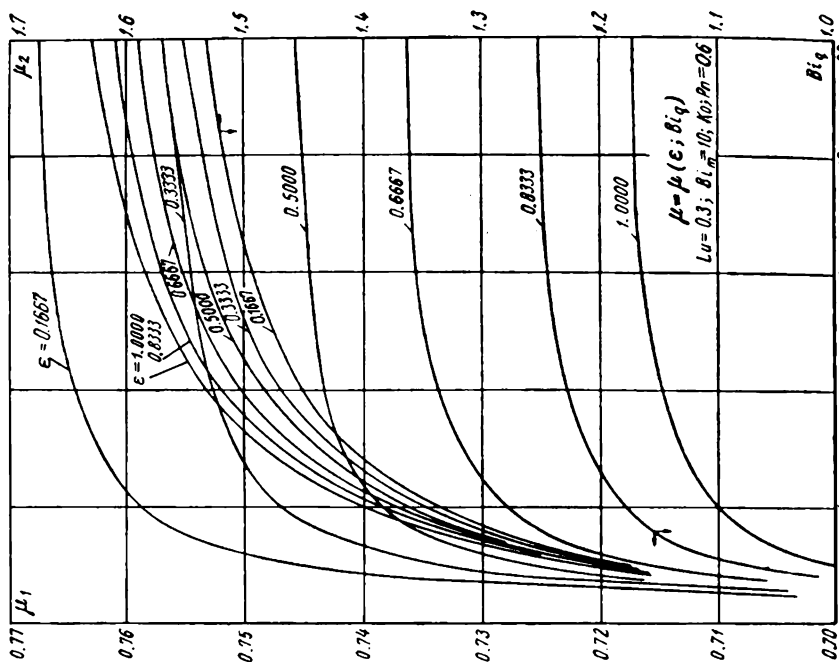


FIGURE 6-18. Characteristic-equation roots as functions of  $Bi_q$  and  $\epsilon$  (for  $Bi_q$  from 1 to 20), for an infinite plate

an error of 11.3% is introduced when series terms after the first two are dropped for  $Fo \approx 0.1$ , this error is only 0.52% for  $Fo = 0.9$ . In Table 6-16 are shown the values of the Fourier number above which the system of equations may be simplified, with constant initial conditions, without introducing an error greater than 1.0% or 0.5%.

TABLE 6-16  
Values of  $Fo$  above which the calculations of heat and mass transfer may be simplified ( $Lu = 0.3$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ;  $Bi_m = 10$ )

Type of problem	$Bi_q$	$Fo$								Error $ \Delta $ , %
		$1-T$		$1-\Theta$		$1-\bar{T}$	$1-\bar{\Theta}$	$\frac{d\bar{T}}{dFo}$	$\frac{d\bar{\Theta}}{dFo}$	
		$x=0$	$x=1$	$x=0$	$x=1$					
Plate, constant initial conditions	1	0.75 0.79	0.66 0.81	0.79 0.94	$\sim 1.06$ $\sim 1.19$	0.48 0.61	0.59 0.74	1.03 1.19	1.13 1.31	1 0.5
	10	0.73 0.89	0.95 1.08	0.80 0.95	1.06 1.19	0.11 0.17	0.57 0.73	0.29 0.37	1.06 1.19	1 0.5
Sphere, constant initial conditions	10		0.24 0.27		0.27 0.31	0.17 0.21	0.13 0.15	0.26 0.30	0.22	1 0.5
Plate, parabolic initial conditions	10	0.66 0.84	0.87 1.04	0.73 0.88	1.00 1.11	0.13 0.16	0.50 0.64	0.25 0.33	0.33 1.15	1 0.5

The simplifications make it possible to represent the general solutions in a form convenient for practical application. The Fourier numbers above which the calculations may be simplified ("initial" Fourier numbers) vary with the combination of dimensionless numbers entering into the solution.

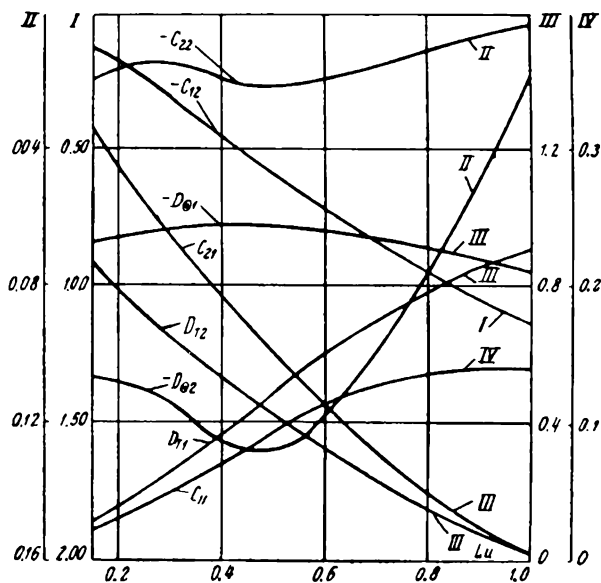


FIGURE 6-19. Coefficients  $C_n$ ,  $D_{Tn}$ , and  $D_{\Theta n}$  as functions of  $Lu$ , for an infinite plate

TABLE 6-17

The constant coefficients as functions of  $Lu$  (for  $Bl_q = 10$ ;  $Bl_m = 10$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ), for an infinite plate

Lu	$C_{ni}$				$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22	$n = 1$	$n = 2$	$n = 1$	$n = 2$
0.15	0.0231	1.258	0.134	0.0198	0.114	0.865	-0.924	0.107
0.3	0.0488	0.938	0.308	0.0145	0.257	0.650	-0.916	0.114
0.5	0.0914	0.606	0.584	0.0216	0.482	0.423	-0.974	0.127
0.7	0.1270	0.315	0.849	0.0155	0.694	0.221	-0.938	0.099
1.0	0.1399	0.034	1.135	0.0034	0.901	0.024	-0.844	0.015
2.0	0.1118	-0.183	1.452	-0.0405	1.100	-0.123	-0.646	-0.133

TABLE 6-18

The constant coefficients as functions of  $Bl_q$  (for  $Lu = 0.3$ ;  $Bl_m = 10$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ), for an infinite plate

$Bl_q$	$C_{ni}$				$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22	$n = 1$	$n = 2$	$n = 1$	$n = 2$
1.0	1.3480	-0.391	0.299	0.0092	1.473	-0.331	-0.765	-0.082
5.0	0.1563	0.860	0.310	0.0084	0.358	0.629	-0.953	0.109
7.0	0.0929	0.912	0.309	0.0109	0.298	0.649	-0.958	0.113
10	0.0488	0.938	0.308	0.0145	0.257	0.650	-0.962	0.114
15	0.0152	0.952	0.307	0.0156	0.225	0.647	-0.964	0.115
20	-0.0026	0.963	0.307	0.0163	0.208	0.647	-0.964	0.116

TABLE 6-19

The constant coefficients as functions of  $Bl_m$  (for  $Lu = 0.3$ ;  $Bl_q = 10$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ), for an infinite plate

$Bl_m$	$C_{ni}$				$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22	$n = 1$	$n = 2$	$n = 1$	$n = 2$
0.1	16.65	0.312	2.558	-0.1000	12.83	0.194	0.178	-0.004
1.5	0.2408	0.360	0.250	-0.0219	0.466	0.539	-1.014	0.021
5	0.0037	0.947	0.297	0.0315	0.220	0.664	-0.991	0.109
7	0.0259	0.943	0.303	0.0217	0.238	0.657	-0.975	0.112
10	0.0488	0.938	0.308	0.0145	0.267	0.650	-0.962	0.114
15	0.0732	0.932	0.312	0.0094	0.278	0.642	-0.950	0.114
20	0.0818	0.928	0.314	0.0069	0.284	0.638	-0.941	0.114

TABLE 6-20

The constant coefficients as functions of  $\epsilon$  (for  $Lu = 0.3$ ;  $Bl_q = 10$ ;  $Bl_m = 10$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ), for an infinite plate

$\epsilon$	$C_{ni}$				$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22	$n = 1$	$n = 2$	$n = 1$	$n = 2$
0.1667	0.0855	1.112	0.117	0.0065	0.159	0.764	-0.990	0.143
0.3333	0.0671	1.018	0.219	0.0114	0.213	0.702	-0.975	0.127
0.5000	0.0488	0.938	0.308	0.0145	0.257	0.650	-0.962	0.114
0.6667	0.0348	0.864	0.387	0.0175	0.298	0.600	-0.948	0.102
0.8333	0.0273	0.795	0.459	0.0200	0.340	0.554	-0.940	0.092
1.0000	0.0167	0.734	0.524	0.0197	0.374	0.514	-0.930	0.083

TABLE 6-21

The [constant] coefficients as functions of  $Ko$  (for  $Lu = 0.3$ ;  $Bi_g = 10$ ;  $Bi_m = 10$ ;  $\epsilon = 0.5$ ;  $Pn = 0.5$ ), for an infinite plate

$Ko$	$C_{ni}$				$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22	$n = 1$	$n = 2$	$n = 1$	$n = 2$
0.4	0.0141	1.133	0.117	0.0064	0.094	0.786	-0.999	0.147
0.8	0.0319	1.029	0.219	0.0103	0.180	0.715	-0.978	0.129
1.2	0.0488	0.938	0.308	0.0145	0.257	0.650	-0.962	0.114
1.6	0.0627	0.866	0.388	0.0163	0.324	0.600	-0.947	0.102
2.0	0.0753	0.807	0.459	0.0183	0.385	0.559	-0.934	0.093

TABLE 6-22

The constant coefficients as functions of  $Pn$  (for  $Lu = 0.3$ ;  $Bi_g = 10$ ;  $Bi_m = 10$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ), for an infinite plate

$Pn$	$C_{ni}$				$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22	$n = 1$	$n = 2$	$n = 1$	$n = 2$
0.1667	0.0462	0.916	0.3188	0.0052	0.262	0.635	-0.904	0.039
0.3333	0.0472	0.929	0.3132	0.0091	0.259	0.645	-0.933	0.078
0.5000	0.0488	0.938	0.3080	0.0145	0.257	0.650	-0.962	0.114
0.6667	0.0488	0.952	0.3028	0.0181	0.253	0.660	-0.986	0.150
0.8333	0.0483	0.969	0.2978	0.0225	0.249	0.671	-1.010	0.186

As an analysis of the calculations shows, the conditions for simplification (the "initial" Fourier numbers) are affected most by the value of  $Lu$ . As  $Lu$  increases, the "initial"  $Fo$  for simplification decreases. The possibility of simplification is affected considerably less by  $\epsilon$ ,  $Ko$ , and  $Pn$ .

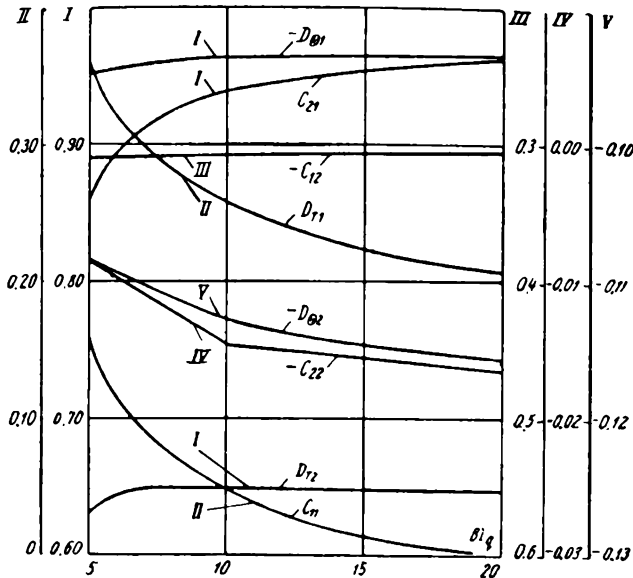


FIGURE 6-20. Coefficients  $C_{ni}$ ,  $D_{Tn}$ , and  $D_{\theta n}$  as functions of  $Bi_g$ , for an infinite plate

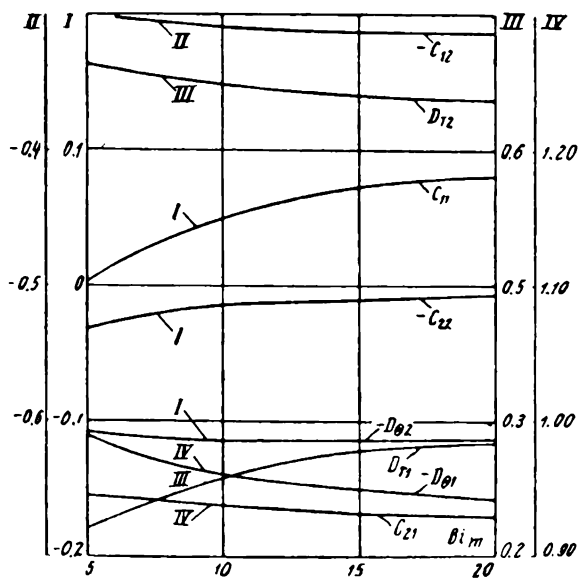


FIGURE 6-21. Coefficients  $C_n$ ,  $D_{Tn}$ , and  $D_{\Theta n}$  as functions of  $Bi_m$ , for an infinite plate

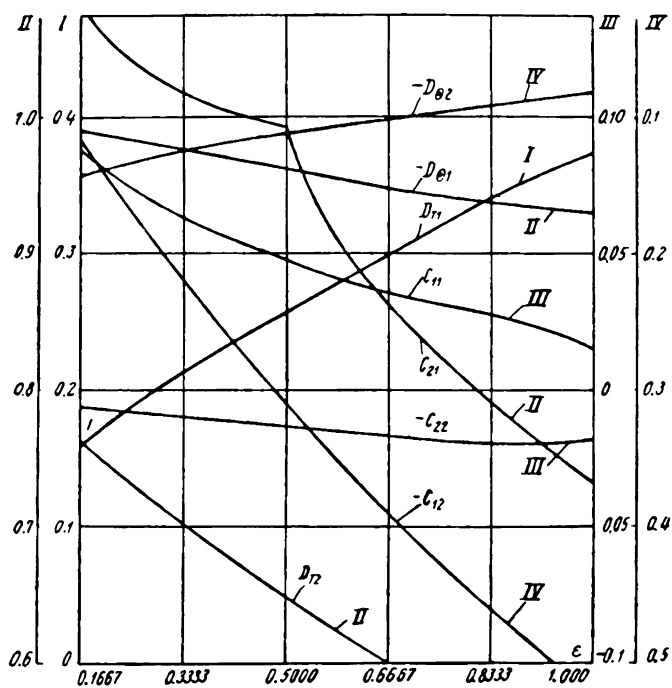


FIGURE 6-22. Coefficients  $C_n$ ,  $D_{Tn}$ , and  $D_{\Theta n}$  as functions of  $\epsilon$ , for an infinite plate

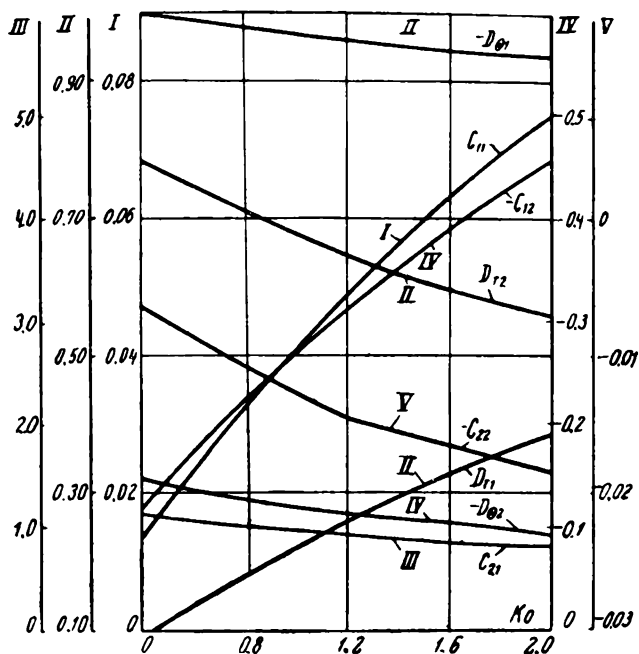


FIGURE 6-23. Coefficients  $C_{n1}$ ,  $D_{Tn1}$ , and  $D_{\theta n1}$  as functions of  $Ko$ , for an infinite plate

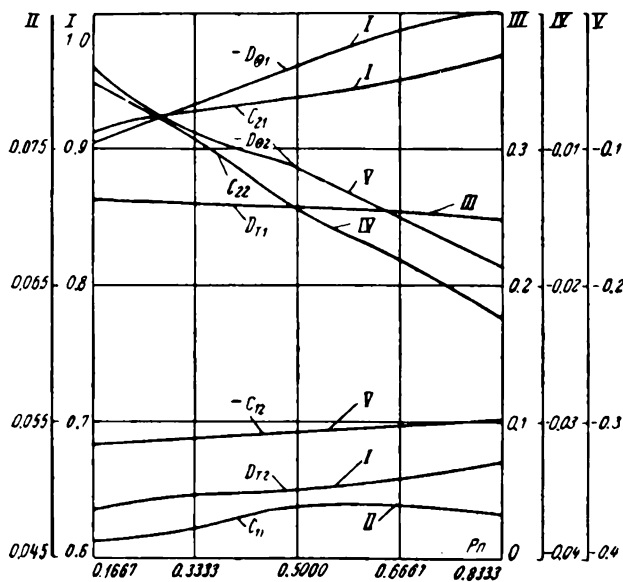


FIGURE 6-24. Coefficients  $C_{n1}$ ,  $D_{Tn1}$ , and  $D_{\theta n1}$  as functions of  $Pn$ , for an infinite plate



As Table 6-16 shows, depending on  $Bi_q$  the "initial"  $Fo$  increases with  $Bi_q$  for local values and decreases for average values. The value of the Biot number has an especially strong effect on the dimensionless heat-transfer potential. Similarly, the "initial" Fourier number for the dimensionless mass-transfer potential depends especially strongly on  $Bi_m$ . On the whole, the value of the "initial" Fourier number for simplification is lower for heat exchange than for mass exchange.

Some numerical values of constant coefficients  $C_{ni}$  are given in Tables 6-17 through 6-22 and to facilitate calculations the corresponding curves are plotted in Figures 6-19 through 6-24.

In terms of average dimensionless transfer potentials solutions (6-3-1) and (6-3-2) become

$$\bar{T}(Fo) = 1 - \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} \exp(-\mu_n^2 Fo); \quad (6-3-12)$$

$$\bar{\theta}(Fo) = 1 + \frac{1}{\epsilon Ko} \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} (1 - \nu_i^2) \exp(-\mu_n^2 Fo), \quad (6-3-13)$$

or, in another form,

$$\bar{T}(Fo) = 1 - \sum_{n=1}^{\infty} D_{Tn} \exp(-\mu_n^2 Fo); \quad (6-3-14)$$

$$\bar{\theta}(Fo) = 1 + \sum_{n=1}^{\infty} D_{\theta n} \exp(-\mu_n^2 Fo), \quad (6-3-15)$$

where  $D_{ni}$ ,  $D_{Tn}$ , and  $D_{\theta n}$  are defined, as usual, in terms of the constant coefficients  $C_{ni}$  (in the present case the values for an infinite plate are used).

The "initial"  $Fo$ , above which formulas (6-3-12) and (6-3-13) may be simplified, are given in Table 6-16. Values of  $D_{Tn}$  and  $D_{\theta n}$  calculated for the first two characteristic-equation roots are listed in Tables 6-17 through 6-22 and plotted in Figures 6-19 through 6-24.

The dimensionless rate of variation of the heat-transfer and mass-transfer potentials (local and average) can be obtained by differentiating equations (6-3-1), (6-3-2), and (6-3-12) through (6-3-15) with respect to  $Fo$ .

#### Parabolic Initial Conditions

$$T(X, Fo) = \frac{t - t_s}{t_o - t_s} = 1 - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \cos \nu_i \mu_n X \exp(-\mu_n^2 Fo); \quad (6-3-16)$$

$$\theta(X, Fo) = \frac{\theta_s - \theta}{\theta_s - \theta_{eq}} = 1 + \frac{1}{\epsilon Ko} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - \nu_i^2) \cos \nu_i \mu_n X \exp(-\mu_n^2 Fo); \quad (6-3-17)$$

where

$$C_{n1} = 2 \frac{G_1 P_{n1} + G_2 Q_{n1}}{\mu_n \psi_n}; \quad C_{n2} = -2 \frac{G_1 P_{n1} + G_2 Q_{n1}}{\mu_n \psi_n}; \quad (6-3-18)$$

$$G_1 = 1 - \epsilon Ko K_1 - 2W \left( \frac{1}{Bi_q} - \frac{1}{\mu_n^2} \right) + 2\epsilon Ko (Pn W - V) \frac{(1 + K_1) Lu}{\mu_n^2}; \quad (6-3-19)$$

$$G_2 = \epsilon Ko + 2\epsilon Ko (Pn W - V) \left( \frac{1}{Bi_m} - \frac{Lu}{\mu_n^2} \right), \quad (6-3-20)$$

and the quantities  $Q_{ni}$ ,  $P_{ni}$ ,  $\psi_n$ ,  $A_{ni}$ , and  $B_{ni}$  are defined by the previous equations (6-3-8), (6-3-9), and (6-3-5) through (6-3-7). The characteristic-equation roots obey equation (6-3-10).

In parabolic problems the dimensionless numbers  $Ko$  and  $Pn$  (together with the other similarity criteria) are defined as before, except that  $t_0$  and  $\theta_0$  are replaced by  $t_s$  and  $\theta_s$ .

Since the characteristic equation for the parabolic problem is identical to that for a plate with constant initial conditions, the roots  $\mu_n$  plotted in Figures 6-11 through 6-18 are also the roots for the present problem. A comparison of the "initial" values of  $Fo$  above which the calculations may be simplified in the case of parabolic initial conditions (Table 6-16) with the analogous values from the problem with constant initial conditions does not indicate any significant discrepancies. Thus, it may be concluded that the initial conditions only slightly affect the range in which simplification is possible. Some values of the constant heat-transfer and mass-transfer coefficients  $C_{ni}$ , calculated for a plate using equation (6-3-18), are given in Tables 6-23 and 6-24, as functions of the dimensionless parameters  $W$  and  $V$  representing the nonuniformity of the initial distributions.

TABLE 6-23

The constant coefficients as functions of parameter  $W$  (parabolic initial conditions) (for  $Lu = 0.3$ ;  $Bi_q = 10$ ;  $Bi_m = 10$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ;  $V = 0.4$ ), for an infinite plate

$W$	$C_{ni}$				$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22	$n = 1$	$n = 2$	$n = 1$	$n = 2$
0	0.0626	0.837	0.392	0.0130	0.327	0.580	-1.223	0.101
0.2	0.0629	1.026	0.398	0.0158	0.332	0.711	-1.243	0.124
0.4	0.0631	1.214	0.405	0.0187	0.337	0.841	-1.263	0.147
0.6	0.0634	1.403	0.411	0.0215	0.341	0.972	-1.283	0.170
0.8	0.0636	1.591	0.417	0.0244	0.346	1.103	-1.303	0.193
1.0	0.0638	1.779	0.424	0.0272	0.350	1.233	-1.324	0.216

TABLE 6-24

The constant coefficients as functions of parameter  $V$  (parabolic initial conditions) (for  $Lu = 0.3$ ;  $Bi_q = 10$ ;  $Bi_m = 10$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ;  $W = 0.4$ ), for an infinite plate

$V$	$C_{ni}$				$D_{Tn}$		$D_{\theta n}$	
	11	12	21	22	$n = 1$	$n = 2$	$n = 1$	$n = 2$
0	0.0493	1.315	0.321	0.0202	0.266	0.911	-1.002	0.159
0.2	0.0562	1.264	0.362	0.0194	0.301	0.876	-1.133	0.153
0.4	0.0631	1.214	0.405	0.0187	0.337	0.841	-1.263	0.147
0.6	0.0700	1.164	0.447	0.0179	0.372	0.807	-1.394	0.141
0.8	0.0769	1.113	0.488	0.0172	0.407	0.771	-1.524	0.135
1.0	0.0838	1.063	0.530	0.0164	0.442	0.737	-1.655	0.129

The method for obtaining average values of the transfer potentials, or local and average rates of variation of the transfer potentials, differs in no way from that used before. Some values  $D_{Tn}$  and  $D_{\theta n}$  are given in the accompanying tables and graphs.

If in equations (6-3-16) through (6-3-20) the parameters  $W$  and  $V$  describing the nonuniformity of the initial transfer-potential distributions are set

equal to zero, then we obtain the previously derived equations (6-3-1) through (6-3-4) for a system with constant initial conditions.

**b) Solution for an infinite cylinder ( $\Gamma=1$ )**

For constant initial conditions,

$$T(X, Fo) = 1 - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} J_0(v_i \mu_n X) \exp(-\mu_n^2 Fo); \quad (6-3-21)$$

$$\theta(X, Fo) = 1 + \frac{1}{\epsilon K_0} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2) J_0(v_i \mu_n X) \exp(-\mu_n^2 Fo). \quad (6-3-22)$$

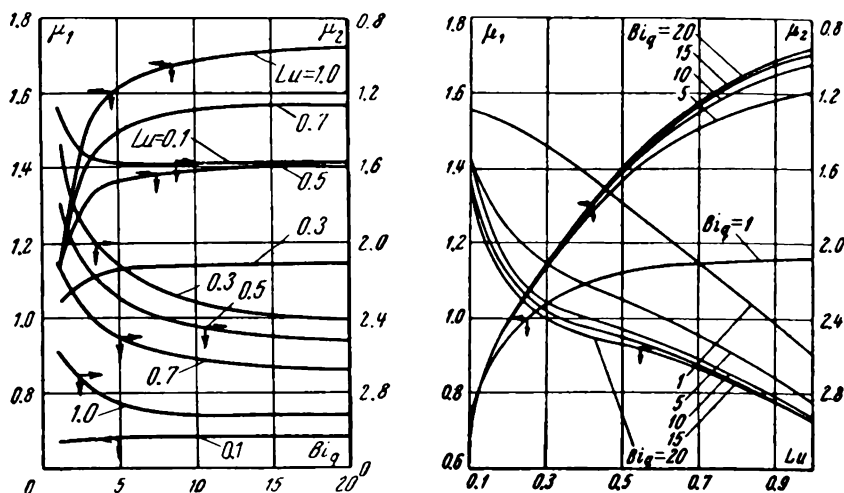


FIGURE 6-25. Characteristic-equation roots as functions of  $Bi_q$  and  $Lu$ , for an infinite cylinder

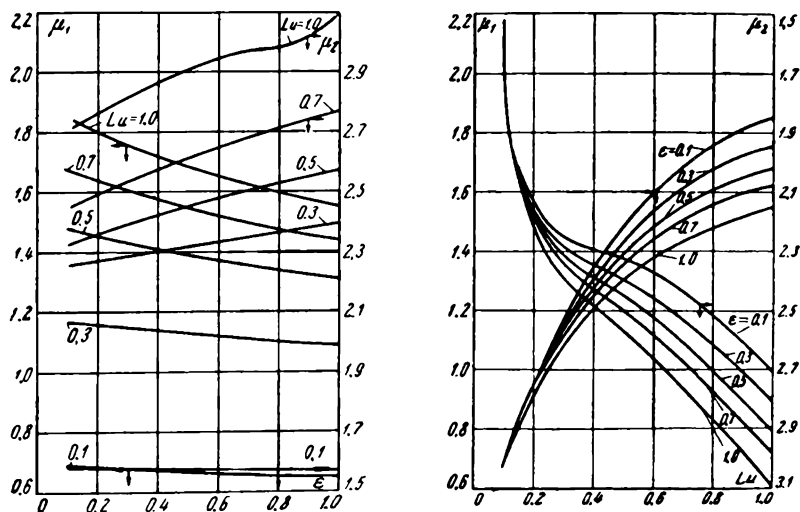


FIGURE 6-26. Characteristic-equation roots as functions of  $\epsilon$  or  $Lu$ , for an infinite cylinder

Quantities  $C_{ni}$  and  $\psi_n$  for solutions (6-3-21) and (6-3-22) are defined by relations (6-3-3) through (6-3-5), where

$$Q_{ni}=[1+(1-v_i^2)K_1]J_0(v_i\mu_n)-\frac{1}{Bl_q}v_i\mu_nJ_1(v_i\mu_n); \tag{6-3-23}$$

$$P_{ni}=(1-v_i^2)J_0(v_i\mu_n)-\frac{(1-v_i^2)+\epsilon Ko Pn}{Bl_m}v_i\mu_n\sin v_i\mu_n; \tag{6-3-24}$$

$$A_{ni}=[1+(1-v_i^2)K_1]J_1(v_i\mu_n)+\frac{1}{Bl_q}v_i\mu_nJ_0(v_i\mu_n); \tag{6-3-25}$$

$$B_{ni}=(1-v_i^2)J_1(v_i\mu_n)+\frac{(1-v_i^2)+\epsilon Ko Pn}{Bl_m}v_i\mu_nJ_0(v_i\mu_n). \tag{6-3-26}$$

The roots  $\mu_n$  obey the characteristic equation

$$P_{n1}Q_{n2}-P_{n2}Q_{n1}=0. \tag{6-3-27}$$

TABLE 6-25  
Infinite cylinder. Roots of characteristic equation (6-3-27) as functions of  $Lu$  and  $Bl_q$   
(for  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ;  $Bl_m = 10$ )

Lu	Bl <sub>q</sub>	$\mu_n$			
		n = 1	n = 2	n = 3	n = 4
0.1	1	0.6762	1.2934	1.5609	2.4703
	5	0.6796	1.5715	2.0252	2.4729
	10	0.6800	1.5703	2.2161	2.4758
	15	0.6801	1.5700	2.2862	2.4783
	20	0.6801	1.5699	2.3217	2.4805
0.3	1	1.0427	1.4701	2.5679	4.0130
	5	1.1293	2.1452	2.5843	4.1166
	10	1.1356	2.3212	2.6022	4.1336
	15	1.1376	2.3805	2.6159	4.1392
	20	1.1385	2.4084	2.6259	4.1419
0.5	1	1.1283	1.7812	3.0454	4.5825
	5	1.3713	2.2974	3.1445	5.0702
	10	1.3946	2.4476	3.1973	5.0946
	15	1.4018	2.5004	3.2222	5.0978
	20	1.4052	2.5268	3.2364	5.0991
0.7	1	1.1530	2.1012	3.2500	5.0158
	5	1.5068	2.4996	3.4556	5.5955
	10	1.5534	2.6146	3.5558	5.7970
	15	1.5682	2.6548	3.5994	5.7205
	20	1.5754	2.6749	3.6234	5.7100
1.0	1	1.1673	2.5839	3.3134	5.5154
	5	1.6093	2.8543	3.6699	6.0932
	10	1.6858	2.9086	3.8458	6.2380
	15	1.7117	2.9261	3.9193	6.2492
	20	1.7247	2.9346	3.9586	6.2526

The remarks made previously about simplifying solutions (6-3-1) and (6-3-2) also apply to solutions (6-3-21) and (6-3-22). Values of

the first four characteristic-equation roots  $\mu_n$  (calculated using a high-speed electronic computer) as functions of  $Lu$ ,  $Bi_q$ ,  $Bi_m$ ,  $\epsilon$ ,  $Ko$  and  $Pn$  are listed in Tables 6-25 through 6-27, and values of the first two

TABLE 6-26

Infinite cylinder. Roots of characteristic equation (6-3-27) as functions of  $Lu$  and  $\epsilon$   
(for  $Ko = 1.2$ ;  $Pn = 0.5$ ;  $Bi_q = 10$ ;  $Bi_m = 10$ )

Lu	$\epsilon$	$\mu_n$			
		$n = 1$	$n = 2$	$n = 3$	$n = 4$
0.1	0.1	0.6871	1.5881	2.1976	2.5017
	0.3	0.6835	1.5791	2.2070	2.4886
	0.5	0.6800	1.5703	2.2161	2.4768
	0.7	0.6765	1.5617	2.2250	2.4634
	1.0	0.6714	1.5492	2.2375	2.4456
0.3	0.1	1.1760	2.2566	2.6944	4.3025
	0.3	1.1551	2.2897	2.6467	4.2130
	0.5	1.1356	2.3212	2.6022	4.1336
	0.7	1.1174	2.3513	2.5604	4.0614
	1.0	1.0921	2.3948	2.5016	3.9633
0.5	0.1	1.4831	2.3264	3.4012	7.4926
	0.3	1.4355	2.3905	3.2917	7.1708
	0.5	1.3946	2.4476	3.1973	5.0946
	0.7	1.3585	2.4994	3.1140	4.9360
	1.0	1.3112	2.5699	3.0042	4.7482
0.7	0.1	1.6865	2.4442	3.8624	
	0.3	1.6128	2.5369	3.6926	
	0.5	1.5534	2.6146	3.5558	5.7970
	0.7	1.5032	2.6826	3.4397	5.4656
	1.0	1.4394	2.7725	3.2914	5.2066
0.1	0.1	1.8523	2.7050	4.2198	
	0.3	1.7592	2.8171	4.0119	
	0.5	1.6858	2.9086	3.8458	6.2380
	0.7	1.6245	2.9875	3.7057	5.9000
	1.0	1.5477	3.0909	3.5266	5.5902

roots are plotted in Figures 6-25 through 6-28. Values of constant coefficients  $C_{ni}$  are given in Tables 6-28 through 6-32 and in Figures 6-29 through 6-33.

In terms of average dimensionless transfer potentials solutions (6-3-21) and (6-3-22) will have the form (6-3-12) and (6-3-13), where

$$D_{ni} = 2C_{ni} \frac{J_i(\nu_i \mu_n)}{\nu_i \mu_n}.$$

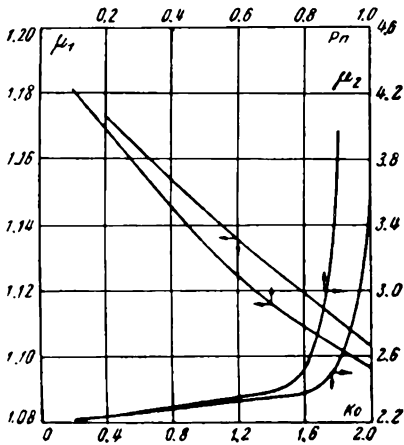
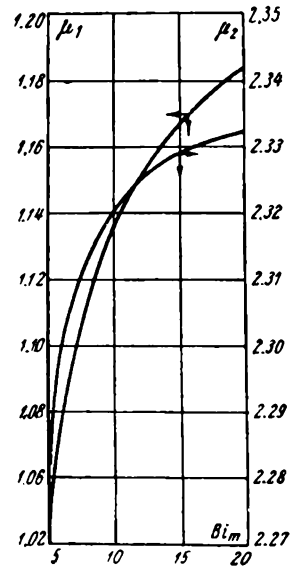
Values of coefficients  $D_{ni}$  for various combinations of dimensionless numbers are listed in Tables 6-28 through 6-32 and plotted in Figures 6-29 through 6-33.

TABLE 6-27

Infinite cylinder. Roots of characteristic equation (6-3-27) as functions of  $Ko$ ,  $Pn$ , and  $Bi_m$ 

Dimensionless number		$\mu_n$			
		$n = 1$	$n = 2$	$n = 3$	$n = 4$
$Ko$	0.4	1.1728	2.2262	2.7034	4.2781
	0.8	1.1536	2.2730	2.6525	4.2034
	1.2	1.1356	2.3212	2.6022	4.1336
	1.6	1.1189	2.3752	2.5481	4.0680
	2.0	1.1032	4.0062	6.9963	
$Pn$	0.1	1.1810	2.2076	2.7244	4.3094
	0.3	1.1573	2.2636	2.6625	4.2179
	0.5	1.1356	2.3212	2.6022	4.1336
	0.7	1.1157	2.3880	2.5355	4.0554
	0.9	1.0971	3.9825	6.9521	
$Bi_m$	5	1.0485	2.2797	2.4683	3.9659
	10	1.1356	2.3212	2.6022	4.1336
	15	1.1674	2.3292	2.6716	4.2197
	20	1.1838	2.3325	2.7108	4.2696

Note: The combination of dimensionless-number values taken as a reference for the calculation was:  $Lu = 0.3$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ;  $Bi_g = 10$ ;  $Bi_m = 10$ .

FIGURE 6-27. Characteristic-equation roots as functions of  $Pn$  and  $Ko$ , for an infinite cylinderFIGURE 6-28. Characteristic-equation roots as functions of  $Bi_m$ , for an infinite cylinder

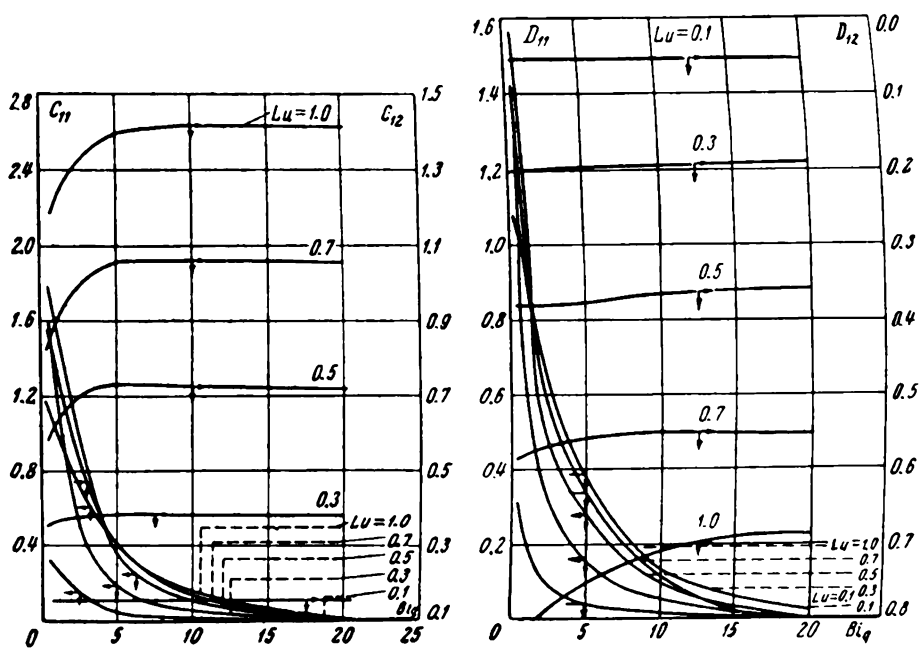


FIGURE 6-29. Coefficients  $C_{n1}$  and  $D_{n1}$  as functions of  $Bi_q$  and  $Lu$ , for an infinite cylinder

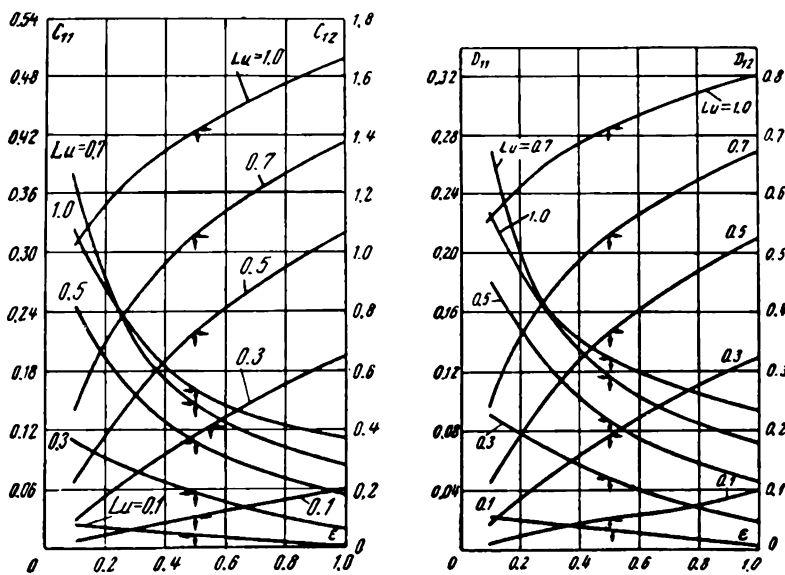


FIGURE 6-30. Coefficients  $C_{n1}$  and  $D_{n1}$  as functions of  $\epsilon$  and  $Lu$ , for an infinite cylinder

TABLE 6-28

Constant coefficients  $C_{nt}$  and  $D_{nt}$  as functions of  $Lu$  and  $Bl_\theta$  (for  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pr = 0.5$ ;  $Bl_m = 10$ ), for an infinite cylinder

$Lu$	$Bl_\theta$	$C_{nt}$				$D_{nt}$			
		11	12	21	22	11	12	21	22
0.1	1	0.33525	0.10565	1.36741	0.00062	0.31705	0.05440	1.10870	-0.00004
	5	0.04258	0.10492	0.08832	-0.05846	0.04024	0.05361	0.06443	0.00771
	10	0.01369	0.10482	0.02804	-0.05885	0.01294	0.05352	0.01682	0.00775
	15	0.00437	0.10478	0.00683	-0.05896	0.00413	0.05348	0.00506	0.00777
	20	-0.00023	0.10476	-0.00036	-0.05901	-0.00022	0.05347	-0.00026	0.00777
0.3	1	1.61032	0.35547	-0.46416	0.01794	1.42312	0.20285	-0.36109	0.00498
	5	0.17996	0.38487	1.33583	0.03558	0.15559	0.19663	0.75885	-0.00208
	10	0.05534	0.38228	1.39463	0.08206	0.04776	0.19362	0.70960	-0.00835
	15	0.01703	0.38132	1.28929	0.10253	0.01469	0.19252	0.63006	-0.01148
	20	-0.00154	0.38083	1.18315	0.11016	-0.00133	0.19212	0.56698	-0.01278
0.5	1	1.78553	0.58049	-0.95886	0.10516	1.56765	0.37822	-0.68517	0.03019
	5	0.33731	0.73118	0.72155	0.02340	0.27773	0.37790	0.40213	0.00106
	10	0.10691	0.72543	0.85752	0.05425	0.08742	0.36532	0.43675	-0.00043
	15	0.03494	0.72229	0.85411	0.06444	0.02851	0.36051	0.42049	-0.00157
	20	0.00000	0.72052	0.84012	0.06886	0.00000	0.35851	0.40647	-0.00221
0.7	1	1.52919	0.83244	-0.96350	0.12828	1.35553	0.58234	-0.63360	0.02950
	5	0.41674	1.05824	0.33142	0.01724	0.33859	0.55664	0.17926	0.00111
	10	0.14483	1.05740	0.48915	0.04617	0.11608	0.53104	0.24764	0.00115
	15	0.05358	1.05360	0.50766	0.05564	0.04276	0.52116	0.25085	0.00068
	20	0.00831	1.05106	0.50893	0.05972	0.00661	0.51602	0.24639	0.00036
1.0	1	1.17113	1.14861	-0.77218	0.05024	1.05878	0.84373	-0.45311	0.00548
	5	0.40096	1.40159	0.10478	0.01288	0.33000	0.75395	0.05380	0.00025
	10	0.15978	1.41472	0.16466	0.02634	0.12894	0.71015	0.08209	0.00011
	15	0.07074	1.41417	0.16765	0.02886	0.05669	0.69262	0.08277	-0.00002
	20	0.02491	1.41279	0.16602	0.02956	0.01989	0.68332	0.08158	-0.00009



TABLE 6-29

Constant coefficients  $C_{n,t}$  and  $D_{n,t}$  as functions of  $Lu$  and  $s$  (for  $Ko = 1.2$ ;  $Pn = 0.5$ ;  $Bl_q = 10$ ;  $Bl_m = 10$ ) for an infinite cylinder

$Lu$	$s$	$C_{n,t}$				$D_{n,t}$			
		11	12	21	22	11	12	21	22
0.1	0.1	0.02468	0.02162	0.04260	-0.01195	0.02326	0.01108	0.03058	0.00157
	0.3	0.01901	0.06386	0.03239	-0.03558	0.01794	0.03267	0.02345	0.00469
	0.5	0.01369	0.10482	0.02804	-0.05885	0.01294	0.05352	0.01682	0.00775
	0.7	0.00869	0.14455	0.01445	-0.08173	0.00823	0.07366	0.01063	0.01077
0.3	1.0	0.00174	0.20200	0.00284	-0.11532	0.00165	0.10264	0.00211	0.01521
	0.1	0.10704	0.08970	1.65782	0.01690	0.08998	0.04616	0.82098	-0.00104
	0.3	0.07731	0.24727	1.51444	0.04841	0.06592	0.12617	0.76082	-0.00408
	0.5	0.05534	0.38228	1.39463	0.08206	0.04776	0.19362	0.70960	-0.00835
0.5	0.7	0.03860	0.50015	1.29749	0.12651	0.03367	0.25167	0.66775	-0.01452
	1.0	0.02004	0.65256	1.22252	0.27374	0.01772	0.32556	0.63865	-0.03481
0.7	0.1	0.24451	0.22017	1.23572	0.02636	0.18619	0.11473	0.59663	0.00291
	0.3	0.15346	0.51912	1.01502	0.04662	0.12178	0.26517	0.50532	0.00203
	0.5	0.10691	0.72543	0.85752	0.05425	0.08742	0.36532	0.43675	-0.00043
	0.7	0.07896	0.88307	0.73635	0.05928	0.06609	0.43984	0.38203	-0.00284
0.9	1.0	0.05348	1.06639	0.60139	-0.06932	0.04597	0.52437	0.31921	-0.00629
	0.1	0.37766	0.46678	0.79787	0.05586	0.26921	0.24817	0.36980	0.01116
	0.3	0.20727	0.84623	0.60855	0.05380	0.15884	0.43391	0.29803	0.00525
	0.5	0.14483	1.05740	0.48915	0.04617	0.11608	0.53104	0.24764	0.00115
1.0	0.7	0.11198	1.20703	0.40465	0.04171	0.09265	0.59748	0.21000	-0.00121
	1.0	0.08408	1.37452	0.32814	0.04218	0.07198	0.66983	0.17530	-0.00356
1.0	0.1	0.32205	1.02070	0.25121	0.08527	0.22533	0.55499	0.10945	0.01773
	0.3	0.20606	1.26574	0.20529	0.04058	0.15799	0.65264	0.09769	0.00343
	0.5	0.15978	1.41472	0.16466	0.02634	0.12894	0.71015	0.08209	0.00011
	0.7	0.13375	1.52888	0.14653	0.02213	0.11174	0.75300	0.09052	-0.00114
1.0	1.0	0.11016	1.65865	0.19325	0.03344	0.09539	0.80307	0.10338	-0.00347

TABLE 6 - 3 0

Constant coefficients  $C_{n,t}$  and  $D_{n,t}$  as functions of  $Pn$  (for  $Lu = 0.3$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Bl_m = 10$ ;  $Bl_q = 10$ ), for an infinite cylinder

$Pn$	$C_{n,t}$				$D_{n,t}$			
	11	12	21	22	11	12	21	22
0.1	0.05586	0.39878	1.20390	0.01034	0.04688	0.20392	0.61710	-0.00049
0.3	0.05566	0.39035	1.26840	0.03652	0.04743	0.19861	0.64870	-0.00285
0.5	0.05534	0.38228	1.39463	0.08206	0.04776	0.19362	0.70960	-0.00835
0.7	0.05495	0.37456	1.82936	0.22454	0.04796	0.18894	0.91923	-0.02688
0.9	0.05453	0.36718	-0.07005	0.17820	0.04805	0.18452	-0.00349	0.01053

TABLE 6 - 3 1

Constant coefficients  $C_{n,t}$  and  $D_{n,t}$  as functions of  $Ko$  (for  $Lu = 0.3$ ;  $\epsilon = 0.5$ ;  $Pn = 0.5$ ;  $Bl_q = 10$ ;  $Bl_m = 10$ ), for an infinite cylinder

$Ko$	$C_{n,t}$				$D_{n,t}$			
	11	12	21	22	11	12	21	22
0.4	0.02046	0.14519	1.44504	0.02130	0.01727	0.07411	0.74032	-0.00124
0.8	0.03877	0.27118	1.37765	0.04525	0.03311	0.13786	0.70414	-0.00373
1.2	0.05534	0.38228	1.39463	0.08206	0.04776	0.19362	0.70960	-0.00835
1.6	0.07049	0.48150	1.64858	0.17562	0.06141	0.24304	0.83106	-0.02056
2.0	0.08447	0.57103	-0.10351	0.26550	0.07420	0.28732	-0.00405	0.01567

TABLE 6 - 3 2

Constant coefficients  $C_{n,t}$  and  $D_{n,t}$  as functions of  $Bl_m$  (for  $Lu = 0.3$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ;  $Bl_q = 10$ ), for an infinite cylinder

$Bl_m$	$C_{n,t}$				$D_{n,t}$			
	11	12	21	22	11	12	21	22
5	-0.00271	0.36011	1.15017	0.19746	-0.00239	0.20407	0.60136	-0.01843
10	0.05534	0.38228	1.30463	0.08206	0.04776	0.19362	0.70960	-0.00835
15	0.07866	0.38940	1.40536	0.04226	0.06730	0.18862	0.71122	-0.00436
20	0.09100	0.39276	1.40405	0.02468	0.07752	0.18578	0.70901	-0.00256

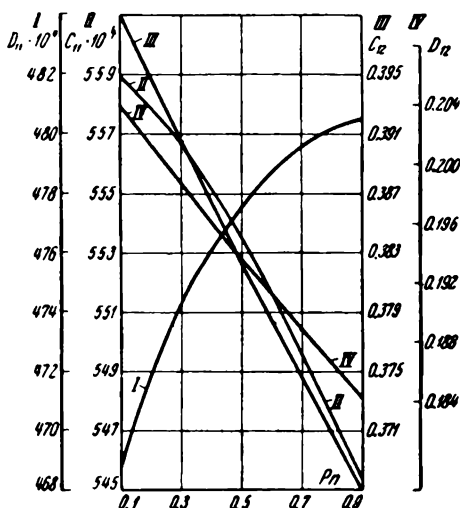


FIGURE 6-31. Coefficients  $C_{n1}$  and  $D_{n1}$  as functions of  $P_n$ , for an infinite cylinder

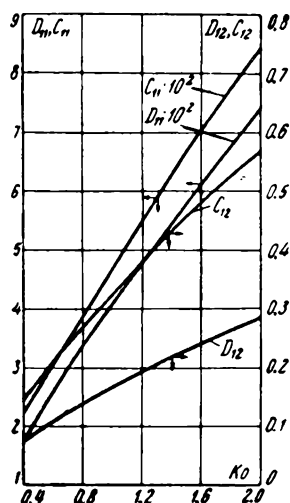


FIGURE 6-32. Coefficients  $C_{n1}$  and  $D_{n1}$  as functions of  $K_0$ , for an infinite cylinder

### Parabolic initial conditions

$$T(X, Fo) = \frac{t - t_s}{t_o - t_s} = 1 - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} J_0(v_i \mu_n X) \exp(-\mu_n^2 Fo); \quad (6-3-28)$$

$$\theta(X, Fo) = \frac{\theta_s - \theta}{\theta_s - \theta_{eq}} = 1 + \frac{1}{\epsilon K_0} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2) J_0(v_i \mu_n X) \exp(-\mu_n^2 Fo), \quad (6-3-29)$$

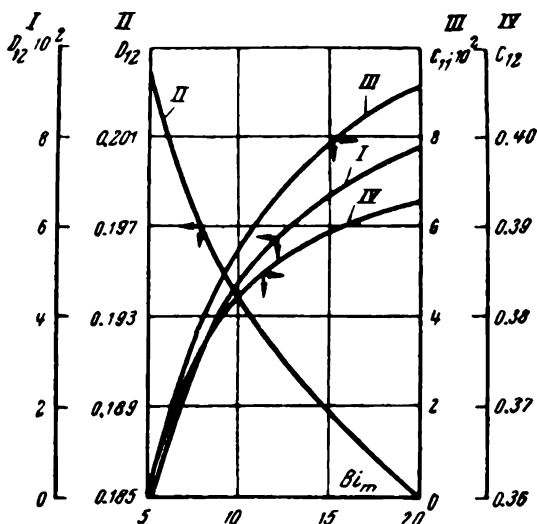


FIGURE 6-33. Coefficients  $C_{n1}$  and  $D_{n1}$  as functions of  $Bi_m$ , for an infinite cylinder

where  $C_{ni}$  are defined by expressions (6-3-18), in which

$$G_1 = 1 - \epsilon \text{Ko} K_1 - 2W \left( \frac{1}{\text{Bi}_q} - \frac{1}{\mu_n^2} \right) + 4\epsilon \text{Ko} (\text{Pn} W - V) \frac{(1 + K_1) Lu}{\mu_n^2}; \quad (6-3-30)$$

$$G_2 = \epsilon \text{Ko} + 2\epsilon \text{Ko} (\text{Pn} W - V) \left( \frac{1}{\text{Bi}_m} - \frac{2 Lu}{\mu_n^2} \right), \quad (6-3-31)$$

while  $Q_{ni}$ ,  $P_{ni}$ ,  $\psi_n$ ,  $A_{ni}$ , and  $B_{ni}$  remain as before. The roots  $\mu_n$  obey characteristic equation (6-3-27).

c) Solution for a sphere ( $\Gamma=2$ )

For constant initial conditions,

$$T(X, \text{Fo}) = 1 + \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \frac{\sin v_i \mu_n X}{X} \exp(-\mu_n^2 \text{Fo}); \quad (6-3-32)$$

$$\theta(X, \text{Fo}) = 1 - \frac{1}{\epsilon \text{Ko}} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2) \frac{\sin v_i \mu_n X}{X} \exp(-\mu_n^2 \text{Fo}), \quad (6-3-33)$$

where  $C_{ni}$  and  $\psi_n$  are defined by equations (6-3-3) through (6-3-5), and

$$Q_{ni} = \left[ 1 - \frac{1}{\text{Bi}_q} + (1 - v_i^2) K_1 \right] \sin v_i \mu_n + \frac{1}{\text{Bi}_q} v_i \mu_n \cos v_i \mu_n; \quad (6-3-34)$$

$$P_{ni} = (1 - v_i^2) \sin v_i \mu_n + \frac{(1 - v_i^2) + \epsilon \text{Ko} \text{Pn}}{\text{Bi}_m} (v_i \mu_n \cos v_i \mu_n - \sin v_i \mu_n); \quad (6-3-35)$$

$$A_{ni} = [1 + (1 - v_i^2) K_1] \cos v_i \mu_n - \frac{1}{\text{Bi}_q} v_i \mu_n \sin v_i \mu_n; \quad (6-3-36)$$

$$B_{ni} = (1 - v_i^2) \cos v_i \mu_n - \frac{(1 - v_i^2) + \epsilon \text{Ko} \text{Pn}}{\text{Bi}_m} v_i \mu_n \sin v_i \mu_n, \quad (6-3-37)$$

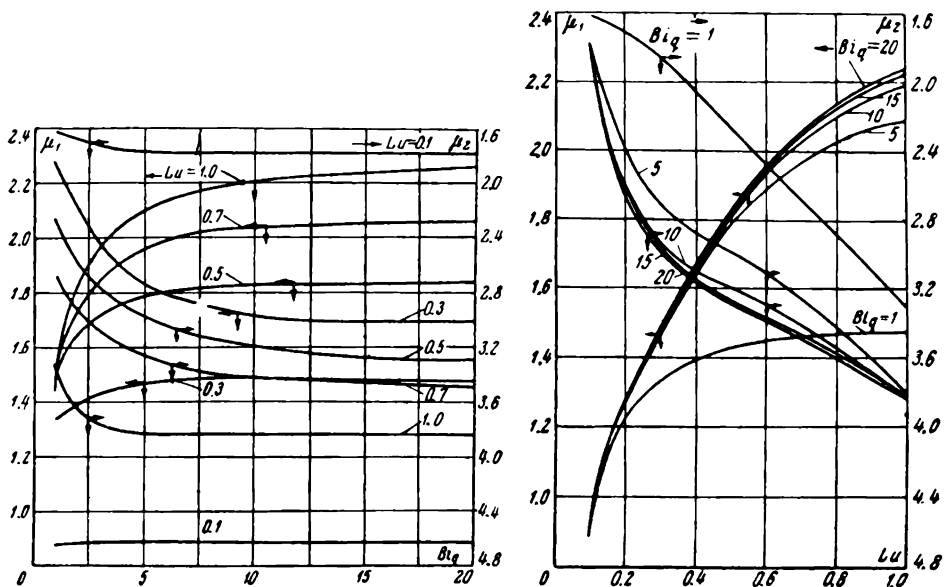


FIGURE 6-34. Characteristic-equation roots as functions of  $\text{Bi}_q$  and  $Lu$ , for a sphere

TABLE 6-33

Sphere. Roots of characteristic equation (6-3-38) as functions of  $Lu$  and  $Bi_q$  (for  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pr = 0.5$ ;  $Bi_m = 10$ )

$Lu$	$Bi_q$	$\mu_n$			
		$n=1$	$n=2$	$n=3$	$n=4$
0.1	1	0.8798	1.6202	1.7714	2.6881
	5	0.8846	1.7841	3.6251	4.5726
	10	0.8851	1.7836	2.7066	2.8718
	15	0.8852	1.7834	2.7006	2.9772
	20	0.8853	1.7833	2.6990	3.0297
0.3	1	1.3413	1.8552	2.9306	4.4007
	5	1.4701	2.741	4.4733	5.7429
	10	1.4788	2.975	4.4946	7.6040
	15	1.4814	3.0075	3.0671	4.5028
	20	1.4827	2.9811	3.1443	4.5072
0.5	1	1.4348	2.2685	3.4996	5.1359
	5	1.7845	2.9899	3.5529	5.4716
	10	1.8166	3.2037	3.6126	5.5277
	15	1.8263	3.2675	3.6523	5.5421
	20	1.8309	3.2955	3.6777	5.5486
0.7	1	1.4605	2.6800	3.7602	5.5923
	5	1.9594	3.2621	3.8992	6.0763
	10	2.0239	3.4217	4.0194	6.1816
	15	2.0440	3.4681	4.0814	6.2023
	20	2.0537	3.4890	4.1168	6.2106
1.0	1	1.4753	3.2987	3.8529	6.0609
	5	2.0906	3.8079	4.0596	6.5239
	10	2.1967	3.8267	4.3272	6.7084
	15	2.2320	3.8297	3.4369	6.7667
	20	2.2495	3.8309	4.4943	6.7930

TABLE 6-34

Sphere. Roots of characteristic equation (6-3-38) as functions of  $Lu$  and  $\epsilon$  (for  $Ko = 1.2$ ;  $Pr = 0.5$ ;  $Bi_q = 10$ ;  $Bi_m = 10$ )

$Lu$	$\epsilon$	$\mu_n$			
		$n=1$	$n=2$	$n=3$	$n=4$
0.1	0.1	0.8941	1.8034	2.7544	2.8294
	0.3	0.8896	1.7934	2.7276	2.8535
	0.5	0.8851	1.7836	2.7066	2.8718
	0.7	0.8806	1.7740	2.6880	2.8878
	1.0	0.8741	1.7599	2.6626	2.9092

TABLE 6-34 (continued)

Lu	$\epsilon$	$\mu_n$			
		$n=1$	$n=2$	$n=3$	$n=4$
0.3	0.1	1.5305	3.006	4.6749	5.8879
	0.3	1.5037	4.5798	7.7599	
	0.5	1.4788	2.975	4.4946	7.6040
	0.7	1.4554	4.4169	6.0002	6.0642
	1.0	1.4229	2.8687	3.0850	4.3113
0.5	0.1	1.9296	3.0468	3.8458	7.9625
	0.3	1.8690	3.1288	3.7219	5.7507
	0.5	1.8166	3.2037	3.6126	5.5277
	0.7	1.7703	3.2772	3.5103	5.3644
	1.0	1.7093	3.607	5.1633	
0.7	0.1	2.1936	3.2039	4.3661	
	0.3	2.0999	3.3212	4.1760	
	0.5	2.0239	3.4217	4.0194	6.1816
	0.7	1.9593	3.5146	3.8808	5.9338
	1.0	1.8771	4.015	5.6598	
1.0	0.1	2.4083	3.5610	4.7533	
	0.3	2.2906	3.7037	4.5215	
	0.5	2.1967	3.8267	4.3272	6.7084
	0.7	2.1179	3.9614	4.1328	6.4043
	1.0	2.0187	4.075	6.0755	

TABLE 6-35

Sphere. Roots of characteristic equation (6-3-38) as functions of  $Ko$ ,  $Pn$ , and  $Bl_m$ .

Dimensionless number		$\mu_n$			
		$n=1$	$n=2$	$n=3$	$n=4$
$Ko$	0.4	1.5266	2.9069	3.0608	4.6541
	0.8	1.5018	2.942	4.5717	5.9872
	1.2	1.4788	2.975	4.4946	7.6040
	1.6	1.4572	4.4221	7.4747	
	2.0	1.4369	2.9639	3.0289	4.3533
$Pn$	0.1	1.5371	2.8769	3.0904	4.6887
	0.3	1.5066	2.982	4.5877	5.9470
	0.5	1.4788	2.975	4.4946	7.6040
	0.7	1.4530	3.007	4.4080	7.4496
	0.9	1.4291	2.9304	3.0667	4.3267
$Bl_m$	5	1.3567	2.7975	2.9700	4.3210
	10	1.4788	2.975	4.4946	7.6040
	15	1.5226	4.5889	7.7016	
	20	1.5448	4.6446	7.7701	

Note. The combination of dimensionless-number values taken as a reference for the calculation was:  $Lu = 0.3$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ;  $Bl_q = 10$ ;  $Bl_m = 10$ .

TABLE 6-36

Constant coefficients  $C_{n,t}$  and  $D_{n,t}$  as functions of  $Lu$  and  $Bl_q$  (for  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ;  $Bl_m = 10$ ), for a sphere

Lu	$Bl_q$	$C_{n,t}$				$D_{n,t}$			
		11	12	21	22	11	12	21	22
0.1	1	-0.43581	-0.04583	-1.38859	-0.00061	-0.34971	-0.05155	-1.69969	0.00023
	5	-0.03525	-0.04528	-0.04053	0.01924	-0.04292	-0.05060	-0.05151	-0.00950
	10	-0.01706	-0.04522	-0.01138	0.01924	-0.01376	-0.05050	-0.01445	-0.00950
	15	-0.00544	-0.04519	-0.00349	0.01924	-0.00439	-0.05046	-0.00443	-0.00950
	20	0.00029	-0.04518	0.00018	0.01924	0.00024	-0.05044	0.00023	-0.00950
0.3	1	-1.48459	-0.15844	0.33064	-0.00963	-1.59483	-0.19318	0.41980	-0.00629
	5	-0.14578	-0.15604	0.07107	-0.04508	-0.16591	-0.18550	0.01325	-0.01247
	10	-0.04410	-0.16402	0.02709	-0.04647	-0.05036	-0.18192	0.00466	-0.01318
	15	-0.01350	-0.16333	0.05210	-0.03132	-0.01544	-0.18075	0.05835	-0.01568
	20	0.00122	-0.16298	0.00163	-0.01947	0.00140	-0.18017	0.00184	-0.00967
0.5	1	-1.53663	-0.28214	-0.50632	-0.04619	-1.66574	-0.36575	0.66205	-0.03023
	5	-0.24216	-0.31768	-0.48230	-0.01822	-0.29659	-0.35828	-0.57418	0.00262
	10	-0.07434	-0.31047	-0.48496	-0.04335	-0.09176	-0.34262	-0.53679	0.01363
	15	-0.02404	-0.30763	-0.40831	-0.04641	-0.02974	-0.33716	-0.44027	0.01651
	20	0.00000	-0.30616	-0.35330	-0.04475	0.00000	-0.33442	-0.37636	0.01665
0.7	1	-1.35636	-0.43837	0.42810	-0.05187	-1.42727	-0.57362	0.55582	-0.02538
	5	-0.29146	-0.46448	-0.30587	-0.01850	-0.36181	-0.53131	-0.35579	0.00169
	10	-0.09681	-0.45249	-0.29677	-0.03462	-0.12170	-0.49808	-0.32696	0.00759
	15	-0.03526	-0.44703	-0.24985	-0.03411	-0.04448	-0.48568	-0.27038	0.00862
	20	-0.00542	-0.44406	-0.22030	-0.03221	-0.00685	-0.47932	-0.23641	0.00860
1.0	1	-1.12538	-0.64748	0.16540	-0.01332	-1.11332	-0.84224	0.20573	-0.00153
	5	-0.28833	-0.62330	-0.65300	-0.13413	-0.35304	-0.72546	-0.70989	0.03770
	10	-0.10823	-0.60629	-0.20408	-0.04493	-0.13530	-0.66702	-0.22036	0.01314
	15	-0.04686	-0.59815	-0.12055	-0.02683	-0.05893	-0.64410	-0.13002	0.00789
	20	-0.01631	-0.59358	-0.08627	-0.01929	-0.02057	-0.63208	-0.09302	0.00569

TABLE 6 - 37

Constant coefficients  $C_{nl}$  and  $D_{nl}$  as functions of  $l_u$  and  $\epsilon$  (for  $K_0 = 1.2$ ;  $P_n = 0.5$ ;  $Bl_q = 10$ ;  $Bl_m = 10$ ), for a sphere

$l_u$	$\epsilon$	$C_{nl}$				$D_{nl}$			
		11	12	21	22	11	12	21	22
0.1	0.1	-0.03008	-0.00935	-0.02030	0.00393	-0.02474	-0.01047	-0.02598	-0.00194
	0.3	-0.02343	-0.02757	-0.01572	0.01167	-0.01908	-0.03085	-0.02004	-0.00575
	0.5	-0.01706	-0.04522	-0.01138	0.01924	-0.01376	-0.05050	-0.01445	-0.00950
	0.7	-0.01094	-0.06230	-0.00726	0.02655	-0.00874	-0.06946	-0.00919	-0.01318
0.3	1.0	-0.00222	-0.08694	-0.00146	0.03748	-0.00175	-0.09671	-0.00184	-0.01856
	0.1	-0.07945	-0.03875	0.04659	-0.01024	-0.09479	-0.04356	-0.00363	-0.00281
	0.3	-0.05953	-0.10643	0.03543	-0.02929	-0.06949	-0.11879	0.00178	-0.00818
	0.5	-0.04410	-0.16402	0.02709	-0.04647	-0.05036	-0.18192	0.00466	-0.01318
0.5	0.7	-0.03177	-0.21398	0.02038	-0.06198	-0.03550	-0.23601	0.00583	-0.01783
	1.0	-0.01727	-0.27814	-0.04531	0.09413	-0.01867	-0.30451	-0.05548	-0.04717
	0.1	-0.15008	-0.09561	-0.59298	-0.01387	-0.19416	-0.10854	-0.62663	-0.00058
	0.3	-0.10075	-0.22353	-0.51881	-0.02877	-0.12758	-0.24965	-0.56354	0.00470
0.7	0.5	-0.07434	-0.31047	-0.48496	-0.04335	-0.09176	-0.34262	-0.53679	0.01363
	0.7	-0.05779	-0.37613	-0.52143	-0.07878	-0.06941	-0.41115	-0.58464	0.03308
	1.0	-0.04191	-0.45161	0.02715	-0.13200	-0.04827	-0.48812	0.00398	-0.03971
	0.1	-0.21264	-0.20503	-0.37779	-0.02720	-0.27825	-0.23635	-0.38404	-0.00866
1.0	0.3	-0.12859	-0.36556	-0.31817	-0.03086	-0.16597	-0.40947	-0.34065	-0.00910
	0.5	-0.09681	-0.45249	-0.29577	-0.03462	-0.12170	-0.49808	-0.32696	0.00759
	0.7	-0.07976	-0.51309	-0.26434	-0.05127	-0.09722	-0.55776	-0.36498	0.01910
	1.0	-0.06504	-0.58001	0.04308	-0.17088	-0.07548	-0.62182	0.00483	-0.06214
1.0	0.1	-0.17783	-0.45591	-0.13973	-0.04933	-0.23254	-0.53422	-0.13259	-0.01631
	0.3	-0.12809	-0.54977	-0.15538	-0.03515	-0.16529	-0.61872	-0.16110	0.00176
	0.5	-0.10823	-0.60629	-0.20408	-0.04493	-0.13530	-0.66702	-0.22036	0.01314
	0.7	-0.09736	-0.64861	-0.24762	-0.18029	-0.11726	-0.70271	-0.60212	-0.08012
1.0	1.0	-0.08799	-0.69813	-0.05802	-0.20588	-0.09995	-0.74405	0.01330	-0.06329



TABLE 6-38

Constant coefficients  $C_{n,t}$  and  $D_{n,t}$  as functions of  $P_n$  (for  $Lu = 0.3$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Bl_m = 10$ ;  $Bl_q = 10$ ), for a sphere

	$C_{n,t}$				$D_{n,t}$			
	11	12	21	22	11	12	21	22
$P_n$								
0.1	-0.04145	-0.17227	-0.63239	-0.01066	-0.04857	-0.19264	-0.70743	0.00436
0.3	-0.04285	-0.16801	0.02506	-0.04542	-0.05007	-0.18707	0.00113	-0.01280
0.5	-0.04410	-0.16402	0.02709	-0.04647	-0.05036	-0.18192	0.00466	-0.01318
0.7	-0.04525	-0.16027	0.03032	-0.04733	-0.05250	-0.17713	0.00884	-0.01364
0.9	-0.04632	-0.15672	-0.05215	0.01959	-0.05054	-0.17265	-0.06240	-0.00987

TABLE 6-39

Constant coefficients  $C_{n,t}$  and  $D_{n,t}$  as functions of  $Ko$  (for  $Lu = 0.3$ ;  $\epsilon = 0.5$ ;  $P_n = 0.5$ ;  $Bl_q = 10$ ;  $Bl_m = 10$ ), for a sphere

	$C_{n,t}$				$D_{n,t}$			
	11	12	21	22	11	12	21	22
$Ko$								
0.4	-0.01533	-0.06246	-0.73209	-0.02581	-0.01820	-0.06974	-0.81659	0.01127
0.8	-0.02998	-0.11650	0.01793	-0.03227	-0.03490	-0.12963	0.00120	-0.00911
1.2	-0.04410	-0.16402	0.02709	-0.04647	-0.05036	-0.18192	0.00466	-0.01318
1.6	-0.05779	-0.20633	0.03705	-0.05947	-0.06475	-0.22817	0.01010	-0.01693
2.0	-0.07116	-0.24440	-0.39804	0.11168	-0.07823	-0.26952	-0.46895	-0.05631

TABLE 6-40

Constant coefficients  $C_{n,t}$  and  $D_{n,t}$  as functions of  $Bl_m$  (for  $Lu = 0.3$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $P_n = 0.5$ ;  $Bl_q = 10$ ), for a sphere

	$C_{n,t}$				$D_{n,t}$			
	11	12	21	22	11	12	21	22
$Bl_m$								
5	0.00237	-0.16347	-0.00234	-0.01949	0.00257	-0.19762	-0.00282	0.00848
10	-0.04410	-0.16402	0.02709	-0.04647	-0.05036	-0.18192	0.00466	-0.01318
15	-0.06033	-0.16358	0.04454	-0.04980	-0.07006	-0.17443	0.00488	-0.01536
20	-0.06837	-0.16316	0.05485	-0.05122	-0.08004	-0.17021	0.00403	-0.01629

$\mu_n$  being the roots of the characteristic equation

$$P_{n1}Q_{n2} - P_{n2}Q_{n1} = 0. \quad (6-3-38)$$

The infinite sums entering into (6-3-32) and (6-3-33) converge quite rapidly as the Fourier number increases. Thus, once again, for  $Fo$  above a certain value (see Table 6-16) all series terms except the first two may be neglected. A comparison of the "initial" Fourier numbers for simplified calculations for a plate and for a sphere shows that for a sphere the "initial" Fourier numbers are considerably lower than for a plate.

In terms of average transfer potentials solutions (6-3-32) and (6-3-33) become

$$\bar{T}(Fo) = 1 + \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} \exp(-\mu_n^2 Fo); \quad (6-3-39)$$

$$\bar{\theta}(Fo) = 1 - \frac{1}{\epsilon K_0} \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} (1 - v_i^2) \exp(-\mu_n^2 Fo), \quad (6-3-40)$$

where

$$D_{ni} = 3C_{ni} \frac{\sin v_i \mu_n - v_i \mu_n \cos v_i \mu_n}{v_i^2 \mu_n^2}.$$

Some values of characteristic-equation roots  $\mu_n$ , calculated using a high-speed computer, are listed in Tables 6-33 through 6-35 and plotted in Figures 6-34 through 6-37. Constant coefficients  $C_{ni}$  and  $D_{ni}$  are given, for various combinations of dimensionless numbers, in Tables 6-36 through 6-40 and in Figures 6-38 through 6-42.

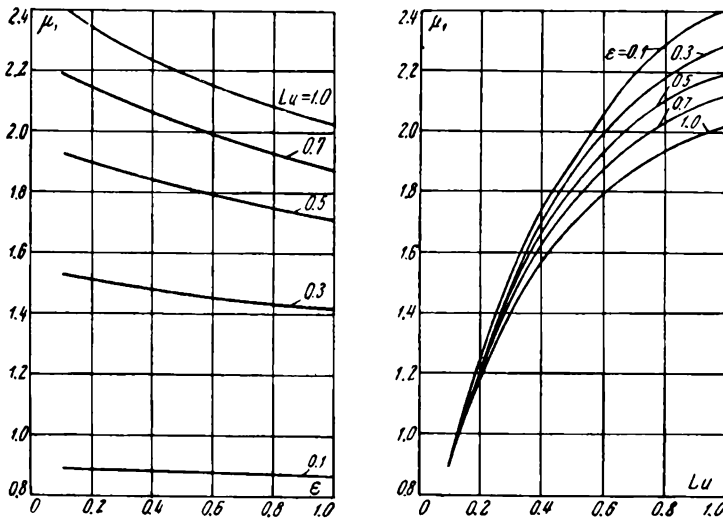


FIGURE 6-35. Characteristic-equation roots as functions of  $\epsilon$  and  $Lu$ , for a sphere

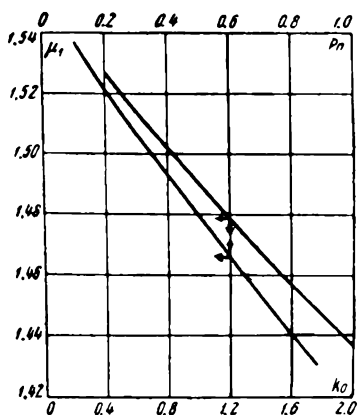


FIGURE 6-36. Characteristic-equation roots as functions of  $P_n$  and  $K_o$ , for a sphere

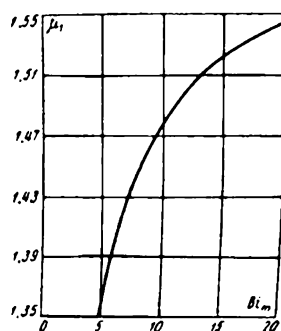


FIGURE 6-37. Characteristic-equation roots as functions of  $Bi_m$ , for a sphere

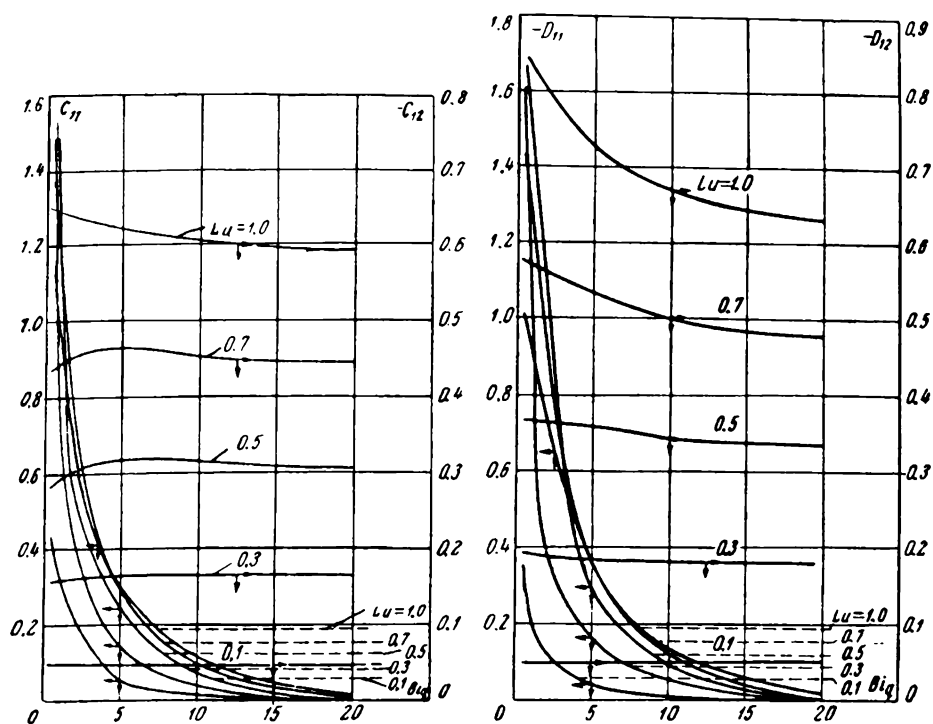
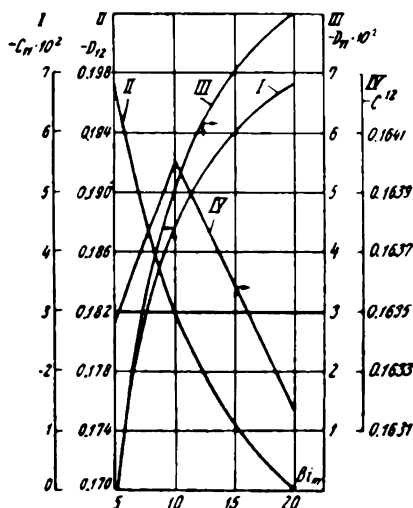
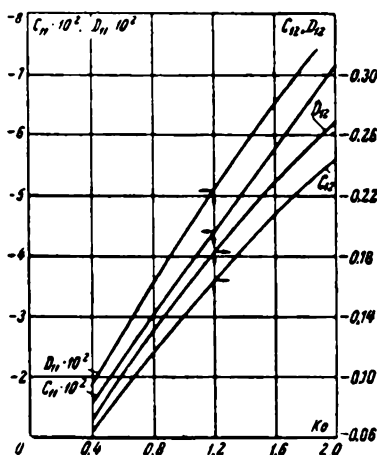


FIGURE 6-38. Coefficients  $C_{n,i}$  and  $D_{n,i}$  as functions of  $Bi_e$  and  $Lu$ , for a sphere





### Parabolic initial conditions

$$T(X, Fo) = \frac{t - t_s}{t_o - t_s} = 1 + \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \frac{\sin v_i \mu_n X}{X} \exp(-\mu_n^2 Fo); \quad (6-3-41)$$

$$\theta(X, F_0) = \frac{\theta_s - \theta}{\theta_s - \theta_{eq}} = 1 - \frac{1}{\epsilon K_0} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2)^{\frac{\sin v_i \mu_n X}{X}} \exp(-\mu_n^2 F_0), \quad (6-3-42)$$

where coefficients  $C_{ni}$  are defined by equations (6-3-18), with

$$G_1 = 1 - \varepsilon K_0 K_1 - 2W \left( \frac{1}{\text{Bl}_q} - \frac{3}{\mu_n^2} \right) + 6\varepsilon K_0 (P_n W - V) \frac{(1 + K_1) Lu}{\mu_n^2}; \quad (6-3-43)$$

and

$$G_s = s K_0 + 2s K_0 (Pn W - V) \left( \frac{1}{Bl_q} - \frac{3 Lu}{\mu_n^2} \right), \quad (6-3-44)$$

and with the rest of the quantities ( $\psi_{ni}$ ,  $Q_{ni}$ ,  $P_{ni}$ ,  $A_{ni}$  and  $B_{ni}$ ) remaining as before. The roots  $\mu_n$  obey characteristic equation (6-3-38). If dimensionless parameters  $\Psi$  and  $V$  are zero, then the solutions (6-3-32) and (6-3-33) for constant initial conditions are obtained.

#### 6-4. Unsteady Fields of Heat-Transfer and Mass-Transfer Potential. Mass Flux at Surface of Body a Function of Time

In this section we will consider several solutions of the system of heat-transfer and mass-transfer equations for an infinite plate ( $\Gamma=0$ ); various types of boundary conditions of the third kind will be assumed. In all the problems the dimensionless mass flux at the surface of the body will be an implicit function of time. The initial conditions will be functions of the spatial coordinates:  $T(X, 0) = F_1(X)$  and  $\Theta(X, 0) = F_2(X)$ . As a result, the dimensionless transfer potentials must be written as

$$T(X, Fo) = \frac{t - t_*}{t_*} \quad \text{and} \quad \Theta = \frac{\theta_* - \theta}{\theta_*}.$$

Contrary to the preceding sections (especially §§ 5-2 and 5-3), in order to abbreviate the transformations involved, the heat-transfer and mass-transfer potentials will be denoted as  $\Theta_l$  ( $l=1, 2$ ), where  $\Theta_1 = T$  and  $\Theta_2 = \Theta$ .

##### a) Solution for a symmetrical infinite plate

Let us first solve system of heat-transfer and mass-transfer equations (5-1-2) and (5-1-3) for the boundary conditions

$$\frac{\partial \Theta_1(l, Fo)}{\partial X} - Bi_q [1 - \Theta_1(l, Fo)] + (1 - \epsilon) Lu Ko Ki_m(Fo) = 0; \quad (6-4-1)$$

$$-\frac{\partial \Theta_2(l, Fo)}{\partial X} + Pn \frac{\partial \Theta_1(l, Fo)}{\partial X} + Ki_m(Fo) = 0, \quad (6-4-2)$$

the initial conditions

$$\Theta_l(X, 0) = F_l(X), \quad (0 \leq X \leq 1), \quad l = 1, 2, \quad (6-4-3)$$

and the symmetry conditions

$$\frac{\partial \Theta_l(0, Fo)}{\partial X} = 0. \quad (6-4-4)$$

Following Prudnikov /7/, let us consider an auxiliary problem. We are given the unknown functions  $\Theta_1(l, Fo) = \varphi_1(Fo)$  and  $\Theta_2(l, Fo) = \varphi_2(Fo)$ , also written as

$$\Theta_k(l, Fo) = \varphi_k(Fo), \quad (k=1, 2), \quad (6-4-5)$$

which will be defined later. The solution of the auxiliary problem reduces to a determination of the dimensionless potentials  $\Theta_l(X, Fo)$  which will satisfy system of equations (4-1-2) and (4-1-3), initial conditions (6-4-3), symmetry conditions (6-4-4), and boundary conditions (6-4-5). The latter will be solved by means of a combined application of Fourier and Laplace integral transformations. First, with respect to the variable  $X$ , we apply the finite Fourier cosine transformation

$$\{\Theta_l(p, Fo)\}_c = \int_0^1 \Theta_l(X, Fo) \cos\left(n + \frac{1}{2}\right) \pi X dX. \quad (6-4-6)$$

The inverse transform of the transfer potential is determined using the formula

$$\Theta_l(X, Fo) = 2 \sum_{n=1}^{\infty} \{\Theta_l(p, Fo)\}_c \cos\left(n + \frac{1}{2}\right) \pi X. \quad (6-4-7)$$

In order to transform the second derivative, we use the formula

$$\left\{ \frac{\partial^2 \theta_l(p, Fo)}{\partial X^2} \right\}_e = - \left( n + \frac{1}{2} \right)^2 \pi^2 \{ \theta_l(p, Fo) \}_e + \\ + (-1)^n \left( n + \frac{1}{2} \right) \pi \theta_l(1, Fo) - \frac{\partial \theta_l(0, Fo)}{\partial X}. \quad (6-4-8)$$

The Laplace transformation is applied with respect to the variable Fourier number:

$$\{ \theta_l(X, s) \}_L = \int_0^\infty \theta_l(X, Fo) \exp(-s Fo) dFo, \quad (6-4-9)$$

and also

$$\left\{ \frac{\partial \theta_l(X, s)}{\partial Fo} \right\}_L = s \{ \theta_l(X, s) \}_L - \theta_l(X, 0). \quad (6-4-10)$$

By successive applications of integral transformations (6-4-6) and (6-4-9) (just as was done in Chapter V), and by taking into account formulas (6-4-8) and (6-4-10), initial conditions (6-4-3), and boundary conditions (6-4-5), we reduce the original problem to the system of algebraic equations

$$\left. \begin{aligned} [s + (1 + \epsilon Ko Pn Lu) \Lambda_n^2] \{ \theta_l \}_{eL} - \epsilon Ko Lu \Lambda_n^2 \{ \theta_s \}_{eL} = \\ = \{ F_l \}_e + (-1)^n \Lambda_n (1 + \epsilon Ko Pn Lu) \{ \varphi_l \}_L - (-1)^n \Lambda_n \epsilon Ko Lu \{ \varphi_s \}_L; \\ - Lu Pn \Lambda_n^2 \{ \theta_l \}_{eL} + (s + Lu \Lambda_n^2) \{ \theta_s \}_{eL} = \{ F_s \}_e - (-1)^n \Lambda_n Lu Pn \{ \varphi_l \}_L + \\ + (-1)^n \Lambda_n Lu \{ \varphi_s \}_L, \end{aligned} \right\} \quad (6-4-11)$$

where

$$\{ \varphi_k \}_L = \int_0^\infty \varphi_k(Fo) \exp(-s Fo) dFo; \\ \{ F_l \}_e = \{ F_l(p) \}_e = \int_0^1 F_l(X) \cos \Lambda_n X dX, \\ \Lambda_n = \left( n + \frac{1}{2} \right) \pi.$$

When system (6-4-11) is solved for the dimensionless potentials and inverse Laplace transformations are applied to the latter, then we obtain

$$\{ \theta_l(p, Fo) \}_e = \sum_{s=1}^2 \sum_{i=1}^2 [A_{si}^l \{ P_{si} \}_e + B_{si}^l \{ Q_{si} \}_e] \quad (6-4-12) \\ (l=1, 2),$$

where

$$\{ P_{si} \}_e = \{ F_s(p) \}_e \exp[-\Lambda_n^2 v_i^2 Lu Fo]; \quad (6-4-13)$$

$$\{ Q_{si} \}_e = (-1)^n \Lambda_n \int_0^{Fo} \varphi_s(Fo^*) \exp[-\Lambda_n^2 v_i^2 Lu (Fo - Fo^*)] dFo^*; \quad (6-4-14)$$

$$v_i^2 = \frac{1}{2} \left[ \left( 1 + \epsilon Ko Pn + \frac{1}{Lu} \right) + (-1)^i \sqrt{\left( 1 + \epsilon Ko Pn + \frac{1}{Lu} \right)^2 - \frac{4}{Lu}} \right].$$

The constant coefficients  $A_{si}^l$  and  $B_{si}^l$  ( $l, s, i=1, 2$ ) are defined as

$$A_{1i}^1 = (-1)^i \frac{1 - v_i^2}{v_1^2 - v_2^2}; \quad B_{1i}^1 = (-1)^i \frac{1 - v_i^2 + \epsilon \text{Ko Pn Lu } v_i^2}{v_1^2 - v_2^2};$$

$$A_{2i}^1 = (-1)^i \frac{\epsilon \text{Ko}}{v_1^2 - v_2^2}; \quad B_{2i}^1 = (-1)^i \frac{\epsilon \text{Ko Lu } v_i^2}{v_1^2 - v_2^2};$$

$$A_{1i}^2 = (-1)^i \frac{\text{Pn}}{v_1^2 - v_2^2}; \quad B_{1i}^2 = (-1)^i \frac{\text{Pn Lu } v_i^2}{v_1^2 - v_2^2};$$

$$A_{2i}^2 = (-1)^i \frac{1/\text{Lu} - v_i^2 + \epsilon \text{Ko Pn}}{v_1^2 - v_2^2}; \quad B_{2i}^2 = (-1)^i \frac{\text{Lu}(1/\text{Lu} - v_i^2)}{v_1^2 - v_2^2}.$$

These expressions for coefficients  $A_{si}^l$  and  $B_{si}^l$  are, with the exception of coefficients  $B_{1i}^1$ , identical to the corresponding expressions for  $A_{ki}$  and  $B_{ki}$  in Chapter V (equations (5-2-11)). Thus, the values of these constants as functions of  $\epsilon \text{Ko Pn}$  and  $\text{Lu}$  (with  $\text{Ko}^* = \epsilon \text{Ko}$ ) may be found listed in Tables 5-1 and 5-2 and plotted in Figures 5-1 through 5-6. Values of coefficients  $B_{1i}^1$  are given in Table 6-41.

TABLE 6-41  
Values of coefficients  $B_{1i}^1$  and  $\frac{1}{\text{Pn}} B_{1i}^2$

Lu	i	$\epsilon \text{Ko Pn}=0.1$		$\epsilon \text{Ko Pn}=0.4$		$\epsilon \text{Ko Pn}=0.7$		$\epsilon \text{Ko Pn}=1.0$	
		$B_{1i}^1$	$\frac{1}{\text{Pn}} B_{1i}^2$	$B_{1i}^1$	$\frac{1}{\text{Pn}} B_{1i}^2$	$B_{1i}^1$	$\frac{1}{\text{Pn}} B_{1i}^2$	$B_{1i}^1$	$\frac{1}{\text{Pn}} B_{1i}^2$
0.2	1	0.0106	0.0470	0.0354	0.0398	0.0529	0.0343	0.0658	0.0300
	2	-0.0188	-0.2471	-0.0764	-0.2398	-0.1351	-0.2342	-0.1942	-0.2300
0.4	1	0.0569	0.2185	0.1405	0.1417	0.1748	0.1039	0.1907	0.0811
	2	-0.0268	-0.6184	-0.1329	-0.5417	-0.2506	-0.5039	-0.3716	-0.4812
0.6	1	0.1678	0.5351	0.2801	0.2560	0.3009	0.1675	0.3024	0.1225
	2	0.0008	-1.1351	-0.1647	-0.8560	-0.3535	-0.7674	-0.5426	-0.7225
0.8	1	0.3679	0.9005	0.4097	0.3454	0.4024	0.2133	0.3862	0.1512
	2	0.1879	-1.7005	-0.1867	-1.1454	-0.4561	-1.0132	-0.7162	-0.9512
1.0	1	0.5360	1.1398	0.7479	0.4046	0.4781	0.2444	0.4472	0.1708
	2	0.2080	-2.1398	-0.2126	-1.4047	-0.5641	-1.2444	-0.8944	-0.1708

The application of inversion formula (6-4-7) to equation (6-4-12) next gives

$$\theta_i(X, \text{Fo}) = \sum_{s=1}^2 \sum_{i=1}^2 [A_{si}^l P_{si} + B_{si}^l Q_{si}] \quad (l=1, 2), \quad (6-4-15)$$

where

$$P_{si} = 2 \sum_{n=0}^{\infty} \exp[-\Lambda_n^2 v_i^2 \text{Lu Fo}] \int_0^1 F_s(\xi) \cos \Lambda_n \xi d\xi \cos \Lambda_n X; \quad (6-4-16)$$

$$Q_{si} = 2 \sum_{n=0}^{\infty} \exp[-\Lambda_n^2 v_i^2 \text{Lu Fo}] \text{Fo}^0 (-1)^n \Lambda_n \Phi_s(\text{Fo}) \cos \Lambda_n X. \quad (6-4-17)$$



Here the symbol  $F_0$  before a function denotes the convolution operation:

$$\begin{aligned} f(F_0) \overset{F_0}{\circ} g(F_0) &= \int_0^{F_0} f(F_0^*) g(F_0 - F_0^*) dF_0^* = \\ &= \int_0^{F_0} f(F_0 - F_0^*) g(F_0^*) dF_0^* = g(F_0) \overset{F_0}{\circ} f(F_0). \end{aligned}$$

Equations (6-4-16) and (6-4-17) can be simplified by introducing the theta function /21/. Then, after the necessary transformations we obtain

$$\begin{aligned} P_{si} &= \frac{1}{2} \int_0^1 \left\{ \vartheta_2 \left[ \frac{X-\xi}{2}, v_i^2 \text{Lu } F_0 \right] + \right. \\ &\quad \left. + \vartheta_2 \left[ \frac{X+\xi}{2}, v_i^2 \text{Lu } F_0 \right] \right\} F_s(\xi) d\xi; \end{aligned} \quad (6-4-18')$$

$$Q_{si} = \int_0^{F_0} \frac{\partial}{\partial X} \vartheta_2 \left[ \frac{1-X}{2}, v_i^2 \text{Lu } (F_0 - F_0^*) \right] \varphi_s(F_0^*) dF_0^*; \quad (6-4-18)$$

$$\vartheta_2(X, F_0) = \frac{1}{\sqrt{\pi F_0}} \sum_{n=-\infty}^{\infty} (-1)^n \exp \left[ -\frac{(X+n)^2}{F_0} \right]. \quad (6-4-19)$$

In order to solve the problem at hand, we have to determine the unknown functions  $\varphi_k(F_0)$  ( $k=1, 2$ ) from boundary conditions (6-4-1) and (6-4-2). If we differentiate (6-4-15) with respect to  $X$  and then replace the second derivative of  $\vartheta_2$  with respect to  $X$  [in (6-4-18') and (6-4-18)] by its derivative with respect to  $F_0$ , we obtain

$$\frac{\partial \vartheta_l(X, F_0)}{\partial X} = \sum_{s=1}^2 \sum_{i=1}^2 (A'_{si} M_{si} + B'_{si} N_{si}) \quad (l=1, 2), \quad (6-4-20)$$

where

$$\begin{aligned} M_{si}(X, F_0) &= \frac{1}{2} \int_0^1 \frac{\partial}{\partial X} \left\{ \vartheta_2 \left[ \frac{X-\xi}{2}, v_i^2 \text{Lu } F_0 \right] + \right. \\ &\quad \left. + \vartheta_2 \left[ \frac{X+\xi}{2}, v_i^2 \text{Lu } F_0 \right] \right\} F_s(\xi) d\xi; \\ N_{si}(X, F_0) &= \\ &= \frac{1}{v_i^2 \text{Lu}} \int_0^{F_0} \frac{\partial}{\partial (F_0 - F_0^*)} \vartheta_2 \left[ \frac{1-X}{2}, v_i^2 \text{Lu } (F_0 - F_0^*) \right] \varphi_s(F_0^*) dF_0^*. \end{aligned} \quad (6-4-21)$$

From (6-4-19) it is clear that the derivative  $\frac{\partial}{\partial (F_0 - F_0^*)} \vartheta_2$ , which exists for all  $X \neq 1$ , goes to infinity as  $F_0^{-3/2}$  when  $F_0 \rightarrow 0$  (provided that  $F_0 \rightarrow F_0^*$  for  $X=1$ ). To avoid this difficulty, let us replace boundary conditions (6-4-1) and (6-4-2) by certain conditions which can be obtained from the latter by integrating them with respect to  $F_0$ . The integral conditions are

$$\begin{aligned} \int_0^{F_0} \frac{\partial \vartheta_1(1, F_0^*)}{\partial X} dF_0^* - \text{Bi}_q \int_0^{F_0} [1 - \vartheta_1(1, F_0^*)] dF_0^* + \\ + (1 - \varepsilon) \text{Ko Lu} \int_0^{F_0} \text{Ki}_m(F_0^*) dF_0^* = 0; \end{aligned} \quad (6-4-22)$$

$$-\int_0^{Fo} \frac{\partial \theta_2(l, Fo^*)}{\partial X} dFo^* + Pn \int_0^{Fo} \frac{\partial \theta_1(l, Fo^*)}{\partial X} dFo^* + \int_0^{Fo} Ki_m(Fo^*) dFo^* = 0. \quad (6-4-23)$$

Also, by integrating (6-4-20) with respect to  $Fo$ , we obtain

$$\int_0^{Fo} \frac{\partial \theta_l(X, Fo^*)}{\partial X} dFo^* = \sum_{s=1}^2 \sum_{i=1}^2 (A_{si}^l M_{si}^* + B_{si}^l N_{si}^*) \quad (l=1, 2), \quad (6-4-24)$$

where

$$M_{si}^*(X, Fo) = \int_0^{Fo} M_{si}(X, Fo^*) dFo^*, \quad N_{si}^*(X, Fo) = \int_0^{Fo} N_{si}(X, Fo^*) dFo^*.$$

In the integration of (6-4-21) with respect to  $Fo$  the order of integration has to be changed using Dirichlet's formula, and then we obtain

$$\begin{aligned} N_{si}^*(X, \omega) &= \frac{1}{v_i^2 Lu} \int_0^{\omega} dFo \int_0^{Fo} \varphi_s(Fo^*) \times \\ &\times \frac{\partial}{\partial(Fo - Fo^*)} \theta_2 \left[ \frac{1-X}{2}, v_i^2 Lu(Fo - Fo^*) \right] dFo^* = \\ &= \frac{1}{v_i^2 Lu} \int_0^{\omega} \varphi_s(Fo^*) dFo^* \int_{Fo^*}^{\omega} \frac{\partial}{\partial(Fo - Fo^*)} \theta_2 \left[ \frac{1-X}{2}, v_i^2 Lu(Fo - Fo^*) \right] dFo = \\ &= \frac{1}{v_i^2 Lu} \int_0^{\omega} \varphi_s(Fo^*) dFo^* \left\{ \theta_2 \left[ \frac{1-X}{2}, v_i^2 Lu(Fo - Fo^*) \right] - \theta_2 \left[ \frac{1-X}{2}, 0 \right] \right\}. \end{aligned}$$

From this equation it follows that for all  $X < 1$

$$N_{si}^*(X, \omega) = \frac{1}{v_i^2 Lu} \int_0^{\omega} \varphi_s(Fo^*) \theta_2 \left[ \frac{1-X}{2}, v_i^2 Lu(\omega - Fo^*) \right] dFo^*. \quad (6-4-25)$$

Calculations show that (6-4-25) is also valid for  $X=1$ , that is,

$$N_{si}^*(1, \omega) = \frac{1}{v_i^2 Lu} \int_0^{\omega} \varphi_s(Fo^*) \theta_2[0, v_i^2 Lu(\omega - Fo^*)] dFo^*. \quad (6-4-26)$$

The last integral converges, since according to (6-4-19) for  $Fo^* \rightarrow \omega$  the function  $\theta_2$  goes to infinity as  $\omega^{-1/2}$ , provided  $\omega \rightarrow 0$ . Equations (6-4-24) through (6-4-26) may be used to obtain the respective terms in modified boundary conditions (6-4-22) and (6-4-23). Then, taking into account (6-4-5), we obtain a system of Volterra integral equations of the first kind:

$$\sum_{\beta=1}^2 \int_0^{Fo} K_{\alpha\beta}(Fo - Fo^*) \varphi_{\beta}(Fo^*) dFo^* = g_{\alpha}(Fo) \quad (\alpha=1, 2), \quad (6-4-27)$$

where functions  $K_{\alpha\beta}(Fo - Fo^*)$  and  $g_{\alpha}(Fo)$  have a form which makes it possible to reduce system (6-4-27) to a system of generalized Abel integral equations and to reduce the latter to a system of Volterra integral equations of the second kind. The method used for this transformation will be illustrated later in a problem (see subsection 6-4, b following). Once these integral equations are solved it will be possible to determine the desired functions

$\Phi_k(Fo)$ , whose substitution into equations (6-4-15) through (6-4-17) represents a final answer to the problem at hand.

For the particular case of no thermal-gradient mass transfer in the material ( $Pn = 0$ ) and a constant initial distribution of the transfer potentials, Smirnov /14/ obtained the following solution:

$$\begin{aligned} \Theta(X, Fo) = \frac{\theta_0 - \theta}{\theta_0} = & \int_0^{Fo} Lu Ki_m(Fo^*) dFo^* - Ki_m(Fo) \frac{1}{6} (1 - 3X^*) + \\ & + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n^2 \pi^2} \cos n\pi X \exp(-n^2 \pi^2 Lu Fo) \times \\ & \times \left\{ Ki_m(0) + \int_0^{Fo} \exp(n^2 \pi^2 Lu Fo^*) \frac{d Ki_m(Fo^*)}{d Fo^*} dFo^* \right\}, \end{aligned} \quad (6-4-28)$$

where  $Ki_m(0)$  is the Kirpichev mass-exchange number at the moment  $Fo = 0$ . After a certain amount of time has elapsed, that is, for  $Fo > Fo_1$ , it is possible to neglect the infinite series in (6-4-28), leaving only the first two terms.

The temperature field for an infinite plate may thus be written as

$$\begin{aligned} T(X, Fo) = & 1 - Ki_m(Fo) Ko Lu \frac{t_0}{t_0 - t_*} \left[ 0.5 * (1 - X^*) + \frac{1}{Bi_q} \right] - \\ & - \left[ 1 - Ki_m(0) (1 - *) \frac{Ko Lu}{Bi_q} \cdot \frac{t_0}{t_0 - t_*} \right] \sum_{n=1}^{\infty} A_n \cos \mu_n X \exp(-\mu_n^2 Fo) + \\ & + Ki_m(0) * Ko Lu \frac{t_0}{t_0 - t_*} \sum_{n=1}^{\infty} A_n \frac{1}{\mu_n^2} \cos \mu_n X \exp(-\mu_n^2 Fo) + \\ & + * Ko Lu \frac{t_0}{t_0 - t_*} \sum_{n=1}^{\infty} A_n \frac{1}{\mu_n^2} \cos \mu_n X \exp(-\mu_n^2 Fo) \times \\ & \times \int_0^{Fo} \frac{d Ki_m(Fo^*)}{d Fo^*} \exp(\mu_n^2 Fo^*) dFo^* + (1 - *) Ko Lu \frac{1}{Bi_q} \cdot \frac{t_0}{t_0 - t_*} \times \\ & \times \sum_{n=1}^{\infty} A_n \cos \mu_n X \exp(-\mu_n^2 Fo) \int_0^{Fo} \frac{d Ki_m(Fo^*)}{d Fo^*} \exp(\mu_n^2 Fo^*) dFo^*. \end{aligned} \quad (6-4-29')$$

If  $Fo$  is high enough, the following approximate solution for the temperature at any point on the plate may be used:

$$T(X, Fo) = \frac{t - t_0}{t_0 - t_*} = 1 - Ki_m(Fo) Ko Lu \frac{t_0}{t_0 - t_*} \left[ 0.5 * (1 - X^*) + \frac{1}{Bi_q} \right]. \quad (6-4-29)$$

The average values of the transfer potentials are

$$\begin{aligned} \bar{\Theta}(Fo) = & \int_0^{Fo} Lu Ki_m(Fo^*) dFo^*; \\ \bar{T}(Fo) = & 1 - Ko Lu Ki_m(Fo) \frac{t_0}{t_0 - t_*} \left[ \frac{1}{3} * + \frac{1}{Bi_q} \right]. \end{aligned}$$

From the latter formulas it is possible to obtain the following rate of variation of mass-transfer potential (for example, during a drying process this is the drying rate):

$$\frac{d\bar{\Theta}}{dFo} = \left[ \frac{t_0 - \bar{t}(\tau)}{Ko t_0 \left( \frac{1}{3} * + \frac{1}{Bi_q} \right)} \right]. \quad (6-4-30)$$

Expression (6-4-30) is useful in practice, since it makes it possible to determine, from the average temperature of the body and from the coefficients of heat and mass transfer, the rate of variation of mass-transfer potential. In other words, the calculation of the mass-exchange rate is reduced to a calculation of the heat exchange. From the above relations it is also possible to derive a formula for the Kirpichev mass-exchange number:

$$Ki_m(\tau) = \frac{t_c - \bar{t}}{Ko Lu t_c \left( \frac{1}{3} \epsilon + \frac{1}{Bi_q} \right)},$$

which indicates that the vaporization rate is a function of the phase-transition ratios (the Kossovich and Biot numbers) and the inertia number  $Lu$ .

**b) Solution for a symmetrical infinite plate, with generalized boundary conditions of the third kind**

System of heat-transfer and mass-transfer equations (4-1-2) and (4-1-3) will now be solved with the generalized boundary conditions

$$\frac{\partial \theta_1(l, Fo)}{\partial X} + Bi_q \theta_1(l, Fo) - (1 - \epsilon) Lu Ko Bi_m \theta_2(l, Fo) = \chi_1(Fo); \quad (6-4-31)$$

$$-\frac{\partial \theta_2(l, Fo)}{\partial X} + Pn \frac{\partial \theta_1(l, Fo)}{\partial X} - Bi_m \theta_2(l, Fo) = \chi_2(Fo). \quad (6-4-32)$$

The initial conditions and symmetry conditions are given by (6-4-3) and (6-4-4). The functions  $F_l(X)$  and  $\chi_i(Fo)$  are arbitrary bounded integrable functions of their arguments.

The first part of the solution is exactly the same as for the solution of the previous problem. As before, we assume that the dimensionless potentials at the surface are unknown functions of time:

$$\theta_k(l, Fo) = \varphi_k(Fo) \quad (k=1,2). \quad (6-4-5)$$

This auxiliary problem is solved by applying Fourier and Laplace integral transformations. As a result, we obtain expressions (6-4-15) through (6-4-17) or, provided the theta function is introduced, expressions (6-4-15), (6-4-18'), and (6-4-18).

The purpose of the second part of the solution is to determine the unknown functions  $\varphi_k(Fo)$  from the boundary conditions (in the present case from (6-4-31) and 6-4-32)). For the reasons given in the foregoing subsection (6-4, a), it is necessary to consider, instead of boundary conditions (6-4-31) and (6-4-32), certain conditions which are obtained from these by integration with respect to the Fourier number, namely

$$\begin{aligned} \int_0^{Fo} \frac{\partial \theta_1(l, Fo^*)}{\partial X} dFo^* + Bi_q \int_0^{Fo} \varphi_1(Fo^*) dFo^* - (1 - \epsilon) Lu Ko Bi_m \int_0^{Fo} \varphi_2(Fo^*) dFo^* = \\ = \int_0^{Fo} \chi_1(Fo^*) dFo^*; \end{aligned} \quad (6-4-33)$$

$$\begin{aligned} \int_0^{Fo} \frac{\partial \theta_2(l, Fo^*)}{\partial X} dFo^* - Pn \int_0^{Fo} \frac{\partial \theta_1(l, Fo^*)}{\partial X} dFo^* + Bi_m \int_0^{Fo} \varphi_2(Fo^*) dFo^* = \\ = - \int_0^{Fo} \chi_2(Fo^*) dFo^*. \end{aligned} \quad (6-4-34)$$

We may now determine the individual terms in modified boundary conditions (6-4-33) and (6-4-34) by taking into account (6-4-5). Thus we obtain a system of Volterra integral equations of the first kind:

$$\sum_{\beta=1}^2 \int_0^{Fo} K_{\alpha\beta}(Fo - Fo^*) \varphi_{\beta}(Fo^*) dFo^* = g_{\alpha}(Fo) \quad (6-4-35)$$

( $\alpha = 1, 2$ ),

where

$$\begin{aligned} K_{11}(Fo - Fo^*) &= \frac{B_{11}^1}{v_1^2 Lu} \vartheta_2[0, v_1^2 Lu(Fo - Fo^*)] + \\ &+ \frac{B_{21}^1}{v_2^2 Lu} \vartheta_2[0, v_2^2 Lu(Fo - Fo^*)] + Bi_q; \\ K_{12}(Fo - Fo^*) &= \frac{B_{12}^1}{v_1^2 Lu} \vartheta_2[0, v_1^2 Lu(Fo - Fo^*)] + \\ &+ \frac{B_{22}^1}{v_2^2 Lu} \vartheta_2[0, v_2^2 Lu(Fo - Fo^*)] - (1 - \varepsilon) Lu Ko Bi_m; \\ K_{21}(Fo - Fo^*) &= \frac{B_{11}^2 - Pn B_{11}^1}{v_1^2 Lu} \vartheta_2[0, v_1^2 Lu(Fo - Fo^*)] + \\ &+ \frac{B_{21}^2 - Pn B_{21}^1}{v_2^2 Lu} \vartheta_2[0, v_2^2 Lu(Fo - Fo^*)]; \\ K_{22}(Fo - Fo^*) &= \frac{B_{12}^2 - Pn B_{12}^1}{v_1^2 Lu} \vartheta_2[0, v_1^2 Lu(Fo - Fo^*)] + \\ &+ \frac{B_{22}^2 - Pn B_{22}^1}{v_2^2 Lu} \vartheta_2[0, v_2^2 Lu(Fo - Fo^*)] + Bi_m; \\ g_1(Fo) &= \int_0^{Fo} \chi_1(Fo^*) dFo^* - \sum_{s=1}^2 \sum_{l=1}^2 A_{s,l}^1 M^*_{s,l}(1, Fo); \\ g_2(Fo) &= - \int_0^{Fo} \chi_2(Fo^*) dFo^* - \sum_{s=1}^2 \sum_{l=1}^2 (A_{s,l}^1 - Pn A_{s,l}^2) M^*_{s,l}(1, Fo). \end{aligned}$$

In these equations the notation

$$K_{11}(Fo - Fo^*) = \frac{G_1(Fo - Fo^*)}{\sqrt{Fo - Fo^*}} \quad \text{and} \quad K_{22}(Fo - Fo^*) = \frac{G_2(Fo - Fo^*)}{\sqrt{Fo - Fo^*}} \quad (6-4-36)$$

is used, where

$$\begin{aligned} G_1(Fo - Fo^*) &= \frac{x_1}{\sqrt{\pi}} + \frac{B_{11}^1}{v_1^2 Lu \sqrt{v_1^2 Lu \pi}} \sum_{n=-\infty}^{\infty} ' (-1)^n \exp \left[ -\frac{n^2}{v_1^2 Lu (Fo - Fo^*)} \right] + \\ &+ \frac{B_{21}^1}{v_2^2 Lu \sqrt{v_2^2 Lu \pi}} \sum_{n=-\infty}^{\infty} ' (-1)^n \exp \left[ -\frac{n^2}{v_2^2 Lu (Fo - Fo^*)} \right] + \\ &+ Bi_q \sqrt{Fo - Fo^*}; \end{aligned}$$

$$\begin{aligned}
G_2(F_0 - F_0^*) &= \frac{x_2}{\sqrt{\pi}} + \frac{B_{12}^2 - P_n B_{22}^1}{v_1^2 L_u \sqrt{v_1^2 L_u \pi}} \sum_{n=-\infty}' (-1)^n \exp \left[ -\frac{n^2}{v_1^2 L_u (F_0 - F_0^*)} \right] + \\
&+ \frac{B_{22}^2 - P_n B_{22}^1}{v_2^2 L_u \sqrt{v_2^2 L_u \pi}} \sum_{n=-\infty}' (-1)^n \exp \left[ -\frac{n^2}{v_2^2 L_u (F_0 - F_0^*)} \right] - \\
&- (1 - \varepsilon) L_u K_0 B_{im} \sqrt{F_0 - F_0^*}; \\
x_1 &= \frac{B_{11}^1}{v_1^2 L_u \sqrt{v_1^2 L_u}} + \frac{B_{21}^1}{v_2^2 L_u \sqrt{v_2^2 L_u}}; \\
x_2 &= \frac{B_{12}^2 - P_n B_{12}^1}{v_1^2 L_u \sqrt{v_1^2 L_u}} + \frac{B_{22}^2 - P_n B_{22}^1}{v_2^2 L_u \sqrt{v_2^2 L_u}}.
\end{aligned}$$

The summation  $\Sigma'$  in the above equations extends over all integral values of  $n$  except  $n=0$ . The substitution of (6-4-36) into (6-4-35) gives a system of generalized Abel integral equations, which can be reduced to a system of Volterra integral equations of the second kind:

$$\varphi_\alpha(F_0) + \sum_{\beta=1}^2 \int_0^{F_0} \varphi_{\alpha\beta}(F_0, F_0^*) \varphi_\beta(F_0^*) dF_0^* = \gamma_\alpha(F_0) \quad (6-4-37)$$

( $\alpha = 1, 2$ ),

where

$$\begin{aligned}
\varphi_{\alpha\beta}(F_0, F_0^*) &= \frac{1}{x_\alpha \sqrt{\pi}} \frac{d}{dF_0} L_{\alpha\beta}(F_0, F_0^*); \\
\gamma_\alpha(F_0) &= \frac{1}{x_\alpha \sqrt{\pi}} \frac{d}{dF_0} S_\alpha(F_0); \\
L_{\alpha\beta}(F_0, F_0^*) &= \int_0^{F_0} \frac{G_\alpha(\omega - F_0^*)}{\sqrt{(F_0 - \omega)(\omega - F_0^*)}} d\omega \quad (\alpha = \beta); \\
L_{\alpha\beta}(F_0, F_0^*) &= \int_{F_0^*}^{F_0} \frac{K_{\alpha\beta}(\omega - F_0^*)}{\sqrt{F_0 - \omega}} d\omega \quad (\alpha \neq \beta); \\
S_\alpha(F_0) &= \int_0^{F_0} \frac{g_\alpha(\omega)}{\sqrt{F_0 - \omega}} d\omega \\
&(\alpha = 1, 2; \beta = 1, 2).
\end{aligned}$$

A final solution is obtained once functions  $\varphi_\alpha(F_0)$  are determined from system (6-4-37) and substituted into (6-4-15) through (6-4-17). A more detailed account of the solution method is given by Datsev / 22/.

### c) Solution for an unsymmetrical infinite plate, with generalized boundary conditions of the third kind

In principle the methods for solving this problem, as was shown by Prudnikov / 8/, are no different from those for the previous problems. The initial conditions are given by (6-4-3).

Let us assume that the boundary conditions may be written as

$$K_{11}\theta_1(0, F_0) + K_{12}\theta_2(0, F_0) + K_{13} \frac{\partial \theta_1(0, F_0)}{\partial X} + K_{14} \frac{\partial \theta_2(0, F_0)}{\partial X} = \chi_1(F_0); \quad (6-4-38)$$

$$K_{21}\theta_1(0, Fo) + K_{22}\theta_2(0, Fo) + K_{23}\frac{\partial\theta_1(0, Fo)}{\partial X} + K_{24}\frac{\partial\theta_2(0, Fo)}{\partial X} = \chi_2(Fo); \quad (6-4-39)$$

$$K_{31}\theta_1(1, Fo) + K_{32}\theta_2(1, Fo) + K_{33}\frac{\partial\theta_1(1, Fo)}{\partial X} + K_{34}\frac{\partial\theta_2(1, Fo)}{\partial X} = \chi_3(Fo); \quad (6-4-40)$$

$$K_{41}\theta_1(1, Fo) + K_{42}\theta_2(1, Fo) + K_{43}\frac{\partial\theta_1(1, Fo)}{\partial X} + K_{44}\frac{\partial\theta_2(1, Fo)}{\partial X} = \chi_4(Fo), \quad (6-4-41)$$

where  $K_{km}$  are combinations of dimensionless numbers which remain constant during the entire process of heat and mass transfer, and where  $F_l(X)$  and  $\chi_k(Fo)$  ( $l=1, 2$ ;  $k, m=1, 2, 3, 4$ ) are arbitrary bounded integrable functions of their arguments.

Let us assume that functions

$$\theta_i(0, Fo) = \varphi_i(Fo) \quad \text{and} \quad \theta_i(1, Fo) = \psi_i(Fo) \quad (6-4-42)$$

are given. Then, by a combined application of Fourier and Laplace integral transformations, a solution of system of heat-transfer and mass-transfer equations (4-1-2) and (4-1-3) with boundary conditions (6-4-42) and initial conditions (6-4-3) can be obtained in the form

$$\theta_i(X, Fo) = \sum_{s=1}^2 \sum_{l=1}^2 (A_{sl}^i P_{s,l} + B_{sl}^i Q_{s,l}), \quad (6-4-43)$$

where  $A_{sl}^i$  and  $B_{sl}^i$  are constant coefficients, and where

$$\begin{aligned} P_{s,l} = & - \int_0^{Fo} \frac{\partial}{\partial X} \vartheta_s \left[ \frac{X}{2}, \nu_l^2 \text{Lu}(Fo - Fo^*) \right] \varphi_s(Fo^*) dFo^* + \\ & + \int_0^{Fo} \frac{\partial}{\partial X} \vartheta_s \left[ \frac{1-X}{2}, \nu_l^2 \text{Lu}(Fo - Fo^*) \right] \psi_s(Fo^*) dFo^*; \\ Q_{s,l} = & \frac{1}{2} \int_0^1 \left\{ \vartheta_s \left[ \frac{X-\xi}{2}, \nu_l^2 \text{Lu} Fo \right] - \vartheta_s \left[ \frac{X+\xi}{2}, \nu_l^2 \text{Lu} Fo \right] \right\} F_s(\xi) d\xi; \\ \vartheta_s(X, Fo) = & \frac{1}{\sqrt{\pi Fo}} \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{(X+n)^2}{Fo} \right]. \end{aligned}$$

In addition, functions  $\varphi_i(Fo)$  and  $\psi_i(Fo)$  may be determined from boundary conditions (6-4-38) through (6-4-41). When (6-4-43) is differentiated with respect to  $X$  and the second derivative with respect to  $X$  of the theta function  $\vartheta_s$  is replaced by its derivative with respect to  $Fo$ , we obtain

$$\frac{\partial\theta_i(X, Fo)}{\partial X} = \sum_{s=1}^2 \sum_{l=1}^2 (A_{sl}^i M_{s,l} + B_{sl}^i N_{s,l}), \quad (6-4-44)$$

where

$$\begin{aligned} M_{s,l} = & - \frac{1}{\nu_l^2 \text{Lu}} \int_0^{Fo} \frac{\partial}{\partial(Fo - Fo^*)} \vartheta_s \left[ \frac{X}{2}, \nu_l^2 \text{Lu}(Fo - Fo^*) \right] \varphi_s(Fo^*) dFo^* + \\ & + \frac{1}{\nu_l^2 \text{Lu}} \int_0^{Fo} \frac{\partial}{\partial(Fo - Fo^*)} \vartheta_s \left[ \frac{1-X}{2}, \nu_l^2 \text{Lu}(Fo - Fo^*) \right] \psi_s(Fo^*) dFo^*; \end{aligned} \quad (6-4-45)$$

$$N_{s,l} = \frac{1}{2} \int_0^1 \frac{\partial}{\partial X} \left\{ \vartheta_s \left[ \frac{X-\xi}{2}, \nu_l^2 \text{Lu} Fo \right] - \vartheta_s \left[ \frac{X+\xi}{2}, \nu_l^2 \text{Lu} Fo \right] \right\} F_s(\xi) d\xi. \quad (6-4-46)$$

As in the previous problems, it can be shown that the derivative  $\frac{\partial}{\partial(Fo - Fo^*)} \vartheta_s$ , which exists for all  $X \neq 0$  in the first integral and for all  $X \neq 1$  in the second integral on the right sides of (6-4-45) and (6-4-46), goes to infinity as  $Fo^{-3/2}$  when  $Fo \rightarrow 0$  (provided that  $Fo \rightarrow Fo^*$  for  $X = 0$  in the first integral and for  $X = 1$  in the second integral). Thus, boundary conditions (6-4-38) through (6-4-41) may be replaced by conditions derived from them by integration with respect to  $Fo$ . The integration of (6-4-44) with respect to  $Fo$  gives

$$\int_0^{Fo} \frac{\partial \Theta_i(X, Fo^*)}{\partial X} dFo^* = \sum_{s=1}^2 \sum_{i=1}^2 (A_{si} M_{si}^* + B_{si} N_{si}^*), \quad (6-4-47)$$

where

$$M_{si}^* = \int_0^{Fo} M_{si}(X, Fo^*) dFo^* \quad \text{and} \quad N_{si}^* = \int_0^{Fo} N_{si}(X, Fo^*) dFo^*.$$

When (6-4-45) is integrated with respect to  $Fo$ , the order of integration being reversed according to the Dirichlet formula, we obtain, for all values  $0 < X < 1$  and in particular for  $X = 0$  and  $X = 1$ ,

$$\begin{aligned} M_{si}^*(X, \omega) = & -\frac{1}{v_i^2 Lu} \int_0^\omega \varphi_s(Fo^*) \vartheta_s \left[ \frac{X}{2}, v_i^2 Lu (\omega - Fo^*) \right] dFo^* + \\ & + \frac{1}{v_i^2 Lu} \int_0^\omega \psi_s(Fo^*) \vartheta_s \left[ \frac{1-X}{2}, v_i^2 Lu (\omega - Fo^*) \right] dFo^*. \end{aligned} \quad (6-4-48)$$

When equations (6-4-47) are written for these values of  $X$  and the corresponding expressions substituted into the equations obtained by integrating (6-4-38) through (6-4-41) with respect to  $Fo$ , we obtain a system of Volterra integral equations of the first kind:

$$\sum_{\beta=1}^4 \int_0^{Fo} K_{\alpha\beta}(Fo, Fo^*) \varphi_\beta(Fo^*) dFo^* = g_\alpha(Fo) \quad (\alpha = 1, 2, 3, 4).$$

This system may in turn be reduced to a system of generalized Abel integral equations and finally to a system of Volterra integral equations of the second kind. Functions  $\varphi_i(Fo)$  and  $\psi_i(Fo)$  may be found from this system.

## 6-5. Some Particular Cases of Heat and Mass Transfer

Unsteady fields of heat-transfer and mass-transfer potentials corresponding to particular transfer cases are best obtained from solutions of the complete system of equations (4-1-2) and (4-1-3) by taking limits. When passing in this way from the general solution of the system of equations to a particular solution, the form of the characteristic equation after taking the limits and the "form" of the roots thus obtained are quite important. Thus, for an infinite plate characteristic equation (6-3-10) becomes, as a result of taking limits as  $Pn \rightarrow 0$ ,

$$\left( \cos \mu - \frac{1}{Bi_q} \mu \sin \mu \right) \left( \cos \frac{\mu}{\sqrt{Lu}} - \frac{1}{Bi_m} \frac{\mu}{\sqrt{Lu}} \sin \frac{\mu}{\sqrt{Lu}} \right) = 0, \quad (6-5-1)$$

so that the previous single form of the root  $\mu_{n0}$  which obeys the complete system of equations must now be written in the dual form

$$\mu = \mu_n \quad \text{and} \quad \frac{\mu}{\sqrt{Lu}} = \mu_m.$$



Therefore, in order to pass to a particular system of heat-transfer and mass-transfer equations by taking limits, the original infinite sum in the general solution should be repeated term by term as many times as there are "forms" of the roots of the transformed characteristic equation. For example, the infinite sum of (6-3-1) must be repeated twice. In the first sum  $\mu = \mu_n$  and in the second sum  $\frac{\mu}{\sqrt{Lu}} = \mu_m$  are substituted for  $\mu_{n0}$ , since we have two "forms" of the roots for characteristic equation (6-5-1). After this, the limits may be taken and the particular solution obtained in the usual manner.

The taking of limits makes it possible to verify the correctness of solutions of the complete system of equations. Also, various solutions of incomplete systems are obtained easily and rapidly on the basis of one general solution, and new interesting features and properties of a process become evident. For example, the development of methods involving the taking of limits in the system of heat-transfer and mass-transfer equations has led us to discover the new physical significance of the dimensionless number  $Lu$ , as a measure of the coupling between heat transfer and mass transfer. In relation to this, it has become possible, from the formal mathematical point of view, to calculate the heat and mass transfer in an incomplete system ( $Pn = 0$ ,  $Ko = 0$ ,  $\varepsilon = 0$ ) in terms of mass-exchange quantities only or heat-exchange quantities only. It is interesting to note that for certain heat-transfer and mass-transfer processes, the quantity  $\sqrt{Lu}$  is to a certain extent analogous to the quantity  $c^2$  in Einstein's relation  $E = c^2 m$  determining the correspondence between energy and mass.

In the following subsections (6-5, a and 6-5, b) several particular solutions will be obtained by means of the above method of taking limits.

#### a) No thermal mass transfer in the material

The generalized solution for the dimensionless transfer potentials, with constant initial conditions, has the form

$$T(X, Fo) = 1 - \sum_{n=1}^{\infty} A_n \left( 1 + \frac{\varphi_n}{\psi_n} \right) \Phi_T(\mu_n X) \exp(-\mu_n^2 Fo) - \sum_{m=1}^{\infty} A_m Ko \left[ \frac{\varepsilon Lu}{1-Lu} \Phi_T(\mu_m X) + \frac{\varphi_m}{\psi_m} \Phi_T(\sqrt{Lu} \mu_m X) \right] \exp(-\mu_m^2 Lu Fo); \quad (6-5-2)$$

$$\Theta(X, Fo) = 1 - \sum_{m=1}^{\infty} A_m \Phi_T(\mu_m X) \exp(-\mu_m^2 Lu Fo). \quad (6-5-3)$$

For an infinite plate,

$$\Phi_0(\mu_n X) = \cos \mu_n X; \quad \Phi_0(\mu_m X) = \cos \mu_m X; \quad \Phi_0(\sqrt{Lu} \mu_m X) = \cos \sqrt{Lu} \mu_m X;$$

$$\varphi_n = Ko \left[ \frac{Bi_m}{Bi_q} \left( 1 - \varepsilon - \frac{\varepsilon}{Lu-1} \right) \sqrt{Lu} \mu_n \sin \frac{\mu_n}{\sqrt{Lu}} + \frac{\varepsilon Lu Bi_m}{Lu-1} \cos \frac{\mu_n}{\sqrt{Lu}} \right];$$

$$\varphi_m = Lu \left[ \left( 1 - \varepsilon - \frac{\varepsilon}{Lu-1} \right) \mu_m \sin \mu_m + \frac{\varepsilon Bi_q}{Lu-1} \cos \mu_m \right];$$

$$\psi_n = Bi_m \cos \frac{\mu_n}{\sqrt{Lu}} - \frac{\mu_n}{\sqrt{Lu}} \sin \frac{\mu_n}{\sqrt{Lu}}; \quad \psi_m = Bi_q \cos \sqrt{Lu} \mu_m - \sqrt{Lu} \mu_m \sin \sqrt{Lu} \mu_m;$$

where  $\mu_n$  and  $\mu_m$  are the roots of the characteristic equation

$$\cot \mu_l = \frac{1}{Bi_l} \mu_l \quad \left( l = \begin{cases} n & \text{or} \\ m \end{cases} \right); \quad (6-5-4)$$

and coefficients  $A_n$  and  $A_m$  are defined by the expressions

$$A_l = \frac{2 \sin \mu_l}{\mu_l + \sin \mu_l \cos \mu_l} = (-1)^{l+1} \frac{2BI_l \sqrt{BI_l^2 + \mu_l^2}}{\mu_l (BI_l^2 + BI_l + \mu_l^2)} \quad (6-5-5)$$

Values of characteristic-equation roots  $\mu_l$  are given in Table 6-42 and values of coefficients  $A_l$  are given in Table 6-43.

TABLE 6-42

Roots of characteristic equation  $\cot \mu_l = \frac{1}{BI_l \mu_l}$

$BI_l$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_5$	$\mu_6$
0	0.0000	3.1416	6.2832	9.4248	12.5664	15.7080
0.001	0.0316	3.1419	6.2833	9.4249	12.5665	15.7080
0.002	0.0447	3.1422	6.2835	9.4250	12.5665	15.7081
0.004	0.0632	3.1429	6.2838	9.4252	12.5667	15.7082
0.006	0.0774	3.1435	6.2841	9.4254	12.5668	15.7083
0.008	0.0893	3.1441	6.2845	9.4256	12.5670	15.7085
0.01	0.0998	3.1448	6.2848	9.4258	12.5672	15.7086
0.02	0.1410	3.1479	6.2864	9.4269	12.5680	15.7092
0.04	0.1987	3.1543	6.2895	9.4290	12.5696	15.7105
0.06	0.2425	3.1606	6.2927	9.4311	12.5711	15.7118
0.08	0.2791	3.1668	6.2959	9.4333	12.5727	15.7131
0.1	0.3111	3.1731	6.2991	9.4354	12.5743	15.7143
0.2	0.4328	3.2039	6.3148	9.4459	12.5823	15.7207
0.3	0.5218	3.2341	6.3305	9.4565	12.5902	15.7270
0.4	0.5932	3.2636	6.3461	9.4670	12.5981	15.7334
0.5	0.6533	3.2923	6.3616	9.4775	12.6060	15.7397
0.6	0.7051	3.3204	6.3770	9.4879	12.6139	15.7460
0.7	0.7506	3.3477	6.3923	9.4983	12.6218	15.7524
0.8	0.7910	3.3744	6.4074	9.5087	12.6296	15.7587
0.9	0.8274	3.4003	6.4224	9.5190	12.6375	15.7650
1.0	0.8603	3.4256	6.4373	9.5293	12.6453	15.7713
1.5	0.9882	3.5422	6.5097	9.5801	12.6841	15.8026
2.0	1.0769	3.6436	6.5783	9.6296	12.7223	15.8336
3.0	1.1925	3.8088	6.7040	9.7240	12.7966	15.8945
4.0	1.2646	3.9352	6.8140	9.8119	12.8678	15.9536
5.0	1.3138	4.0336	6.9096	9.8928	12.9352	16.0107
6.0	1.3496	4.1116	6.9924	9.9667	12.9988	16.0654
7.0	1.3766	4.1746	7.0640	10.0339	13.0584	16.1177
8.0	1.3978	4.2264	7.1263	10.0949	13.1141	16.1675
9.0	1.4149	4.2694	7.1806	10.1502	13.1660	16.2147
10.0	1.4289	4.3058	7.2281	10.2003	13.2142	16.2594
15.0	1.4729	4.4255	7.3959	10.3898	13.4078	16.4474
20.0	1.4961	4.4915	7.4954	10.5117	13.5420	16.5864
30.0	1.5202	4.5615	7.6057	10.6543	13.7085	16.7691
40.0	1.5325	4.5979	7.6647	10.7334	13.8048	16.8794
50.0	1.5400	4.6202	7.7012	10.7832	11.8666	16.9519
60.0	1.5451	4.6353	7.7259	10.8172	13.9094	17.0026
80.0	1.5514	4.6543	7.7573	10.8606	13.9644	17.0686
100.0	1.5552	4.6658	7.7764	10.8871	13.9981	17.1093
$\infty$	1.5708	4.7124	7.8540	10.9956	14.1372	17.2788

TABLE 6-43

Values of coefficients  $A_l$  for an infinite plate

$BI_l$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$
0	1.0000	-0.0000	0.0000	-0.0000	0.0000	-0.0000
0.001	1.0002	-0.0002	0.0000	-0.0000	0.0000	-0.0000
0.002	1.0004	-0.0004	0.0001	-0.0000	0.0000	-0.0000
0.004	1.0008	-0.0008	0.0002	-0.0001	0.0000	-0.0000
0.006	1.0012	-0.0012	0.0003	-0.0001	0.0001	-0.0000
0.008	1.0015	-0.0016	0.0004	-0.0002	0.0001	-0.0001
0.01	1.0020	-0.0020	0.0005	-0.0002	0.0001	-0.0001
0.02	1.0030	-0.0040	0.0010	-0.0004	0.0003	-0.0002
0.04	1.0065	-0.0080	0.0020	-0.0009	0.0005	-0.0003
0.06	1.0099	-0.0119	0.0030	-0.0013	0.0007	-0.0004

TABLE 6-43 (continued)

$Bi_L$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$
0.08	1.0130	-0.0158	0.0040	-0.0018	0.0010	-0.0006
0.10	1.0159	-0.0197	0.0050	-0.0022	0.0013	-0.0008
0.20	1.0312	-0.0381	0.0100	-0.0045	0.0025	-0.0016
0.30	1.0450	-0.0555	0.0148	-0.0067	0.0038	-0.0024
0.40	1.0581	-0.0719	0.0196	-0.0089	0.0050	-0.0032
0.50	1.0701	-0.0873	0.0243	-0.0110	0.0063	-0.0040
0.60	1.0813	-0.1025	0.0289	-0.0132	0.0075	-0.0048
0.70	1.0918	-0.1154	0.0335	-0.0153	0.0087	-0.0056
0.80	1.1016	-0.1282	0.0379	-0.0175	0.0100	-0.0064
0.90	1.1107	-0.1403	0.0423	-0.0196	0.0112	-0.0072
1.00	1.1192	-0.1517	0.0466	-0.0217	0.0124	-0.0080
1.5	1.1537	-0.2013	0.0667	-0.0318	0.0184	-0.0119
2.0	1.1784	-0.2367	0.0848	-0.0414	0.0241	-0.0157
3.0	1.2102	-0.2881	0.1154	-0.0589	0.0351	-0.0231
4.0	1.2287	-0.3215	0.1396	-0.0750	0.0451	-0.0300
5.0	1.2403	-0.3442	0.1588	-0.0876	0.0543	-0.0366
6.0	1.2478	-0.3604	0.1740	-0.0991	0.0626	-0.0427
7.0	1.2532	-0.3722	0.1861	-0.1089	0.0701	-0.0483
8.0	1.2569	-0.3812	0.1959	-0.1174	0.0763	-0.0535
9.0	1.2598	-0.3880	0.2039	-0.1246	0.0828	-0.0583
10.0	1.2612	-0.3934	0.2104	-0.1309	0.0881	-0.0676
15.0	1.2677	-0.4084	0.2320	-0.1514	0.1072	-0.0795
20.0	1.2699	-0.4147	0.2394	-0.1621	0.1182	-0.0901
30.0	1.2717	-0.4198	0.2472	-0.1718	0.1291	-0.1015
40.0	1.2723	-0.4217	0.2502	-0.1759	0.1340	-0.1069
50.0	1.2727	-0.4227	0.2517	-0.1779	0.1365	-0.1098
60.0	1.2728	-0.4232	0.2526	-0.1791	0.1379	-0.1115
80.0	1.2730	-0.4237	0.2535	-0.1803	0.1394	-0.1132
100.0	1.2731	-0.4239	0.2539	-0.1808	0.1405	-0.1141
$\infty$	1.2732	-0.4244	0.2546	-0.1819	0.1415	-0.1157

TABLE 6-44

Roots of characteristic equation  $\frac{J_0(\mu_i)}{J_1(\mu_i)} = \frac{1}{Bi_L} \mu_i$

$Bi_L$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_5$	$\mu_6$
0.0	0.0000	3.8317	7.0156	10.1735	13.3237	16.4706
0.01	0.1412	3.8343	7.0170	10.1745	13.3244	16.4712
0.02	0.1995	3.8369	7.0184	10.1754	13.3252	16.4718
0.04	0.2814	3.8421	7.0213	10.1774	13.3267	16.4731
0.06	0.3438	3.8473	7.0241	10.1794	13.3282	16.4743
0.08	0.3960	3.8525	7.0270	10.1813	13.3297	16.4755
0.10	0.4417	3.8577	7.0298	10.1833	13.3312	16.4767
0.15	0.5376	3.8706	7.0369	10.1882	13.3349	16.4797
0.20	0.6170	3.8835	7.0440	10.1931	13.3387	16.4828
0.30	0.7465	3.9091	7.0582	10.2029	13.3462	16.4888
0.40	0.8516	3.9344	7.0723	10.2127	13.3537	16.4949
0.50	0.9408	3.9594	7.0864	10.2225	13.3611	16.5010
0.60	1.0184	3.9841	7.1004	10.2322	13.3686	16.5070
0.70	1.0873	4.0085	7.1143	10.2419	13.3761	16.5131
0.80	1.1490	4.0325	7.1282	10.2519	13.3835	16.5191
0.90	1.2048	4.0562	7.1421	10.2613	13.3910	16.5251
1.0	1.2558	4.0795	7.1558	10.2710	13.3984	16.5312
1.5	1.4569	4.1902	7.2233	10.3188	13.4353	16.5612
2.0	1.5994	4.2910	7.2884	10.3658	13.4719	16.5910
3.0	1.7887	4.4634	7.4103	10.4566	13.5434	16.6499

TABLE 6-44 (continued)

$Bi_1$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_5$	$\mu_6$
4.0	1.9081	4.6018	7.5201	10.5423	13.6125	16.7073
5.0	1.9898	4.7131	7.6177	10.6223	13.6786	16.7630
6.0	2.0490	4.8033	7.7039	10.6964	13.7414	16.8168
7.0	2.0937	4.8772	7.7797	10.7646	13.8008	16.8684
8.0	2.1286	4.9384	7.8464	10.8271	13.8566	16.9179
9.0	2.1566	4.9897	7.9051	10.8842	13.9090	16.9650
10	2.1795	5.0332	7.9569	10.9363	13.9580	17.0099
15	2.2509	5.1773	8.1422	11.1367	14.1576	17.2008
20	2.2880	5.2568	8.2534	11.2677	14.2983	17.3442
30	2.3261	5.3410	8.3771	11.4221	14.4748	17.5348
40	2.3455	5.3846	8.4432	11.5081	14.5774	17.6508
50	2.3572	5.4112	8.4840	11.5621	14.6433	17.7272
60	2.3651	5.4291	8.5116	11.5990	14.6889	17.7807
80	2.3750	5.4516	8.5466	11.6461	14.7475	17.8502
100	2.3809	5.4652	8.5678	11.6747	14.7834	17.8931
$\infty$	2.4048	5.5201	8.6537	11.7915	14.9309	18.0711

TABLE 6-45

Values of coefficients  $A_i$  for an infinite cylinder

$Bi_1$	$+A_1$	$-A_2$	$+A_3$	$-A_4$	$+A_5$	$-A_6$
0.0	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.01	1.0031	0.0034	0.0013	0.0008	0.0005	0.0004
0.02	1.0049	0.0067	0.0027	0.0015	0.0010	0.0007
0.04	1.0102	0.0135	0.0052	0.0031	0.0021	0.0015
0.06	1.0150	0.0201	0.0081	0.0046	0.0031	0.0023
0.08	1.0199	0.0268	0.0110	0.0062	0.0041	0.0030
0.10	1.0245	0.0333	0.0135	0.0077	0.0051	0.0037
0.15	1.0366	0.0497	0.0202	0.0116	0.0077	0.0056
0.20	1.0482	0.0658	0.0269	0.0154	0.0103	0.0075
0.30	1.0711	0.0972	0.0401	0.0231	0.0155	0.0112
0.40	1.0931	0.1277	0.0582	0.0307	0.0205	0.0150
0.50	1.1142	0.1571	0.0662	0.0383	0.0256	0.0187
0.60	1.1345	0.1857	0.0790	0.0458	0.0307	0.0224
0.70	1.1539	0.2132	0.0917	0.0539	0.0358	0.0261
0.80	1.1724	0.2398	0.1043	0.0608	0.0408	0.0298
0.90	1.1902	0.2654	0.1167	0.0682	0.0459	0.0335
1.0	1.2071	0.2901	0.1289	0.0756	0.0509	0.0372
1.5	1.2807	0.4008	0.1877	0.1117	0.0756	0.0554
2.0	1.3377	0.4923	0.2422	0.1404	0.0998	0.0732
3.0	1.4192	0.6309	0.3384	0.2114	0.1463	0.1084
4.0	1.4698	0.7278	0.4184	0.2699	0.1898	0.1420
5.0	1.5029	0.7973	0.4842	0.3220	0.2301	0.1735
6.0	1.5253	0.8484	0.5382	0.3679	0.2672	0.2038
7.0	1.5409	0.8869	0.5825	0.4080	0.3010	0.2317
8.0	1.5523	0.9225	0.6189	0.4430	0.3316	0.2579
9.0	1.5611	0.9393	0.6491	0.4735	0.3593	0.2826
10	1.5677	0.9575	0.6784	0.5000	0.3843	0.3042
15	1.5853	1.0091	0.7519	0.5901	0.4760	0.3913
20	1.5918	1.0309	0.7889	0.6382	0.5303	0.4461
30	1.5964	1.0488	0.8195	0.6827	0.5853	0.5062
40	1.5988	1.0550	0.8335	0.7018	0.6133	0.5390
50	1.5995	1.0587	0.8396	0.7112	0.6227	0.5544
60	1.6009	1.0589	0.8428	0.7165	0.6301	0.5642
80	1.6012	1.0599	0.8463	0.7212	0.6398	0.5770
100	1.6014	1.0631	0.8505	0.7245	0.6415	0.5850
$\infty$	1.6021	1.0648	0.8558	0.7296	0.6485	0.5896

TABLE 6-46

Roots of characteristic equations  $\tan \mu_l = -\frac{1}{Bi_l - 1} \mu_l$  ( $l = q; m$ )

$Bi_l$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_5$	$\mu_6$
0.0	0.0000	4.4934	7.7253	10.9041	14.0662	17.2208
0.005	0.1221	4.4945	7.7259	10.9046	14.0666	17.2210
0.01	0.1730	4.4956	7.7265	10.9050	14.0669	17.2213
0.02	0.2445	4.4979	7.7278	10.9060	14.0676	17.2219
0.03	0.2991	4.5001	7.7291	10.9069	14.0683	17.2225
0.04	0.3450	4.5023	7.7304	10.9078	14.0690	17.2231
0.05	0.3854	4.5045	7.7317	10.9087	14.0697	17.2237
0.06	0.4217	4.5068	7.7330	10.9096	14.0705	17.2242
0.07	0.4551	4.5090	7.7343	10.9105	14.0712	17.2248
0.08	0.4860	4.5112	7.7356	10.9115	14.0719	17.2254
0.09	0.5150	4.5134	7.7369	10.9124	14.0726	17.2260
0.10	0.5423	4.5157	7.7382	10.9133	14.0733	17.2266
0.15	0.6609	4.5268	7.7447	10.9179	14.0769	17.2295
0.20	0.7593	4.5379	7.7511	10.9225	14.0804	17.2324
0.30	0.9208	4.5601	7.7641	10.9316	14.0875	17.2382
0.40	1.0528	4.5822	7.7770	10.9408	14.0946	17.2440
0.50	1.1656	4.6042	7.7899	10.9499	14.1017	17.2498
0.60	1.2644	4.6261	7.8028	10.9591	14.1088	17.2556
0.70	1.3525	4.6479	7.8156	10.9682	14.1159	17.2614
0.80	1.4320	4.6696	7.8284	10.9774	14.1230	17.2672
0.90	1.5044	4.6911	7.8412	10.9865	14.1301	17.2730
1.0	1.5708	4.7124	7.8540	10.9956	14.1372	17.2788
1.1	1.6320	4.7335	7.8667	11.0047	14.1443	17.2845
1.2	1.6887	4.7544	7.8794	11.0137	14.1513	17.2903
1.3	1.7414	4.7751	7.8920	11.0228	14.1584	17.2961
1.4	1.7906	4.7956	7.9046	11.0318	14.1654	17.3019
1.5	1.8366	4.8158	7.9171	11.0409	14.1724	17.3076
1.6	1.8738	4.8358	7.9295	11.0498	14.1795	17.3134
1.7	1.9203	4.8556	7.9419	11.0588	14.1865	17.3192
1.8	1.9586	4.8751	7.9542	11.0677	14.1935	17.3249
1.9	1.9947	4.8943	7.9665	11.0767	14.2005	17.3306
2.0	2.0288	4.9132	7.9787	11.0856	14.2075	17.3364
2.5	2.1746	5.0037	8.0385	11.1296	14.2421	17.3649
3.0	2.2889	5.0870	8.0962	11.1727	14.2764	17.3932
4.0	2.4557	5.2329	8.2045	11.2560	14.3434	17.4490
5.0	2.5704	5.3540	8.3029	11.3349	14.4080	17.5034
6.0	2.6537	5.4544	8.3914	11.4086	14.4699	17.5562
7.0	2.7165	5.5378	8.4703	11.4773	14.5288	17.6072
8.0	2.7654	5.6078	8.5406	11.5408	14.5847	17.6567
9.0	2.8044	5.6669	8.6031	11.5994	14.6374	17.7032
10.0	2.8363	5.7172	8.6587	11.6532	14.6870	17.7481
11.0	2.8628	5.7606	8.7083	11.7027	14.7335	17.7908
16.0	2.9476	5.9080	9.8898	11.8959	14.9251	17.9742
21.0	2.9930	5.9921	9.0019	12.0250	15.0625	18.1136
31.0	3.0406	6.0831	9.1294	12.1807	15.2380	18.3018
41.0	3.0651	6.1311	9.1987	12.2688	15.3417	18.4180
51.0	3.0801	6.1606	9.2420	12.3247	15.4090	18.4953
61.0	3.0901	6.1805	9.2715	12.3632	15.4559	18.5497
81.0	3.1028	6.2058	9.3089	12.4124	15.5164	18.6209
101.0	3.1105	6.2211	9.3317	12.4426	15.5537	18.6650
$\infty$	3.1416	6.2832	9.4248	12.5664	15.7080	18.8496

TABLE 6-47  
Values of coefficients  $A_i$  for a sphere

$Bi$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$
0.000	1.0000	-0.0000	0.0000	-0.0000	0.0000	-0.0000
0.005	1.0025	-0.0023	0.0013	-0.0009	0.0007	-0.0006
0.01	1.0035	-0.0046	0.0026	-0.0018	0.0014	-0.0012
0.02	1.0055	-0.0091	0.0052	-0.0037	0.0029	-0.0023
0.03	1.0089	-0.0137	0.0078	-0.0055	0.0043	-0.0035
0.04	1.0121	-0.0182	0.0104	-0.0074	0.0057	-0.0047
0.05	1.0147	-0.0227	0.0130	-0.0092	0.0071	-0.0058
0.06	1.0181	-0.0273	0.0156	-0.0110	0.0085	-0.0070
0.07	1.0206	-0.0318	0.0183	-0.0129	0.0100	-0.0081
0.08	1.0239	-0.0363	0.0209	-0.0147	0.0114	-0.0093
0.09	1.0266	-0.0409	0.0235	-0.0166	0.0128	-0.0105
0.10	1.0297	-0.0454	0.0260	-0.0184	0.0142	-0.0116
0.15	1.0443	-0.0679	0.0390	-0.0276	0.0214	-0.0174
0.20	1.0592	-0.0894	0.0520	-0.0368	0.0285	-0.0232
0.30	1.0880	-0.1345	0.0779	-0.0551	0.0427	-0.0349
0.40	1.1164	-0.1781	0.1036	-0.0734	0.0569	-0.0465
0.50	1.1440	-0.2216	0.1292	-0.0916	0.0710	-0.0580
0.60	1.1713	-0.2633	0.1546	-0.1098	0.0852	-0.0696
0.70	1.1978	-0.3048	0.1799	-0.1270	0.0998	-0.0812
0.80	1.2237	-0.3455	0.2050	-0.1460	0.1134	-0.0927
0.90	1.2488	-0.3854	0.2299	-0.1640	0.1275	-0.1042
1.0	1.2732	-0.4244	0.2546	-0.1819	0.1415	-0.1157
1.1	1.2970	-0.4626	0.2792	-0.1997	0.1555	-0.1272
1.2	1.3200	-0.4999	0.3035	-0.2175	0.1694	-0.1387
1.3	1.3424	-0.5364	0.3276	-0.2352	0.1823	-0.1501
1.4	1.3640	-0.5720	0.3515	-0.2528	0.1972	-0.1616
1.5	1.3843	-0.6067	0.3752	-0.2703	0.2110	-0.1730
1.6	1.4051	-0.6405	0.3986	-0.2878	0.2243	-0.1843
1.7	1.4247	-0.6735	0.4218	-0.3051	0.2385	-0.1957
1.8	1.4436	-0.7063	0.4447	-0.3228	0.2522	-0.2078
1.9	1.4618	-0.7368	0.4674	-0.3395	0.2659	-0.2183
2.0	1.4793	-0.7673	0.4899	-0.3565	0.2795	-0.2296
2.5	1.5579	-0.9073	0.5980	-0.4365	0.3449	-0.2855
3.0	1.6223	-1.0288	0.6993	-0.5205	0.4122	-0.3405
4.0	1.7201	-1.2253	0.8811	-0.6719	0.5384	-0.4476
5.0	1.7870	-1.3733	1.0363	-0.8095	0.6570	-0.5501
6.0	1.8338	-1.4860	1.1673	-0.9333	0.7702	-0.6428
7.0	1.8673	-1.5731	1.2776	-1.0437	0.8695	-0.7398
8.0	1.8920	-1.6403	1.3703	-1.1415	0.9633	-0.8264
9.0	1.9106	-1.6949	1.4482	-1.2280	1.0489	-0.9023
10	1.9249	-1.7381	1.5141	-1.3042	1.1269	-0.9077
11	1.9364	-1.7732	1.5698	-1.3713	1.1977	-1.0527
16	1.9663	-1.8766	1.7489	-1.6058	1.4633	-1.3305
21	1.9801	-1.9235	1.8385	-1.7360	1.6256	-1.5149
31	1.9905	-1.9626	1.9186	-1.8616	1.7950	-1.7225
41	1.9948	-1.9780	1.9515	-1.9161	1.8732	-1.8263
51	1.9964	-1.9856	1.9680	-1.9441	1.9145	-1.8802
61	1.9974	-1.9901	1.9773	-1.9601	1.9387	-1.9135
81	1.9985	-1.9942	1.9869	-1.9769	1.9644	-1.9492
101	1.9993	-1.9962	1.9915	-1.9850	1.9767	-1.9667
$\infty$	2.0000	-2.0000	2.0000	-2.0000	2.0000	-2.0000

For an infinite cylinder,

$$\begin{aligned}\Phi_1(\mu_n X) &= J_0(\mu_n X); \quad \Phi_1(\mu_m X) = J_0(\mu_m X); \quad \Phi_1(\sqrt{Lu} \mu_m X) = J_0(\sqrt{Lu} \mu_m X); \\ \varphi_n &= Ko \left[ \frac{Bi_m}{Bi_q} \left( 1 - \epsilon - \frac{\epsilon}{Lu-1} \right) \sqrt{Lu} \mu_n J_1 \left( \frac{\mu_n}{\sqrt{Lu}} \right) + \frac{\epsilon Lu Bi_m}{Lu-1} J_0 \left( \frac{\mu_n}{\sqrt{Lu}} \right) \right]; \\ \varphi_m &= Lu \left[ \left( 1 - \epsilon - \frac{\epsilon}{Lu-1} \right) \mu_m J_1(\mu_m) + \frac{\epsilon Bi_q}{Lu-1} J_0(\mu_m) \right]; \\ \psi_n &= Bi_m J_0 \left( \frac{\mu_n}{\sqrt{Lu}} \right) - \frac{\mu_n}{\sqrt{Lu}} J_1 \left( \frac{\mu_n}{\sqrt{Lu}} \right); \quad \psi_m = Bi_q J_0(\sqrt{Lu} \mu_m) - \sqrt{Lu} \mu_m J_1(\sqrt{Lu} \mu_m);\end{aligned}$$

where  $\mu_n$  and  $\mu_m$  are the roots of the characteristic equation

$$\frac{J_0(\mu_l)}{J_1(\mu_l)} = \frac{1}{Bi_l} \mu_l, \quad (6-5-6)$$

and coefficients  $A_n$  and  $A_m$  are given by the expressions

$$A_l = \frac{2 J_1(\mu_l)}{\mu_l [J_0^2(\mu_l) + J_1^2(\mu_l)]} = \frac{2 Bi_l}{J_0(\mu_l) [\mu_l^2 + Bi_l^2]}, \quad (l = n, m) \quad (6-5-7)$$

The roots of characteristic equation (6-5-6) are given in Table 6-44 and the values of coefficients  $A_l$  are given in Table 6-45.

For a sphere,

$$\begin{aligned}\Phi_2(\mu_n X) &= \frac{\sin \mu_n X}{\mu_n X}; \quad \Phi_2(\mu_m X) = \frac{\sin \mu_m X}{\mu_m X}; \quad \Phi_2(\sqrt{Lu} \mu_m X) = \frac{\sin \sqrt{Lu} \mu_m X}{\mu_m X}; \\ \varphi_n &= Ko Lu \frac{Bi_m}{Bi_q} \left\{ \left[ 1 - \epsilon + \frac{\epsilon}{Lu-1} (Bi_q - 1) \right] \sin \frac{\mu_n}{\sqrt{Lu}} - \left[ 1 - \epsilon - \frac{\epsilon}{Lu-1} \right] \times \right. \\ &\quad \left. \times \frac{\mu_n}{\sqrt{Lu}} \cos \frac{\mu_n}{\sqrt{Lu}} \right\}; \\ \varphi_m &= Lu \left\{ \left[ 1 - \epsilon + \frac{\epsilon}{Lu-1} (Bi_q - 1) \right] \sin \mu_m - \left( 1 - \epsilon - \frac{\epsilon}{Lu-1} \right) \mu_m \cos \mu_m \right\}; \\ \psi_n &= (Bi_m - 1) \sin \frac{\mu_n}{\sqrt{Lu}} + \frac{\mu_n}{\sqrt{Lu}} \cos \frac{\mu_n}{\sqrt{Lu}}; \\ \psi_m &= (Bi_q - 1) \sin \sqrt{Lu} \mu_m + \sqrt{Lu} \mu_m \cos \sqrt{Lu} \mu_m;\end{aligned}$$

where  $\mu_n$  and  $\mu_m$  are the roots of the characteristic equation

$$\tan \mu_l = - \frac{\mu_l}{Bi_l - 1}; \quad (6-5-8)$$

and coefficients  $A_n$  and  $A_m$  are defined by the relations

$$A_l = 2 \frac{\sin \mu_l - \mu_l \cos \mu_l}{\mu_l - \sin \mu_l \cos \mu_l} = (-1)^{l+1} \frac{2 Bi_l \sqrt{\mu_l^2 + (Bi_l - 1)^2}}{(\mu_l^2 + Bi_l^2 - Bi_l)}. \quad (6-5-9)$$

$(l = n, m)$

Values of the roots of characteristic equation (6-5-8) are given in Table 6-46 and values of coefficients  $A_l$  are given in Table 6-47.

#### b) No thermal mass transfer or phase transitions in the material

The generalized solution for this case has the form of (6-5-2) and (6-5-3), but instead of  $\varphi_n$  and  $\varphi_m$  we must substitute  $\varphi_n^*$  and  $\varphi_m^*$ , functions which are expressed as follows:

for an infinite plate

$$\varphi_n^* = Ko \sqrt{Lu} \frac{Bi_m}{Bi_q} \mu_n \sin \frac{\mu_n}{\sqrt{Lu}} \quad \text{and} \quad \varphi_m^* = Ko Lu \mu_m \sin \mu_m;$$

for an infinite cylinder

$$\varphi_n^* = Ko \sqrt{Lu} \frac{Bl_m}{Bl_q} \mu_n J_1 \left( \frac{\mu_n}{\sqrt{Lu}} \right) \quad \text{and} \quad \varphi_m^* = Ko Lu \mu_m J_1(\mu_m);$$

for a sphere

$$\varphi_n^* = Ko Lu \frac{Bl_m}{Bl_q} \left( \sin \frac{\mu_n}{\sqrt{Lu}} - \frac{\mu_n}{\sqrt{Lu}} \cos \frac{\mu_n}{\sqrt{Lu}} \right) \quad \text{and} \quad \varphi_m^* = Ko Lu Bl_m \sin \mu_m.$$

The characteristic-equation roots and coefficients are defined as previously.

In the absence of mass exchange ( $Ko = 0$ ), the above solutions become the ordinary classical solutions for the temperature distribution in a plate, cylinder, or sphere:

$$T(X, Fo) = 1 - \sum_{n=1}^{\infty} A_n \Phi_n(\mu_n X) \exp(-\mu_n^2 Fo). \quad (6-5-10)$$

For approximate calculations it is possible to use the following formula to calculate the first root of the corresponding characteristic equation

$$\mu_1^2 = \mu_{1\infty}^2 \frac{1}{1 + \frac{k}{Bl_q} b}, \quad (6-5-11)$$

where  $\mu_{1\infty}$  is the value of  $\mu_1$  for  $Bl_q = \infty$ . This quantity is a constant; for a plate  $\mu_{1\infty} = \pi/2$ , for a cylinder  $\mu_{1\infty} = 2.405$ , and for a sphere  $\mu_{1\infty} = \pi$ . Constants  $k$  and  $b$  are respectively: for a plate 2.24 and 1.02, for a cylinder 2.45 and 1.04, and for a sphere 2.70 and 1.07.

These considerations related to equation (6-5-10) also apply to the analogous solution for mass conduction, provided we set  $LuFo = Fo_m$ . Equation (6-5-3) then becomes identical to equation (6-5-10).

In order to facilitate the determination of the dimensionless transfer potentials using equation (6-5-10) or the analogous expression for mass transfer, several authors (Budrin, Bakhman, Heisler, Labuntsov, and others) have proposed nomograms. The widest range of Biot numbers (from 0.01 to  $\infty$ ) and Fourier numbers (from 0 to 600) is that given by Heisler [23], whose curves are reproduced in Figure 6-43. Figure 6-43, a gives the temperature at the center as a function of the Fourier number and the reciprocal of the Biot number. Figure 6-43, b provides an auxiliary nomogram in which the dimensionless temperature at a given point of the body is represented as a function of  $1/Bi$  for various values of the dimensionless coordinate.

To determine the temperature at any point of the body the value  $(t_{ce} - t_o)/(t_o - t_e)$  read off on the nomogram of Figure 6-43, a must be multiplied by the value of  $(t_x - t_o)/(t_{ce} - t_o)$  for the corresponding values of  $1/Bi$  and  $X = \frac{x}{R}$  on the nomogram of Figure 6-43, b [as previously,  $t_o$  is the temperature of the surroundings].

It is evident from the figure that the temperature distribution for a plate or a cylinder is approximately uniform for  $1/Bi > 10$  (for  $Bl < 0.1$ ). Therefore, for  $Bl < 0.1$  it is possible to use the formula

$$\frac{t - t_o}{t_o - t_e} = \exp(-Bi Fo).$$

A study of Figure 6-43, a and b, indicates that for low values of  $1/Bi$  and  $Fo$  the nomograms are unsuitable. Therefore, Heisler constructed additional nomograms for values of  $1/Bi$  from 0.05 to  $\infty$  and for low values of  $Fo$  (from 0 to 0.2). The complete nomograms for the temperatures at the center and surface of an infinite plate or cylinder are given in Figure 6-43, c. These graphs are useful for reference calculations of heat exchange in the case of unsteady heat conduction. Once the temperature values for an infinite plate or cylinder are given, it is easy to obtain the values for a parallelepiped or a finite cylinder.

Solution (6-5-10) is very widely applied in the various fields of science and industry, and so it will be convenient to analyze it in more detail. If the Biot number is small ( $Bi_q < 0.1$ ), then all the other terms of the series are negligible in comparison with the first. The first characteristic-equation root may in this case be expressed approximately as

$$\mu_1^2 = (\Gamma + 1) Bl_q.$$

Consequently, solution (6-5-10) has the form

$$T(X, Fo) = 1 - \Phi_r(\sqrt{(\Gamma + 1) Bl_q} X) \exp[-(\Gamma + 1) Bl_q Fo],$$

where

$$\Phi_o = \cos \sqrt{Bl_q} X; \quad \Phi_1 = J_0(\sqrt{2Bl_q} X); \quad \Phi_2 = \frac{\sin \sqrt{3Bl_q} X}{\sqrt{3Bl_q} X}.$$



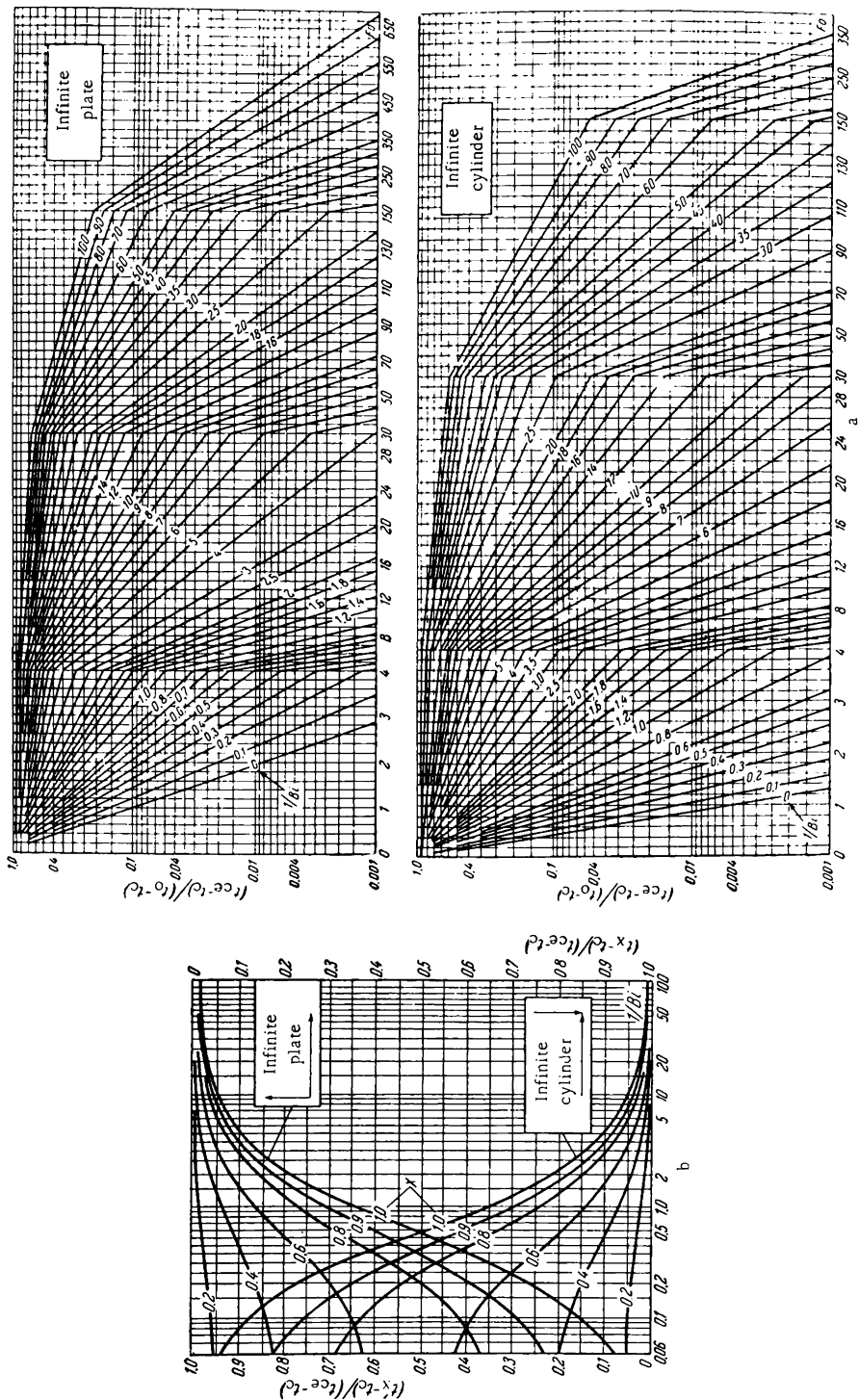


FIGURE 6-43. Heisler's nomograms  
a) Variation of temperature at center of infinite plate or cylinder; b) graph for determining temperature at any point on infinite plate or cylinder

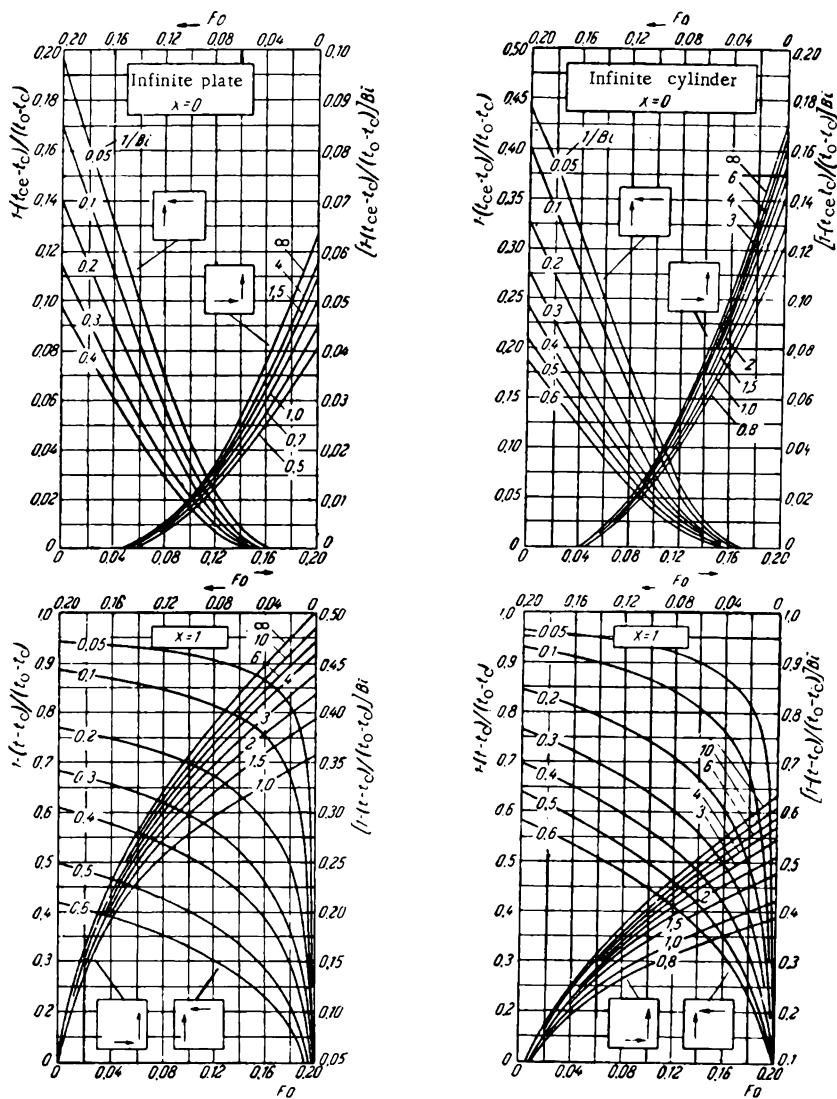


FIGURE 6-43. Heisler's nomograms (continued)

c) graphs for an infinite plate or cylinder, with low values of the Fourier number

In this case the rate of heating of the material

$$\frac{\partial T}{\partial Fo} = Bi_q \Phi_r (\sqrt{(\Gamma + 1) Bi_q X}) \exp [-(\Gamma + 1) Bi_q Fo]$$

is directly proportional to the Biot heat-exchange number  $Bi_q = \frac{\alpha_q R}{\lambda_q}$ , that is, it is directly proportional to the heat-exchange coefficient and inversely proportional to the characteristic dimension of the body. Consequently, the rate of heating is determined by the rate of heat transfer from the surroundings to the surface of the body (external problem).

If the Biot number approaches infinity, then we obtain solutions identical to those considered in Chapter IV, and the boundary conditions of the third kind are transformed into boundary conditions of the first kind.

Solution (6-5-10) is suitable when the Fourier number is high enough that, with sufficient accuracy, only one or two series terms need be retained. For low Fo, when several terms must be kept, certain difficulties are introduced into the calculations. In this case it is more convenient to use a different form of the solution. The corresponding transformations, which were carried out in Chapter IV, lead to the following solutions in terms of inverse transforms:

for an infinite plate

$$T(X, Fo) \approx \text{erfc} \frac{1-X}{2\sqrt{Fo}} - \exp[Bi_q(1-X) + (Bi_q \sqrt{Fo})^2] \text{erfc} \left( \frac{1-X}{2\sqrt{Fo}} + Bi_q \sqrt{Fo} \right) + \\ + \text{erfc} \frac{1+X}{2\sqrt{Fo}} - \exp[Bi_q(1+X) + (Bi_q \sqrt{Fo})^2] \text{erfc} \left( \frac{1+X}{2\sqrt{Fo}} + Bi_q \sqrt{Fo} \right); \quad (6-5-12)$$

for an infinite cylinder

$$T(X, Fo) = 2Bi_q \frac{Fo}{\sqrt{X}} i \text{erfc} \frac{1-X}{2\sqrt{Fo}} + 4Bi_q \frac{Fo}{\sqrt{X}} \left( \frac{1}{8X} + \frac{3}{8} - Bi_q \right) i^2 \text{erfc} \frac{1-X}{2\sqrt{Fo}} + \dots \quad (6-5-13)$$

(for  $X \neq 0$ ;

$$T(0, Fo) = 1 - 4Bi_q Fo \exp \left( -\frac{1}{4Fo} \right) \quad (6-5-14)$$

(for  $X = 0$ ).

Solutions (6-5-13) and (6-5-14) are valid only for low values of  $Bi_q$ . For high  $Bi_q$  it is necessary to use the solution

$$T(1, Fo) \approx \frac{Bi_q}{Bi_q - 1/2} \left\{ 1 - \exp \left[ Fo \left( Bi_q - \frac{1}{2} \right)^{1/2} \right] \text{erfc} \left( Bi_q - \frac{1}{2} \right) \sqrt{Fo} \right\};$$

for a sphere

$$T(X, Fo) \approx (\pm) \frac{Bi_q}{X(Bi_q - 1)} \left\{ \text{erfc} \frac{1 \mp X}{2\sqrt{Fo}} - \exp[(Bi_q - 1)^2 Fo + \right. \\ \left. + (Bi_q - 1)(1 \mp X)] \text{erfc} \left[ \frac{1 \mp X}{2\sqrt{Fo}} + (Bi_q - 1) \sqrt{Fo} \right] \right\}; \quad (6-5-15)$$

here we introduced the notation

$$[(\mp) A(\mp Z)] = +A(-Z) - A(+Z);$$

for  $X = 0$  instead of (6-5-15) it is necessary to use the relation

$$T(0, Fo) \approx 2Bi_q \exp[(Bi_q - 1)^2 Fo + (Bi_q - 1) \text{erfc} \left( \frac{1}{2\sqrt{Fo}} + (Bi_q - 1) \sqrt{Fo} \right)]. \quad (6-5-16)$$

Pöschl /24/ calculated the temperature  $T(1, Fo)$  at the surfaces of the given bodies, for low values of Fo (from 0.0003 to 0.1) and for various values of  $Bi_q$  (from 0.1 to 2000). Equation (6-5-10) was used for the calculations and, as noted by Pöschl, for Fo = 0.0003 it was necessary to use 36 series terms. A similar situation was noted by Stasikov /25/, who, in order to determine accurately the third significant figure for  $Bi_q = 2$  and Fo = 0.0001, had to use the first 40 terms of the series. As  $Bi_q$  increases, the number of summed terms required increases sharply. A comparison of the calculations according to the approximate formulas (6-5-12) through (6-5-16) with the calculations according to the classical formula (6-5-10) shows quite satisfactory

agreement even for the low values of Fo considered by Föschl and Stasikov. This indicates the great advantage of the operational method over the ordinary method, since the operational method leads to several approximate solutions for different values of Fo and Bi<sub>q</sub>.

TABLE 6-48

Values of coefficients  $D_n = \frac{2\text{Bi}_q^2}{\mu_n^2(\mu_n^2 + \text{Bi}_q^2 + \text{Bi}_q)}$

Bi <sub>q</sub>	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	D <sub>5</sub>	D <sub>6</sub>
∞	0.8106	0.0901	0.0324	0.0165	0.0100	0.0067
50.0	0.8260	0.0899	0.0323	0.0161	0.0095	0.0061
30.0	0.8354	0.0893	0.0315	0.0152	0.0086	0.0053
15.0	0.8565	0.0885	0.0279	0.0120	0.0060	0.0033
10.0	0.8743	0.0839	0.0236	0.0090	0.0040	0.0020
9.0	0.8796	0.0821	0.0222	0.0081	0.0035	0.0017
8.0	0.8859	0.0797	0.0205	0.0072	0.0030	0.0015
7.0	0.8932	0.0766	0.0185	0.0062	0.0025	0.0012
6.0	0.9021	0.0723	0.0162	0.0051	0.0020	0.0009
5.0	0.9130	0.0664	0.0135	0.0040	0.0015	0.0007
4.0	0.9264	0.0582	0.0104	0.0029	0.0010	0.0005
3.0	0.9430	0.0468	0.0070	0.0019	0.0006	0.0003
2.0	0.9635	0.0313	0.0037	0.0009	0.0003	0.0001
1.5	0.9749	0.0220	0.0023	0.0005	0.0002	0.0001
1.0	0.9862	0.0124	0.0011	0.0002	0.0001	
0.9	0.9882	0.0105	0.0009	0.0002	0.0001	
0.8	0.9903	0.0088	0.0007	0.0001		
0.7	0.9920	0.0070	0.0006	0.0001		
0.6	0.9939	0.0054	0.0004	0.0001		
0.5	0.9955	0.0040	0.0003			
0.4	0.9973	0.0027	0.0002			
0.3	0.9982	0.0016	0.0001			
0.2	0.9995	0.0007				
0.1	1.0000	0.0002				

Solution (6-5-10) corresponds to a constant initial distribution of the dimensionless temperature. In the general case, when  $T(X, 0) = F(X)$ , the solution becomes more complicated. For example, if  $F(X)$  is an even function the solutions have the following form:

$$T(X, \text{Fo}) = \frac{t - t_s}{t_s} = T_o - \sum_{n=1}^{\infty} B_n \Psi_n \Phi_n(\mu_n X) \exp(-\mu_n^2 \text{Fo}), \quad (6-5-17)$$

where  $T_o = \frac{t_o - t_s}{t_s}$ ;  $\Phi_n$  has the same forms as for (6-5-10); and  $B_n$  and  $\Psi_n$  are defined as follows:

$$\Gamma = 0 \quad B_n = \frac{2\mu_n}{\mu_n + \sin \mu_n \cos \mu_n}; \quad \Psi_n = \int_0^1 F_1(X) \cos \mu_n X dX; \quad (6-5-18)$$

for a cylinder

$$\Gamma = 1 \quad B_n = \frac{2}{J_0^2(\mu_n) + J_1^2(\mu_n)}; \quad \Psi_n = \int_0^1 X F_1(X) J_0(\mu_n X) dX; \quad (6-5-19)$$

for a sphere

$$\Gamma = 2 \quad B_n = \frac{2\mu_n^2}{\mu_n - \sin \mu_n \cos \mu_n}; \quad \Psi_n = \int_0^1 X F_1(X) \sin \mu_n X dX; \quad (6-5-20)$$

with

$$F_1(X) = T_o - F(X).$$

TABLE 6-49

$$\text{Values of coefficients } D_n = \frac{4\text{Bi}_q^2}{\mu_n^2(\mu_n^2 + \text{Bi}_q^2)}$$

$\text{Bi}_q$	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$
$\infty$	0.6917	0.1313	0.0534	0.0288	0.0179	0.0122
50.0	0.7170	0.1309	0.0530	0.0284	0.0172	0.0113
30.0	0.7359	0.1289	0.0529	0.0268	0.0155	0.0097
10.0	0.8041	0.1260	0.0387	0.0152	0.0070	0.0035
9.0	0.8133	0.1229	0.0361	0.0137	0.0061	0.0030
8.0	0.8242	0.1187	0.0331	0.0120	0.0052	0.0025
7.0	0.8375	0.1132	0.0296	0.0103	0.0043	0.0021
6.0	0.8532	0.1057	0.0254	0.0084	0.0034	0.0016
5.0	0.8721	0.0953	0.0207	0.0064	0.0025	0.0012
4.0	0.8984	0.0813	0.0156	0.0045	0.0017	0.0008
3.0	0.9225	0.0625	0.0103	0.0028	0.0010	0.0005
2.0	0.9535	0.0388	0.0053	0.0013	0.0005	0.0002
1.5	0.9694	0.0240	0.0032	0.0008	0.0003	0.0001
1.0	0.9843	0.0136	0.0015	0.0003	0.0001	0.0001
0.90	0.9868	0.0114	0.0012	0.0003	0.0001	
0.80	0.9893	0.0093	0.0010	0.0002	0.0001	
0.70	0.9916	0.0074	0.0007	0.0002	0.0001	
0.60	0.9936	0.0056	0.0006	0.0001		
0.50	0.9955	0.0040	0.0004			
0.40	0.9970	0.0026	0.0003			
0.30	0.9983	0.0015	0.0001			
0.20	0.9992	0.0007				
0.15	0.9995	0.0004				
0.10	0.9998	0.0002				
0.08	0.9999	0.0001				

TABLE 6-50

$$\text{Values of coefficients } D_n = \frac{6\text{Bi}_q^2}{\mu_n^2(\mu_n^2 + \text{Bi}_q^2 - \text{Bi}_q)}$$

$\text{Bi}_q$	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$
$\infty$	0.6079	0.1520	0.0675	0.0380	0.0243	0.0169
51.0	0.6427	0.1518	0.0671	0.0380	0.0236	0.0158
21.0	0.6886	0.1510	0.0652	0.0363	0.0180	0.0108
10.0	0.7667	0.1496	0.0485	0.0196	0.0091	0.0047
9.0	0.7737	0.1453	0.0455	0.0175	0.0079	0.0040
8.0	0.7889	0.1396	0.0408	0.0152	0.0067	0.0033
7.0	0.8068	0.1319	0.0360	0.0128	0.0055	0.0027
6.0	0.8280	0.1215	0.0305	0.0104	0.0044	0.0021
5.0	0.8533	0.1075	0.0245	0.0079	0.0032	0.0015
4.0	0.8829	0.0890	0.0180	0.0055	0.0021	0.0010
3.0	0.9171	0.0655	0.0115	0.0034	0.0013	0.0006
2.5	0.9353	0.0520	0.0085	0.0024	0.0009	0.0004
2.0	0.9534	0.0380	0.0057	0.0016	0.0006	0.0003
1.9	0.9569	0.0352	0.0052	0.0014	0.0005	0.0002
1.8	0.9605	0.0325	0.0047	0.0013	0.0005	0.0002
1.7	0.9641	0.0297	0.0043	0.0011	0.0004	0.0002
1.6	0.9678	0.0270	0.0038	0.0010	0.0004	0.0002
1.5	0.9707	0.0243	0.0034	0.0009	0.0003	0.0001
1.4	0.9739	0.0217	0.0030	0.0008	0.0003	0.0001
1.3	0.9770	0.0192	0.0026	0.0007	0.0003	0.0001
1.2	0.9800	0.0167	0.0022	0.0006	0.0002	0.0001
1.1	0.9828	0.0144	0.0019	0.0005	0.0002	0.0001
1.0	0.9855	0.0122	0.0016	0.0004	0.0001	0.0001
0.90	0.9881	0.0101	0.0013	0.0003	0.0001	
0.80	0.9905	0.0081	0.0010	0.0003	0.0001	
0.70	0.9926	0.0064	0.0008	0.0002	0.0001	
0.60	0.9944	0.0048	0.0005	0.0001		
0.50	0.9959	0.0034	0.0004	0.0001		
0.40	0.9974	0.0022	0.0003	0.0001		
0.30	0.9985	0.0013	0.0001			
0.20	0.9994	0.0006	0.0001			
0.15	0.9996	0.0003				
0.10	0.9997	0.0001				
0.09	1.0000	0.0001				
0.08	1.0000	0.0001				

If we integrate the obtained solutions according to the formula

$$\bar{\psi}(Fo) = (\Gamma + 1) \int_0^1 X^\Gamma \psi(X, Fo) dX,$$

then it is easy to find expressions for the dimensionless transfer potentials averaged over the coordinate. In particular, for equation (6-5-10) we obtain

$$\bar{T}(Fo) = 1 - \sum_{n=1}^{\infty} D_n \exp(-\mu_n^2 Fo),$$

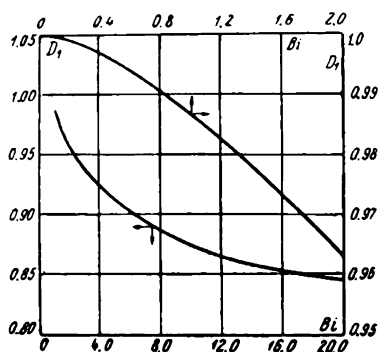


FIGURE 6-44. Coefficient  $D_1$  as a function of the Biot number, for an infinite plate

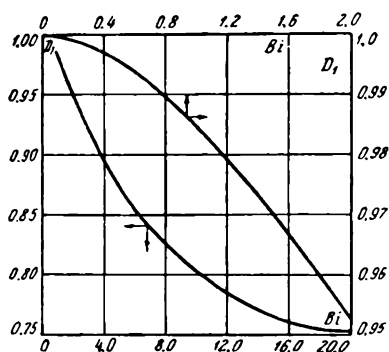


FIGURE 6-45. Coefficient  $D_1$  as a function of the Biot number, for an infinite cylinder

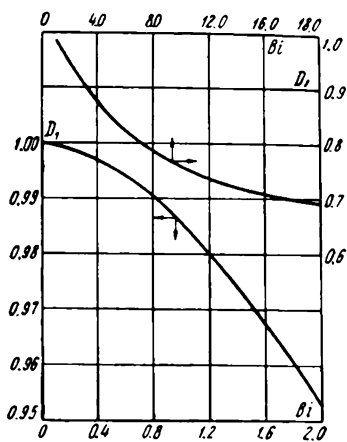


FIGURE 6-46. Coefficient  $D_1$  as a function of the Biot number, for a sphere

where the coefficients  $D_n$  for a plate, cylinder, and sphere are, respectively,

$$\Gamma = 0, \quad D_n = A_n \frac{\sin \mu_n}{\mu_n} = \frac{2Bi_q^2}{\mu_n^2 (\mu_n^2 + Bi_q^2 + Bi_q)}; \quad (6-5-5)'$$

$$\Gamma = 1, \quad D_n = 2A_n \frac{J_1(\mu_n)}{\mu_n} = \frac{4\text{Bi}_q^2}{\mu_n^2(\mu_n^2 + \text{Bi}_q^2)}; \quad (6-5-7)'$$

$$\Gamma = 2, \quad D_n = 3A_n \frac{\sin \mu_n - \mu_n \cos \mu_n}{\mu_n^2} = \frac{6\text{Bi}_q^2}{\mu_n^2(\mu_n^2 + \text{Bi}_q^2 - \text{Bi}_q)}; \quad (6-5-9)'$$

The first six coefficients  $D_n$  are listed in Tables (6-48) through (6-50), for various values of the Fourier number. During calculations it is usually only necessary to use the first coefficient; therefore, the graphs  $D_1 = f(\text{Bi}_q)$  have been plotted in Figures (6-44) through (6-46).

c) Heat and mass transfer with nonsymmetrical conditions of the third kind.  
No thermal mass transfer

Unsteady heat and mass transfer, for the case when the exchange of heat and matter with the surroundings (at temperatures  $t_{c1}$  and  $t_{c2}$ ) takes place at different rates at two surfaces (ends) (the coefficients of heat and mass exchange being different, namely  $\alpha_{q1}$ ,  $\alpha_{q2}$  and  $\alpha_{m1}$ ,  $\alpha_{m2}$ ), is represented by boundary conditions with the following form:

for  $X=0$

$$\frac{\partial T(0, \text{Fo})}{\partial X} - \text{Bi}_{q1}[T_{c1} - T(0, \text{Fo})] + (1 - \varepsilon) \text{Ko Lu Bi}_{m1}[\Theta_{p1} - \Theta(0, \text{Fo})] = 0; \quad (6-5-21)$$

$$\frac{\partial \Theta(0, \text{Fo})}{\partial X} - \text{Bi}_{m1}[\Theta_{p1} - \Theta(0, \text{Fo})] = 0; \quad (6-5-22)$$

for  $X=1$

$$\frac{\partial T(1, \text{Fo})}{\partial X} - \text{Bi}_{q2}[T_{c2} - T(1, \text{Fo})] - (1 - \varepsilon) \text{Ko Lu Bi}_{m2}[\Theta_{p2} - \Theta(1, \text{Fo})] = 0; \quad (6-5-23)$$

$$\frac{\partial \Theta(1, \text{Fo})}{\partial X} - \text{Bi}_{m2}[\Theta_{p2} - \Theta(1, \text{Fo})] = 0, \quad (6-5-24)$$

where

$$T = \frac{t - t_*}{t_* - t_*}; \quad \Theta = \frac{\theta_* - \theta}{\theta_* - \theta_*}; \quad X = \frac{x}{l}; \quad \text{Bi}_{qj} = \frac{\alpha_{qj} l}{\lambda_q}; \quad \text{Bi}_{mj} = \frac{\alpha_{mj} l}{\lambda_m};$$

$$T_{cj} = \frac{t_{cj} - t_*}{t_* - t_*}; \quad \Theta_{pj} = \frac{\theta_* - \theta_{pj}}{\theta_* - \theta_*} \quad (j = 1, 2).$$

The solution of system of equations (4-1-2) and (4-1-3) with boundary conditions (6-5-21) through (6-5-24) may be accomplished by using Laplace integral transformations. For an infinite plate (or an infinite bar with insulated lateral surfaces) the solution, for constant initial conditions  $T(X, 0) = 0$  and  $\Theta(X, 0) = 0$ , has the following form:

$$T(X, \text{Fo}) = \frac{t - t_*}{t_* - t_*} = \frac{T_{c2} \text{Bi}_{q2} - T_{c1} \text{Bi}_{q1} + (T_{c2} - T_{c1}) \text{Bi}_{q1} \text{Bi}_{q2} (1 - X)}{\text{Bi}_{q2} - \text{Bi}_{q1}} +$$

$$+ (1 - \varepsilon) \text{Ko Lu} \times$$

$$\times \frac{\text{Bi}_{m1} (\text{Bi}_{m2} \Theta_{p2} - \text{Bi}_{m1} \Theta_{p1} - \text{Bi}_{m1} \text{Bi}_{m2} \Theta_{p1}) + \text{Bi}_{m1} \text{Bi}_{m2} (\Theta_{p1} - \Theta_{p2}) [2 + (1 - X) (\text{Bi}_{q1} + \text{Bi}_{q2})]}{(\text{Bi}_{q2} - \text{Bi}_{q1}) (\text{Bi}_{m2} - \text{Bi}_{m1} - \text{Bi}_{m1} \text{Bi}_{m2})} +$$

$$+ \sum_{n=1}^{\infty} C_n [\sigma_{1n} \text{Bi}_{q1} \sin \mu_n X - \sigma_{2n} (1 + \text{Bi}_{q1}) \mu_n \cos \mu_n X] \exp(-\mu_n^2 \text{Fo}) +$$

$$+ \sum_{l=1}^{\infty} C_l \left[ \frac{e Z_l}{1 - \text{Lu}} (u_{1l} \cos \mu_l X - u_{2l} \text{Bi}_{m1} \sin \mu_l X) + \right.$$

$$\left. + (\sigma_{1l} \text{Bi}_{q1} \sin \sqrt{\text{Lu}} \mu_l X - \sigma_{2l} (1 + \text{Bi}_{q1}) \sqrt{\text{Lu}} \mu_l \cos \sqrt{\text{Lu}} \mu_l X) \exp(-\mu_l^2 \text{Lu Fo}) \right]$$

$$\Theta(X, \text{Fo}) = \frac{\theta_* - \theta}{\theta_* - \theta_*} = \frac{\text{Bi}_{m2} \Theta_{p2} - \text{Bi}_{m1} \Theta_{p1} - \text{Bi}_{m1} \text{Bi}_{m2} (\Theta_{p1} + (\Theta_{p2} - \Theta_{p1}) X)}{\text{Bi}_{m2} - \text{Bi}_{m1} - \text{Bi}_{m1} \text{Bi}_{m2}} +$$

$$+ \sum_{l=1}^{\infty} C_l \frac{Z_l}{\text{Lu Ko}} [u_{1l} \cos \mu_l X - u_{2l} \text{Bi}_{m1} \sin \mu_l X] \exp(-\mu_l^2 \text{Lu Fo}),$$

where

$$\begin{aligned}
C_n &= 2 \frac{\text{Lu Ko}}{\mu_n \psi_n Z_n}; \quad C_l = 2 \frac{\text{Lu Ko}}{\mu_l \psi_l Z_l}; \\
\psi_n &= (\text{Bi}_{q2} - \text{Bi}_{q1} - \text{Bi}_{q1} \text{Bi}_{q2} - \mu_n^2) \cos \mu_n - (\text{Bi}_{q1} + \text{Bi}_{q2} + \text{Bi}_{q1} \text{Bi}_{q2} + 2) \mu_n \sin \mu_n; \\
Z_n &= (\text{Bi}_{m2} - \text{Bi}_{m1}) \frac{\mu_n}{\sqrt{\text{Lu}}} \cos \frac{\mu_n}{\sqrt{\text{Lu}}} - \left( \text{Bi}_{m1} \text{Bi}_{m2} + \frac{\mu_n^2}{\text{Lu}} \right) \sin \frac{\mu_n}{\sqrt{\text{Lu}}}; \\
\psi_l &= (\text{Bi}_{m2} - \text{Bi}_{m1} - \text{Bi}_{m1} \text{Bi}_{m2} - \mu_l^2) \cos \mu_l - (\text{Bi}_{m1} - \text{Bi}_{m2} + 2) \mu_l \sin \mu_l; \\
Z_l &= (1 + \text{Bi}_{q1}) \sqrt{\text{Lu}} \mu_l x_l (\sqrt{\text{Lu}} \mu_l) - \text{Bi}_{q1} w_l (\sqrt{\text{Lu}} \mu_l); \\
\sigma_{1n} &= \left[ (1 - \varepsilon) \frac{\text{Bi}_{m1}}{\text{Bi}_{q1}} x_n (\mu_n) + \Gamma_{1n} \right] u_{1n} - \text{Bi}_{m1} u_{2n} \Gamma_{2n} - \left[ A_2 - \frac{A_1}{\text{Bi}_{q1}} x_n (\mu_n) \right] \frac{Z_n}{\text{Lu Ko}}; \\
\sigma_{2n} &= \left[ (1 - \varepsilon) \frac{\text{Bi}_{m1}}{1 + \text{Bi}_{q1}} \frac{w_n (\mu_n)}{\mu_n} + \Gamma_{1n} \right] u_{1n} - \text{Bi}_{m1} u_{2n} \Gamma_{2n} - \\
&\quad - \left[ A_2 - \frac{A_1}{1 + \text{Bi}_{q1}} \cdot \frac{w_n (\mu_n)}{\mu_n} \right] \frac{Z_n}{\text{Lu Ko}}; \\
\sigma_{1l} &= \left[ (1 - \varepsilon) \frac{\text{Bi}_{m1}}{\text{Bi}_{q1}} x_l (\sqrt{\text{Lu}} \mu_l) + \Gamma_{1l} \right] u_{1l} - \text{Bi}_{m1} u_{2l} \Gamma_{2l} + \frac{\varepsilon}{1 - \text{Lu}} \cdot \frac{Z_l}{\text{Bi}_{q1} \sqrt{\text{Lu}}} \text{Bi}_{m1} u_{2l}; \\
\sigma_{2l} &= \left[ (1 - \varepsilon) \frac{\text{Bi}_{m1}}{1 + \text{Bi}_{q1}} \cdot \frac{w_l (\sqrt{\text{Lu}} \mu_l)}{\sqrt{\text{Lu}} \mu_l} + \Gamma_{1l} \right] u_{1l} - \text{Bi}_{m1} u_{2l} \Gamma_{2l} + \\
&\quad + \frac{\varepsilon}{1 - \text{Lu}} \cdot \frac{Z_l}{(1 + \text{Bi}_{q1}) \sqrt{\text{Lu}} \mu_l} u_{2l}; \\
\Gamma_{1n} &= \frac{\varepsilon}{1 - \text{Lu}} \left[ x_n \left( \frac{\mu_n}{\sqrt{\text{Lu}}} \right) - x_n (\mu_n) \right] + (1 - \varepsilon) \text{Bi}_{m2} \cos \frac{\mu_n}{\sqrt{\text{Lu}}}; \\
\Gamma_{2n} &= \frac{\varepsilon}{1 - \text{Lu}} \left[ w_n \left( \frac{\mu_n}{\sqrt{\text{Lu}}} \right) - \frac{1}{\sqrt{\text{Lu}}} w_n (\mu_n) \right] + (1 - \varepsilon) \text{Bi}_{m2} \sin \frac{\mu_n}{\sqrt{\text{Lu}}}; \\
\Gamma_{1l} &= \frac{\varepsilon}{1 - \text{Lu}} [x_l (\mu_l) - x_l \sqrt{\text{Lu}} \mu_l] + (1 - \varepsilon) \text{Bi}_{m2} \cos \mu_l; \\
\Gamma_{2l} &= \frac{\varepsilon}{1 - \text{Lu}} \left[ w_l (\mu_l) - \frac{1}{\sqrt{\text{Lu}}} w_l (\sqrt{\text{Lu}} \mu_l) \right] + (1 - \varepsilon) \text{Bi}_{m2} \sin \mu_l; \\
u_{1n} &= \text{Bi}_{m2} \Theta_{p2} \frac{\mu_n}{\sqrt{\text{Lu}}} - \text{Bi}_{m1} \Theta_{p1} \bar{w}_n \left( \frac{\mu_n}{\sqrt{\text{Lu}}} \right); \\
u_{2n} &= \text{Bi}_{m2} \Theta_{p2} - \Theta_{p1} \bar{x}_n \left( \frac{\mu_n}{\sqrt{\text{Lu}}} \right); \\
u_{1l} &= \text{Bi}_{m2} \Theta_{p2} \mu_l - \text{Bi}_{m1} \Theta_{p1} \bar{w}_l (\mu_l); \\
u_{2l} &= \text{Bi}_{m2} \Theta_{p2} - \Theta_{p1} \bar{x}_l (\mu_l); \\
x_n (\xi_n) &= \text{Bi}_{q2} \cos \xi_n - \xi_n \sin \xi_n; \\
\bar{x}_n \left( \frac{\mu_n}{\sqrt{\text{Lu}}} \right) &= \text{Bi}_{m2} \cos \frac{\mu_n}{\sqrt{\text{Lu}}} - \frac{\mu_n}{\sqrt{\text{Lu}}} \sin \frac{\mu_n}{\sqrt{\text{Lu}}}; \\
x_l (\eta_l) &= \text{Bi}_{q2} \cos \eta_l - \eta_l \sin \eta_l; \\
\bar{x}_l (\mu_l) &= \text{Bi}_{m2} \cos \mu_l - \mu_l \sin \mu_l; \\
w_n (\xi_n) &= \text{Bi}_{q2} \sin \xi_n + \xi_n \cos \xi_n; \\
\bar{w}_n \left( \frac{\mu_n}{\sqrt{\text{Lu}}} \right) &= \text{Bi}_{m2} \sin \frac{\mu_n}{\sqrt{\text{Lu}}} + \frac{\mu_n}{\sqrt{\text{Lu}}} \cos \frac{\mu_n}{\sqrt{\text{Lu}}}; \\
w_l (\eta_l) &= \text{Bi}_{q2} \sin \eta_l + \eta_l \cos \eta_l; \\
\bar{w}_l (\eta_l) &= \text{Bi}_{m2} \sin \eta_l + \eta_l \cos \eta_l; \\
A_i &= T_{ci} \text{Bi}_{qi} + (-1)^i (1 - \varepsilon) \text{Ko Lu Bi}_{mi} \Theta_{pi} \quad (i = 1, 2).
\end{aligned}$$

Here



The quantity  $\xi_n$  has the form  $\mu_n/\sqrt{Lu}$  and  $\mu_n$  and  $\eta_l$  have the forms  $\mu_l$  and  $\sqrt{Lu}\mu_l$ .

Roots  $\mu_n$  and  $\mu_l$  are determined by solving the following characteristic equations:

$$\mu_n \cot \mu_n = \frac{Bi_{q1} Bi_{q2} + (1 + Bi_{q1}) \mu_n^2}{Bi_{q2} (1 + Bi_{q1}) - Bi_{q1}};$$

$$\mu_l \cot \mu_l = \frac{Bi_{m1} Bi_{m2} + \mu_l^2}{Bi_{m2} - Bi_{m1}}.$$

The above solution makes it possible to obtain several particular solutions. For example, for mass transfer which is not complicated by heat transfer we obtain the solution given by Alyavin /26/:

$$\theta = \frac{(\theta_{p2} - \theta_{p1})(1 + Bi_{m1}X)}{1 - \frac{Bi_{m1}}{Bi_{m2}} + Bi_{m1}} + \theta_{p1} +$$

$$+ 2 \sum_{l=1}^{\infty} \exp(-\mu_l^2 Lu Fo) \frac{\mu_l \cos \mu_l X + Bi_{m1} \sin \mu_l X}{(\mu_l^2 + Bi_{m1}^2) + (Bi_{m1} - Bi_{m2}) \frac{\mu_l^2 - Bi_{m1} Bi_{m2}}{\mu_l^2 - Bi_{m2}^2}} \Phi(\mu_l),$$

where

$$\Phi(\mu_l) = \frac{\theta_{p2} - \theta_{p1}}{1 - \frac{Bi_{m1}}{Bi_{m2}} + Bi_{m1}} \left( \frac{Bi_{m1}^2}{\mu_l} \cos \mu_l - \frac{Bi_{m1}^2}{\mu_l^2} \sin \mu_l - Bi_{m1} \sin \mu_l - \sin \mu_l \right) +$$

$$+ \theta_{p1} \left( \frac{Bi_{m1}}{\mu_l} \cos \mu_l - \frac{Bi_{m1}}{\mu_l} - \sin \mu_l \right);$$

and  $\mu_l$  are the roots of the characteristic equation

$$\tan \mu_l = \frac{\mu_l (Bi_{m1} - Bi_{m2})}{\mu_l^2 + Bi_{m1} Bi_{m2}}.$$

An analysis of the last solution shows that this distribution of the transfer potentials differs considerably from that for the symmetrical problem. If we assume that the mass exchange at the left side of the plate ( $X=0$ ) is less intensive than that at the right side ( $X=1$ ), then the total mass transfer through the plate during the first stage of the process is determined both by the mass exchange at the left and by that at the right. It is evident that the minima on the curves describing the dimensionless mass-transfer potential will, as time goes by, move more and more from the right to the left. In other words, the region of influence of the right side will continually increase, while that of the left side decreases accordingly. The moment when the minimum on the curve reaches the left boundary plane is the moment when the first stage of the process terminates. After this, a process determined by mass exchange from the right is observed. Up to this moment the shape of the curves has approached more and more the straight line which is characteristic of the steady state.

If the heat transfer is not complicated by mass transfer and if the phase transitions with various rates take place only at the surfaces of the plate, then boundary conditions (6-5-21) and (6-5-23) may be rewritten as

$$-\frac{\partial T(1, Fo)}{\partial X} + Bi_q [T_c - T(1, Fo)] - Ko Lu Kl_{m1} = 0;$$

$$\frac{\partial T(-1, Fo)}{\partial X} + Bi_q [T_c - T(-1, Fo)] - Ko Lu Kl_{m2} = 0.$$

The solution of the heat-transfer equation with these boundary conditions has the form

$$T = \frac{t - t_0}{t_0} = T_c - \frac{Ko Lu}{2 Bi_q} \left\{ Kl_{m1} + Kl_{m2} + \frac{Bi_q}{1 + Bi_q} (Kl_{m1} - Kl_{m2}) X + \right.$$

$$+ \left[ \frac{2 Bi_q T_c}{Ko Lu} - (Kl_{m1} + Kl_{m2}) \right] \sum_{n=1}^{\infty} A_n \cos \mu_n X \exp(-\mu_n^2 Fo) +$$

$$\left. + (Kl_{m2} - Kl_{m1}) \sum_{l=1}^{\infty} A_l \sin \mu_l X \exp(-\mu_l^2 Fo) \right\}, \quad (6-5-25)$$

where

$$A_n = \frac{2 \sin \mu_n}{\mu_n + \sin \mu_n \cos \mu_n} = (-1)^{n+1} \frac{2 \text{Bi}_q (\text{Bi}_q^2 + \mu_n^2)^{1/2}}{\mu_n (\text{Bi}_q^2 + \text{Bi}_q + \mu_n^2)};$$

$$A_l = \frac{2 \cos \mu_l}{\mu_l - \sin \mu_l \cos \mu_l} = (-1)^{l+1} \frac{2 \text{Bi}_q (\text{Bi}_q^2 + \mu_l^2)^{1/2}}{\mu_l (\text{Bi}_q^2 + \text{Bi}_q + \mu_l^2)}; \quad (6-5-26)$$

and  $\mu_n$  and  $\mu_l$  are the roots of the characteristic equations

$$\cot \mu_n = \frac{1}{\text{Bi}_q} \mu_n;$$

$$\tan \mu_l = -\frac{1}{\text{Bi}_l} \mu_l. \quad (6-5-27)$$

The first six values of roots  $\mu_n$  and coefficients  $A_n$  are listed in Tables 6-42 and 6-43; the corresponding values of roots  $\mu_l$  of equation (6-5-27) and coefficients  $A_l$  in (6-5-26) are listed in Tables 6-51 and 6-52.

TABLE 6-51

Roots of characteristic equation $\tan \mu_n = -\frac{1}{\text{Bi}_q} \mu_n$						
$\text{Bi}_q$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_5$	$\mu_6$
0	1.5708	4.7124	7.8540	10.9956	14.1372	17.2788
0.1	1.6320	4.7335	7.8667	11.0047	14.1443	17.2845
0.2	1.6887	4.7544	7.8794	11.0137	14.1513	17.2903
0.3	1.7414	4.7751	7.8920	11.0228	14.1584	17.2961
0.4	1.7906	4.7956	7.9046	11.0318	14.1654	17.3019
0.5	1.8366	4.8158	7.9171	11.0409	14.1724	17.3076
0.6	1.8798	4.8358	7.9295	11.0498	14.1795	17.3134
0.7	1.9203	4.8556	7.9419	11.0588	14.1865	17.3192
0.8	1.9586	4.8751	7.9542	11.0677	14.1935	17.3249
0.9	1.9947	4.8943	7.9665	11.0767	14.2005	17.3306
1.0	2.0288	4.9132	7.9787	11.0856	14.2075	17.3364
1.5	2.1746	5.0037	8.0385	11.1296	14.2421	17.3649
2.0	2.2889	5.0870	8.0962	11.1727	14.2764	17.3932
3.0	2.4557	5.2329	8.2045	11.2560	14.3434	17.4490
4.0	2.5704	5.3540	8.3029	11.3349	14.4080	17.5034
5.0	2.6537	5.4544	8.3914	11.4086	14.4699	17.5562
6.0	2.7165	5.5378	8.4703	11.4773	14.5288	17.6072
7.0	2.7654	5.6078	8.5406	11.5408	14.5847	17.6562
8.0	2.8044	5.6669	8.6031	11.5994	14.6374	17.7032
9.0	2.8363	5.7172	8.6587	11.6532	14.6870	17.7481
10.0	2.8628	5.7606	8.7083	11.7027	14.7335	17.7908
15.0	2.9476	5.9080	8.8898	11.8959	14.9251	17.9742
20.0	2.9930	5.9921	9.0019	12.0250	15.0625	18.1136
30.0	3.0406	6.0831	9.1294	12.1807	15.2380	18.3018
40.0	3.0651	6.1311	9.1986	12.2688	15.3417	18.4180
50.0	3.0801	6.1606	9.2420	12.3247	15.4090	18.4953
60.0	3.0901	6.1805	9.2715	12.3632	15.4559	18.5497
80.0	3.1028	6.2058	9.3089	12.4124	15.5164	18.6209
100.0	3.1105	6.2211	9.3317	12.4426	15.5537	18.6650
$\infty$	3.1416	6.2832	9.4248	12.5664	15.7080	18.8496

For the steady state ( $\text{Fo}=\infty$ ) equation (6-5-25) becomes

$$T = T_c - \frac{\text{Ko Lu}}{2 \text{Bi}_q} \left[ \text{Ki}_{m1} + \text{Ki}_{m2} + \frac{\text{Bi}_q}{1 + \text{Bi}_q} (\text{Ki}_{m1} - \text{Ki}_{m2}) X \right],$$

that is, the temperature is distributed linearly over the thickness of the plate. If  $\text{Ki}_{m1} = \text{Ki}_{m2} = \text{Ki}_m$ , then this relation simplifies to

$$T = T_c - \frac{\text{Ko Lu Ki}_m}{\text{Bi}_q},$$

that is, the temperature at any point of the body is equal to the wet-thermometer temperature  $t(x) = t_w - \frac{\rho_{jm}}{\alpha} = t_w$ , which is constant.

TABLE 6-52

Values of coefficients  $A_i = (-1)^{i+1} \frac{2Bi_l(Bi_l^2 + \mu_l^2)^{1/2}}{\mu_l(Bi_l^2 + Bi_l + \mu_l^2)}$

$Bi_q$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$
0	0.0000	-0.0000	0.0000	-0.0000	0.0000	-0.0000
0.1	0.0721	-0.0089	0.0032	-0.0017	0.0010	-0.0007
0.2	0.1303	-0.0175	0.0064	-0.0033	0.0020	-0.0013
0.3	0.1779	-0.0259	0.0096	-0.0049	0.0030	-0.0020
0.4	0.2172	-0.0341	0.0127	-0.0065	0.0040	-0.0027
0.5	0.2514	-0.0420	0.0158	-0.0082	0.0050	-0.0033
0.6	0.2803	-0.0497	0.0189	-0.0098	0.0059	-0.0040
0.7	0.3055	-0.0571	0.0219	-0.0114	0.0069	-0.0047
0.8	0.3156	-0.0643	0.0249	-0.0129	0.0079	-0.0053
0.9	0.3471	-0.0713	0.0278	-0.0145	0.0089	-0.0060
1.0	0.3646	-0.0781	0.0305	-0.0160	0.0098	-0.0066
1.5	0.4298	-0.1088	0.0446	-0.0237	0.0146	-0.0099
2.0	0.4726	-0.1348	0.0576	-0.0311	0.0193	-0.0129
3.0	0.5254	-0.1756	0.0805	-0.0448	0.0281	-0.0192
4.0	0.5562	-0.2052	0.0957	-0.0571	0.0365	-0.0251
5.0	0.5759	-0.2270	0.1159	-0.0682	0.0442	-0.0310
6.0	0.5892	-0.2435	0.1293	-0.0779	0.0513	-0.0360
7.0	0.5987	-0.2561	0.1404	-0.0865	0.0578	-0.0409
8.0	0.6056	-0.2659	0.1496	-0.0941	0.0638	-0.0455
9.0	0.6108	-0.2736	0.1574	-0.1007	0.0691	-0.0498
10.0	0.6148	-0.2789	0.1639	-0.1065	0.0739	-0.0538
15.0	0.6256	-0.2978	0.1844	-0.1265	0.0920	-0.0694
20.0	0.6300	-0.3056	0.1945	-0.1375	0.1028	-0.0797
30.0	0.6336	-0.3122	0.2034	-0.1479	0.1140	-0.0911
40.0	0.6348	-0.3147	0.2068	-0.1524	0.1191	-0.0966
50.0	0.6354	-0.3160	0.2088	-0.1546	0.1218	-0.0995
60.0	0.6358	-0.3167	0.2098	-0.1559	0.1234	-0.1015
80.0	0.6361	-0.3174	0.2105	-0.1573	0.1250	-0.1034
100.0	0.6363	-0.3177	0.2113	-0.1579	0.1258	-0.1043
$\infty$	0.6366	-0.3183	0.2122	-0.1591	0.1273	-0.1060

As another example of the use of nonsymmetrical boundary conditions of the third kind, let us consider the distribution of dimensionless temperature when heat sources are present in a hollow cylinder. The differential equation is

$$\frac{\partial T}{\partial Fo} = \frac{\partial^2 T}{\partial X^2} + \frac{1}{X} \cdot \frac{\partial T}{\partial X} + Po(X, Fo) \quad (1 \leq X \leq \kappa), \quad (6-5-28)$$

with boundary conditions

$$\frac{\partial T(1, Fo)}{\partial X} + Bi_{q1} T(1, Fo) = q_1(Fo); \quad (6-5-29)$$

$$\frac{\partial T(\kappa, Fo)}{\partial X} + Bi_{q2} T(\kappa, Fo) = q_2(Fo) \quad (6-5-30)$$

and initial conditions

$$T(X, 0) = F(X).$$

This problem was solved by Shimko /27/ using the finite Hankel integral transformation

$$T_H(\rho, Fo) = \int_1^\kappa XT(X, Fo) V_\rho(\mu_n X) dX, \quad (6-5-31)$$

where  $V_0(\mu_n X)$  is determined by the formula

$$V_0(\mu_n X) = [Bi_{q1} Y_0(\mu_n) - \mu_n Y_1(\mu_n)] J_0(\mu_n X) + [\mu_n J_1(\mu_n) - Bi_{q1} J_0(\mu_n)] Y_0(\mu_n X),$$

and roots  $\mu_n$  obey the characteristic equation

$$[Bi_{q1} J_0(\mu) - \mu J_1(\mu)] [Bi_{q2} Y_0(x\mu) - \mu Y_1(x\mu)] - \\ - [Bi_{q2} J_0(x\mu) - \mu J_1(x\mu)] [Bi_{q1} Y_0(\mu) - \mu Y_1(\mu)] = 0.$$

The inversion formula for integral transformation (6-5-31) is

$$T(X, Fo) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{\mu_n^2 [Bi_{q2} J_0(x\mu_n) - \mu_n J_1(x\mu_n)]^2 V_0(\mu_n X) T_H(\mu, Fo)}{(\mu_n^2 + Bi_{q2}^2) [Bi_{q1} J_0(\mu_n) - \mu_n J_1(\mu_n)]^2 - (\mu_n^2 + Bi_{q1}^2) [Bi_{q2} J_0(x\mu_n) - \mu_n J_1(x\mu_n)]^2}. \quad (6-5-32)$$

The solution of equation (6-5-28), with boundary conditions (6-5-29) and (6-5-30), is, in terms of transforms,

$$T_H(\mu_n, Fo) = \exp(-\mu_n^2 Fo) \left\{ \int [K(\mu_n, Fo) + Po_H(\mu_n, Fo)] \exp(\mu_n^2 Fo) dFo + C \right\}, \quad (6-5-33)$$

where

$$K(\mu_n, Fo) = \frac{2}{\pi} \left[ \frac{Bi_{q1} J_0(\mu_n) - \mu_n J_1(\mu_n)}{Bi_{q2} J_0(x\mu_n) - \mu_n J_1(x\mu_n)} \varphi_2(Fo) - \varphi_1(Fo) \right]$$

and

$$C = \int_0^x XF(X) V_0(\mu_n X) - \int [K(\mu_n, 0) + Po_H(\mu_n, 0)] dFo.$$

Then, the substitution of (6-5-33) into (6-5-32) gives the final solution of the problem at hand.

Some particular cases of this solution have been given by Sneddon /28/, Carslaw and Jaeger /29/, Danilova /31/, and others. For example, if,

$$\begin{aligned} T(1, Fo) &= 0, & T(0, Fo) &= 0, \\ \frac{\partial T(x, Fo)}{\partial X} &= 0, & Po(X, Fo) &= Po_1 \exp(-Pd' Fo), \end{aligned}$$

then solution (6-5-32) becomes

$$T = \pi \sum_{n=1}^{\infty} \frac{J_1^2(x, \mu_n)}{J_0^2(\mu_n) - J_1^2(x\mu_n)} [J_0(\mu_n X) Y_0(\mu_n) - J_0(\mu_n) Y_0(\mu_n X)] \times \\ \times \left\{ \exp(-\mu_n^2 Fo) + \frac{Po_1}{\mu_n^2 - Pd'} [\exp(-\mu_n^2 Fo) - \exp(-Pd' Fo)] \right\},$$

where  $\mu_n$  are the roots of the characteristic equation

$$J_1(x\mu) Y_0(\mu) - J_0(\mu) Y_1(x\mu) = 0.$$

#### d) Unsteady potential fields for uncoupled heat and mass transfer

Since the system of differential heat-transfer and mass-transfer equations can be reduced to a system of two uncoupled parabolic-type equations, it will be convenient to consider some typical problems of this kind in which a continuously acting heat or mass source (sink) is present. The differential equation and boundary conditions are

$$\frac{\partial Z(X, Fo)}{\partial Fo} = \frac{\partial^2 Z(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \cdot \frac{\partial Z(X, Fo)}{\partial X} + Po(X, Fo); \quad (6-5-34)$$

$$Z(X, 0) = F(X); \quad (6-5-35)$$

$$\frac{\partial Z(0, Fo)}{\partial X} = 0; \quad (6-5-36)$$

$$-\frac{\partial Z(1, Fo)}{\partial X} + Bi [Z_0 - Z(1, Fo)] = 0. \quad (6-5-37)$$

In equations (6-5-34) through (6-5-37) the notation is that of § 5-5, and the potential [in the surroundings] is

$$Z_0 = \frac{z_0 - z_*}{z_*}.$$

The problems given below have been solved by means of Fourier integral transformations. The solution of differential equation (6-5-34), with limiting conditions (6-5-35) through (6-5-37), can be written in the generalized form:

$$Z(X, Fo) = Z_0 + \sum_{n=1}^{\infty} B_{rn} \Phi_r(\mu_n X) \exp(-\mu_n^2 Fo) \times \left[ \int_0^{Fo} R_r \exp(\mu_n^2 Fo^*) dFo^* - \Psi_r \right], \quad (6-5-38)$$

where  $B_{rn}$  and  $\Psi$  are defined (for a plate, cylinder, and sphere) by equations (6-5-18), (6-5-19), and (6-5-20). The functions  $\Phi_r(\mu_n X)$  and  $R_r$  for the given body shapes are

$$\begin{aligned} \Gamma=0 \quad \Phi_0 &= \cos \mu_n X; \quad R_0 = \int_0^1 Po(X, Fo) \cos \mu_n X dX; \\ \Gamma=1 \quad \Phi_1 &= J_0(\mu_n X); \quad R_1 = \int_0^1 X Po(X, Fo) J_0(\mu_n X) dX; \\ \Gamma=2 \quad \Phi_2 &= \frac{\sin \mu_n X}{\mu_n X}; \quad R_2 = \int_0^1 X Po(X, Fo) \frac{\sin \mu_n X}{\mu_n} dX. \end{aligned}$$

If the initial distributions of the transfer potentials are uniform, then  $F(X, 0) = 0$  and  $F_1(X) = Z_0$ , so that (6-5-38) becomes

$$Z(X, Fo) = Z_0 - Z_0 \sum_{n=1}^{\infty} A_{rn} \Phi_r(\mu_n X) \exp(-\mu_n^2 Fo) + \sum_{n=1}^{\infty} B_{rn} \Phi_r(\mu_n X) \exp(-\mu_n^2 Fo) \int_0^{Fo} R_r \exp(\mu_n^2 Fo^*) dFo^*, \quad (6-5-39)$$

where  $A_{rn}$  and  $\mu_n$  for the plate, cylinder, and sphere are defined, respectively, by the pairs of equations: (6-5-5) and (6-5-4), (6-5-7) and (6-5-6), (6-5-9) and (6-5-8). For  $Po(X, Fo) = 0$ , we obtain the solution (6-5-17).

Let us now consider some particular cases of solution (6-5-39).

1.  $Po(X, Fo) = Po_1$  is constant:

$$\begin{aligned} \frac{z - z_0}{z_0 - z_*} &= 1 + \frac{1}{2(\Gamma + 1)} Po_1 \left( 1 - X^2 + \frac{2}{Bi} \right) - \\ &- \sum_{n=1}^{\infty} \left( 1 + \frac{Po_1}{\mu_n^2} \right) A_{rn} \Phi_r(\mu_n X) \exp(-\mu_n^2 Fo). \end{aligned} \quad (6-5-39a)$$

If we set  $Po_1 = 0$  (no source), then (6-5-39a) simplifies to solution (6-5-10). For the steady state ( $Fo = \infty$ ) we have

$$\frac{z - z_0}{z_0 - z_*} = 1 + \frac{1}{2(\Gamma + 1)} Po_1 \left( 1 - X^2 + \frac{2}{Bi} \right).$$

The average value of the dimensionless potential given by (6-5-39a) is

$$\frac{z-z_0}{z_c-z_0} = 1 + \text{Po}_1 \left[ n_\Gamma + \frac{1}{(\Gamma+1)\text{Bi}} \right] - \sum_{n=1}^{\infty} \left( 1 + \frac{\text{Po}_1}{\mu_n^2} \right) D_{\Gamma n} \exp(-\mu_n^2 \text{Fo}),$$

where  $n_0 = 1/2$  for  $\Gamma = 0$ ;  $n_1 = 1/2$  for  $\Gamma = 1$ ; and  $n_2 = 1/12$  for  $\Gamma = 2$ . The formulas and tabulated values for coefficients  $D_{\Gamma n}$  were given in subsection 6-5, b.

2. For  $\text{Po}(X, \text{Fo}) = \text{Po}_1 \exp(-\text{Pd}' \text{Fo})$ , the dimensionless potential is

$$\begin{aligned} \frac{z-z_0}{z_c-z_0} &= 1 - \frac{\text{Po}_1}{\text{Pd}'} [1 - G_\Gamma(\sqrt{\text{Pd}'} X)] \exp(-\text{Pd}' \text{Fo}) - \\ &- \sum_{n=1}^{\infty} \left( 1 - \frac{\text{Po}_1}{\text{Pd}' - \mu_n^2} \right) A_{\Gamma n} \Phi_\Gamma(\mu_n X) \exp(-\mu_n^2 \text{Fo}), \end{aligned}$$

where for an infinite plate

$$G_0 = \frac{\cos \sqrt{\text{Pd}'} X}{\cos \sqrt{\text{Pd}'} - \frac{1}{\text{Bi}} \sqrt{\text{Pd}'} \sin \sqrt{\text{Pd}'}};$$

for an infinite cylinder

$$G_1 = \frac{J_0(\sqrt{\text{Pd}'} X)}{J_0(\sqrt{\text{Pd}'}) - \frac{1}{\text{Bi}} \sqrt{\text{Pd}'} J_1(\sqrt{\text{Pd}'})};$$

and for a sphere

$$G_2 = \frac{\text{Bi} \frac{\sin \sqrt{\text{Pd}'} X}{X}}{(\text{Bi} - 1) \sin \sqrt{\text{Pd}'} + \sqrt{\text{Pd}'} \cos \sqrt{\text{Pd}'}}.$$

Together with the above, let us give the solution for a semi-infinite body. The limiting conditions may be written as

$$-\frac{\partial Z(l, \text{Fo})}{\partial X} + \text{Bi} [1 - Z(l, \text{Fo})] + \text{Ko Lu Kl}_m(\text{Fo}) = 0,$$

$$Z(X, 0) = 0; \quad \frac{\partial Z(\infty, \text{Fo})}{\partial X} = 0;$$

$$\frac{\partial Z(0, \text{Fo})}{\partial X} + \text{Bi} [1 - Z(0, \text{Fo})] = 0,$$

where

$$Z = \frac{z - z_0}{z_c - z_0}.$$

In this case

$$\begin{aligned} Z &= \text{erfc} \frac{1}{2\sqrt{\text{Fo}_x}} - \exp[\text{Bi}_x + \text{Bi}_x^2 \text{Fo}_x] \text{erfc} \left( \frac{1}{2\sqrt{\text{Fo}_x}} + \text{Bi}_x \sqrt{\text{Fo}_x} \right) + \\ &+ \text{Po}_x \text{Fo}_x \cdot \frac{\tau^{1/2n}}{1 + 1/2n} + \frac{\text{Po}_x}{(-\text{Bi}_x)^{n+1}} \cdot \frac{x}{\frac{1}{2}n-1} \Gamma(1 + 1/2n) \times \\ &\times \left\{ \exp[\text{Bi}_x + \text{Bi}_x^2 \text{Fo}_x] \text{erfc} \left( \frac{1}{2\sqrt{\text{Fo}_x}} + \text{Bi}_x \sqrt{\text{Fo}_x} \right) - \right. \\ &\left. - \sum_{m=0}^{\infty} (-2 \text{Bi}_x \sqrt{\text{Fo}_x})^m i^m \text{erfc} \frac{1}{2\sqrt{\text{Fo}_x}} \right\}, \end{aligned}$$

where  $\Gamma$  is the gamma function, and where

$$\text{Bi}_x = \frac{\alpha}{\lambda} x; \quad \text{Fo}_x = \frac{a\tau}{x^2}; \quad \text{Po}_x = \frac{wx^2}{c\gamma\Delta t}.$$

3. Let us now generalize the solution given in problem 2. Once again it is assumed that equation (6-5-34) is valid, that a continuously acting exponential source term ( $P_0 = P_0 \exp(-Pd'Fo)$ ) is present, and that constant initial conditions  $Z(X, 0) = 0$  and symmetry conditions (6-5-36) are observed. However, in contrast to the solutions given in problem 2, we assume here that the exponential source term also enters into boundary condition (6-5-37), so that we have

$$-\frac{\partial Z(l', Fo)}{\partial X} + Bi [1 - Z(l', Fo)] + Ko Lu Kl_m(Fo) = 0,$$

where

$$Z = \frac{z - z_0}{z_0 - z_0};$$

$$Kl_m(Fo) = (Kl_{m1} - Kl_{m2}) \exp(-Pd_m Fo) + Kl_{m1};$$

$$Kl_{mk} = \frac{j_{mk} R}{\lambda_m \Delta \theta} \quad (k = 1, 2; j_{mk} = \text{constant});$$

$$Pd_m = \frac{kR}{a},$$

the quantity  $k$  in the last equation being the variation rate of the exponential source term in the boundary conditions. For these limiting conditions, the solutions for a plate, cylinder, and sphere have the following form /34/:

$$\begin{aligned} \frac{z - z_0}{z_0 - z_0} = & 1 - \theta_{M2} - \frac{Po_1}{Pd'} [1 - G_r(\sqrt{Pd'X}) \exp(-Pd'Fo) - \\ & - (\theta_{M1} - \theta_{M2}) G_r(\sqrt{Pd_m X}) \exp(-Pd_m Fo) - \\ & - \sum_{n=1}^{\infty} \left[ 1 - \theta_{M2} - \frac{(\theta_{M1} - \theta_{M2}) \mu_n^2}{Pd_m - \mu_n^2} - \frac{Po_1}{Pd' - \mu_n^2} \right] A_{rn} \Phi_{rn}(\mu_n X) \exp(-\mu_n^2 Fo), \end{aligned}$$

where the coefficients  $G_r$  were defined in problem 2 preceding, and where

$$\theta_{Mk} = \frac{Lu Ko Kl_{mk}}{Bi} \quad (k = 1, 2).$$

Generalized solutions (6-5-12) and (6-5-17) describing the propagation of transfer potential were obtained by comparing particular solutions for bodies with plane, cylindrical, and spherical symmetry.

The method by which a general solution of the differential transfer equation

$$\frac{\partial Z(X, Fo)}{\partial X} = \frac{\partial^2 Z(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial Z(X, Fo)}{\partial X} \quad (6-5-40)$$

may be obtained is of great interest, since with such a solution it will no longer be necessary to solve problems for each body shape separately. Such methods have been proposed by several authors, and here we will consider that of Gol'dfarb /32/.

Following Gol'dfarb, we set  $\Gamma = 2\nu + 1$ , so that for a plate  $\nu = -\frac{1}{2}$ , for a cylinder  $\nu = 0$ , and for a sphere  $\nu = \frac{1}{2}$ . The initial distribution over the system cross section will be assumed constant and equal to

$$Z(X, 0) = 0.$$

When a Laplace transformation is applied to equation (6-5-40), we obtain

$$Z''_L + \frac{2\nu + 1}{X} Z'_L - sZ_L = 0. \quad (6-5-41)$$

The general solution of equation (6-5-41) may be written as

$$Z_L = A_1 (\sqrt{sX})^{-1} I_1(\sqrt{sX}) + A_2 (\sqrt{sX})^{-\nu} K_\nu(\sqrt{sX}), \quad (6-5-42)$$

where  $A_i$  ( $i=1, 2$ ) are constants;  $I_\nu$  and  $K_\nu$  are  $\nu$ th-order Bessel functions of the first and second kinds, having purely imaginary arguments:

$$I_\nu(u) = \frac{u^\nu}{2^\nu \Gamma(\nu)} \left[ 1 + \frac{u^2}{2(2\nu+2)} + \frac{u^4}{2 \cdot 4(2\nu+2)(2\nu+4)} + \dots \right] =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu+k)} \left( \frac{u}{2} \right)^{\nu+2k};$$

$$K_\nu(u) = \frac{\pi}{2 \sin \pi \nu} [I_{-\nu}(\mu) - I_\nu(\mu)].$$

$\Gamma$  is the Gauss or gamma function.

Specific solutions may be obtained simply by determining constants  $A_i$  from the boundary conditions and then finding the inverse transform of formula (6-5-42).

1. For boundary conditions of the first kind,

$$Z(1, F_0) = Z_c; \quad \frac{\partial Z(0, F_0)}{\partial X} = 0; \quad Z_c = \frac{z_c - z_0}{z_0}, \quad (6-5-43)$$

solution (6-5-42) has the form

$$Z_L = Z_c \left( \frac{1}{X} \right)^\nu \frac{I_\nu(\sqrt{s}X)}{s I_{\nu+1}(\sqrt{s})}.$$

By means of the expansion theorem we can obtain a solution of the problem in the following final form:

$$\frac{z - z_0}{z_c - z_0} = 1 - \sum_{n=1}^{\infty} \frac{2}{\mu_n} \left( \frac{1}{X} \right)^\nu \frac{J_\nu(\mu_n X)}{J_{\nu+1}(\mu_n)} \exp(-\mu_n^2 F_0), \quad (6-5-43')$$

where  $\mu_n$  are the roots of the characteristic equation

$$J_\nu(\mu_n) = 0;$$

and  $J_\nu$  is a  $\nu$ th-order Bessel function of the first kind, with a real argument:

$$J_\nu(\mu) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu+k)} \left( \frac{\mu}{2} \right)^{\nu+2k}.$$

2. For boundary conditions of the second kind:

$$\frac{\partial Z(1, F_0)}{\partial X} + Ki = 0; \quad \frac{\partial Z(0, F_0)}{\partial X} = 0, \quad (6-5-44)$$

solution (6-5-42) has the form

$$Z_L = Ki \left( \frac{1}{X} \right)^\nu \frac{I_\nu(\sqrt{s}X)}{s \sqrt{s} I_{\nu+1}(\sqrt{s})}.$$

The inverse transform is found, by means of the expansion theorem for multiple roots, to be

$$Z(X, F_0) = \frac{1}{2} Ki \left[ 2(2\nu+2) F_0 + X^2 - \frac{2\nu+2}{2\nu+4} - \sum_{n=1}^{\infty} \frac{4}{\mu_n^2} \left( \frac{1}{X} \right)^\nu \frac{J_\nu(\mu_n X)}{J_{\nu+1}(\mu_n)} \exp(-\mu_n^2 F_0) \right],$$



where  $\mu_n$  are the roots of the characteristic equation

$$J_{\nu+1}(\mu_n) = 0.$$

3. For boundary conditions of the third kind:

$$-\frac{\partial Z(1, Fo)}{\partial X} + Bi[Z_c - Z(1, Fo)] = 0; \quad \frac{\partial Z(0, Fo)}{\partial X} = 0, \quad (6-5-45)$$

solution (6-5-42) becomes

$$Z_L = Z_c \frac{\left(\frac{1}{X}\right)^{\nu} I_{\nu}(V \sqrt{sX})}{s \left[ I_{\nu}(V \sqrt{s}) + \frac{1}{Bi} V \sqrt{s} I_{\nu+1}(V \sqrt{s}) \right]}.$$

The corresponding inverse transform is

$$\frac{z - z_c}{z_c - z_0} = 1 - \sum_{n=1}^{\infty} P_{\nu}(\mu_n) \left(\frac{1}{X}\right)^{\nu} \frac{J_{\nu}(\mu_n X)}{J_{\nu}(\mu_n)} \exp(-\mu_n^2 Fo), \quad (6-5-45')$$

where

$$P_{\nu}(\mu_n) = \frac{2Bi}{(Bi - 2\nu)Bi + \mu_n^2};$$

and  $\mu_n$  are the roots of the characteristic equation

$$\frac{J_{\nu}(\mu_n)}{J_{\nu+1}(\mu_n)} = \frac{1}{Bi} \mu_n.$$

The particular solutions for an infinite plate, cylinder, and sphere may be obtained in their familiar form from the above solutions, provided we substitute the values  $\nu = -1/2$ ; 0;  $+1/2$ , respectively. When this is done, it is important to recall the following relations between Bessel functions of fractional order and the trigonometric functions:

$$J_{-1/2}(u) = \sqrt{\frac{2}{\pi u}} \cos u; \quad J_{1/2}(u) = \sqrt{\frac{2}{\pi u}} \sin u;$$

$$J_{3/2}(u) = \sqrt{\frac{2}{\pi u}} \left( \frac{\sin u}{u} - \cos u \right).$$

## 6-6. The Effects of the Dimensionless Numbers on Heat and Mass Transfer

The solution of the system of equations describing heat and mass transfer indicates in what way the process depends on a large group of heat-exchange and mass-exchange similarity criteria [dimensionless numbers]. For example, the dimensionless temperature function can be written as

$$T = T(X, Fo, Bi_q, Bi_m \text{ (or } Ki_m), Lu, \varepsilon, Ko, Pn, Pd, Po, W, V, \Gamma).$$

However, all the criteria do not affect the development of the process to the same degree. Some of them have their main influence on the heat exchange, while others have their main influence on the mass exchange. A variation in the conditions of interaction between the system material and the surroundings (that is, a variation in the form of the boundary conditions) leads to a variation in the effects which the similarity criteria have on the process.

The aggregate of similarity criteria may be classified into several groups, depending on their role in the heat and mass transfer. They may be criteria of the surface heat and mass exchange (the Biot or Kirpichev numbers), criteria of the heat and mass transfer (the phase-transition ratio; the Kossovich and Posnov numbers), criteria of the coupling between heat and mass transfer (the [Lukomskii] number  $Lu$ ), criteria of the unsteadiness (the Fourier and Predvoditelev numbers), and the dimensionless parameters describing the initial nonuniformity of the distribution.

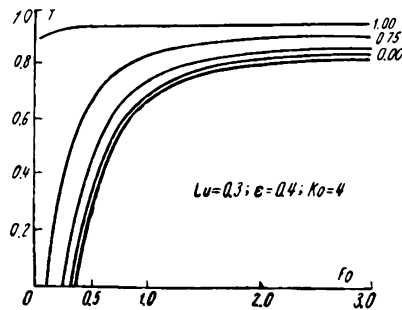


FIGURE 6-47. Unsteady dimensionless-temperature field, for constant mass-exchange rate ( $Lu = 0.3$ ;  $\epsilon = 0.4$ ;  $Ko = 4$ ;  $Pn = 0.25$ ;  $K_{im} = 0.5$ ;  $Bl_q = 10$ )

Let us first consider the effects of individual similarity criteria on heat and mass transfer when the mass-exchange rate is constant. Studies of unsteady potential fields have shown that, in contrast to the apparent simplicity of this process, it is actually controlled by a very complicated mechanism, which is no less varied in its effects than the mechanism behind a mass-exchange process whose rate is a function of the surface mass-transfer potential.

Analytical investigations of unsteady potential fields have indicated that for a constant mass-exchange rate two stages of process development must be distinguished. The first stage is characterized by heating of the system material and by an extremely unstable mass-transfer-potential distribution. At the end of this first stage ( $Fo = 0.5$  to  $0.7$ ) the distribution of mass-transfer potential becomes parabolic. The unsteady fields for this stage must be calculated using the complete solution (6-2-40) and (6-2-41); as usual, the smaller the value of  $Fo$ , the greater are the number of terms of the infinite sum which must be used in the calculation. The second stage of the process is characterized by a further development of the potential fields. This stage proceeds in an ordered or quasisteady way, so that it is permissible to use the simplified solutions (6-2-42) and (6-2-43) in the calculations.

The temperature of the surface layer of the material rises very rapidly from the very beginning of the process, so that appreciable temperature gradients appear in the interior of the material (Figure 6-47). The lower the value of  $Lu$ , the criterion for coupling between the heat and mass transfer, the more intensive is the heating of the material and the more rapid is the establishment of the quasisteady temperature distribution. Each layer has its own constant temperature, so that the distribution of temperature

over the thickness of the material becomes parabolic. The experimental studies of Kazanskii and Lutsik /33/, Polonskaya /12/, Lebedev /20/, and others have confirmed these theoretical conclusions (see Figures 6-48, a and 6-49).

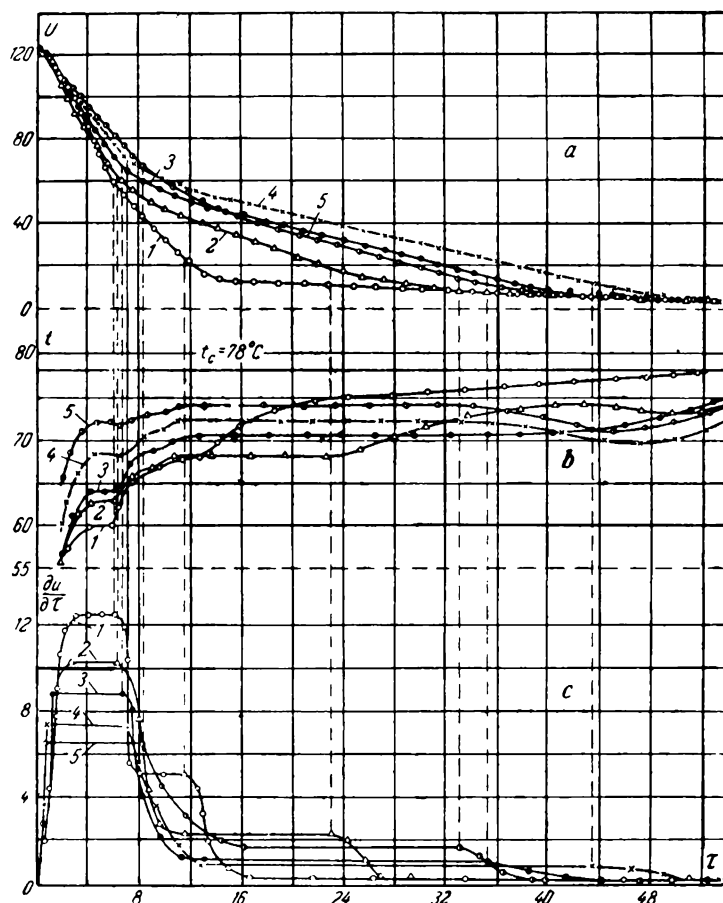


FIGURE 6-48. Experimental curves for the time variation of: a) the mass content  $u = c_m \theta$ ; b) the temperature; and c) the rate of variation of mass content, at various depths in cellulose (according to the experiments of Kazanskii and Lutsik).

1) distance from surface, 5 mm; 2) 15 mm; 3) 25 mm; 4) 35 mm; 5) 45 mm;  
air temperature  $78^\circ\text{C}$

An increase in the Kirpichev mass-exchange number reduces the average and local temperatures in the material according to a linear law (Figure 6-50). On the other hand, an increase in the Biot number raises the temperature of the material (however, for  $Bi_q > 10$  this effect is very slight (Figure 6-51)). Increases in the phase-transition ratio, the Kossovich number, and the Posnov number intensify the first stage of the heating only. After the heating front reaches the center of the material, these

quantities reverse, so that the quasisteady state with respect to temperature is first attained by a material with lower values of these dimensionless numbers.

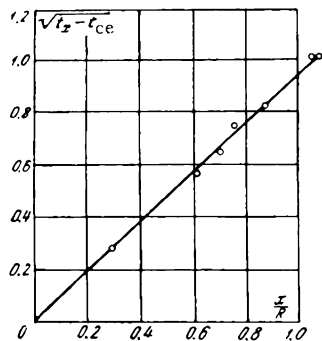


FIGURE 6-49. Verification of the parabolic temperature distribution (according to the data of Polonskaya)

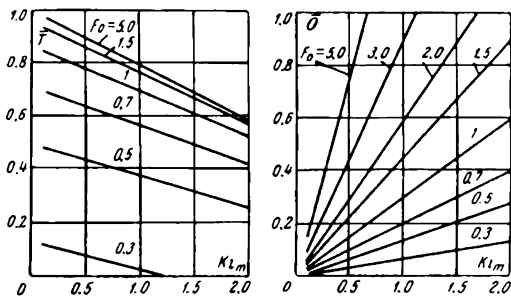


FIGURE 6-50. Effect of the Kirpichev mass-exchange number on  $\bar{T}$  and  $\bar{\theta}$ , for a constant mass-exchange rate

Along with the heating of the material, a removal of mass to the surroundings begins. From the very beginning of the process, the mass transfer affects the total thickness of the material. However, depending on the value of the compound Fedorov number (or the numbers  $\epsilon$ ,  $Ko$ , and  $Pn$ ), as well as on the value of  $Lu$ , this effect may have different forms (see Figures 6-52 and 6-53). The evolution of the distribution of mass-transfer potential for a typical example is shown in Figures 6-52 and 6-53, c.

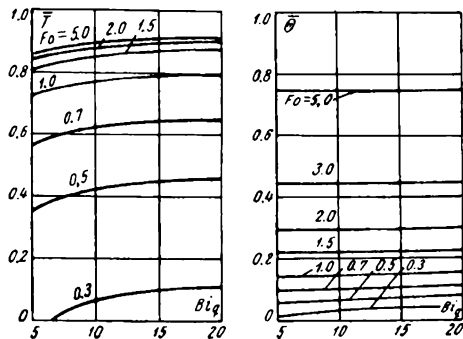


FIGURE 6-51. Effect of the Biot heat-exchange number on  $\bar{T}$  and  $\bar{\theta}$ , for a constant mass-exchange rate

Due to the rapid rise in temperature in the surface layer of the material, vaporization of the bound matter occurs, at a rate which decreases with increasing distance from the surface. On the other hand, due to thermal-gradient transfer, some of the moisture moves into the body interior and increases the mass content, and thus the mass-transfer potential  $\theta$  as well

(but it reduces the value of the dimensionless mass-transfer potential  $\theta = \frac{\theta_s - \theta}{\theta_s - \theta_0}$ ), of the layers adjacent to the surface, in comparison with the initial values. As a result, a complicated distribution of mass-transfer potential, which may have one or several points of maximum or minimum  $\theta$ , is formed in the body. Later, under the influence of the transfer-potential gradients these extrema undergo relaxation. Then, the wave of reduced

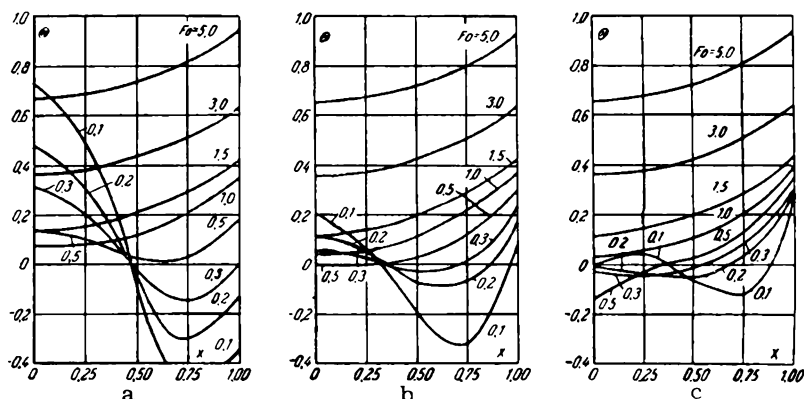


FIGURE 6-52. Distribution of dimensionless mass-transfer potential in the material ( $Lu = 0.3$ )

a)  $\epsilon Ko Pn = 0.3$ ; b)  $\epsilon Ko Pn = 0.4$ ; c)  $\epsilon Ko Pn = 0.5$ .

dimensionless potential  $\theta$  shifts gradually toward the center of the material and decreases in amplitude. The relaxation of the extremum is terminated at values of  $Fo$  from 0.4 to 0.7. The distribution of  $\theta$  then becomes parabolic and it becomes a linear function of  $Fo$ , that is, a quasisteady state is

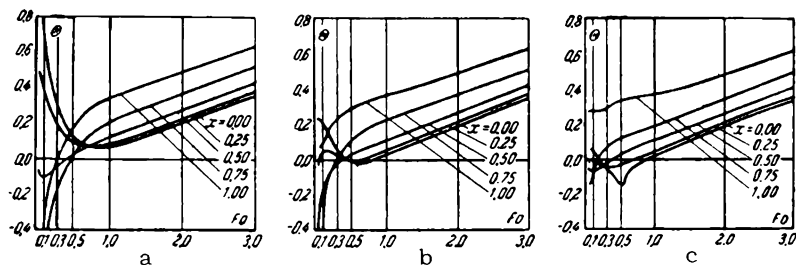


FIGURE 6-53. Variation with time of the dimensionless mass-transfer potential

a)  $\epsilon Ko Pn = 0.3$ ; b)  $\epsilon Ko Pn = 0.4$ ; c)  $\epsilon Ko Pn = 0.5$ .

established. Thus, the rate of mass transfer in the layers also becomes constant, but is different for different layers. The constant rates of mass exchange in the individual layers are established more or less simultaneously, although for different conditions and for materials with different combinations of similarity criteria there may be a slight tendency for the constant rate to be established earlier in deeper layers and later in surface layers or vice versa.

The compound dimensionless number  $Fe$  (or its components  $\epsilon$ ,  $Ko$ , and  $Pn$ ) and the dimensionless numbers  $Lu$  and  $Ki_m$  have the greatest effects on the mass-content distribution or the distribution of the dimensionless potential  $\Theta$ .

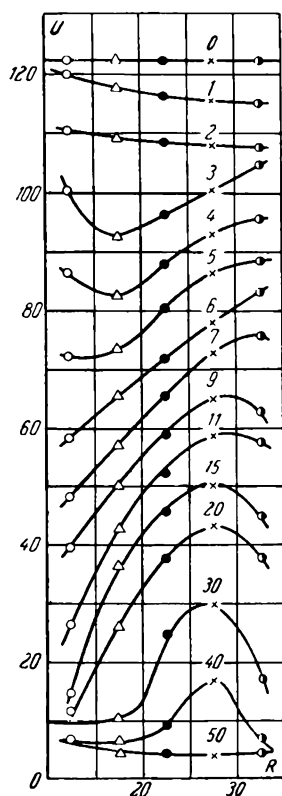


FIGURE 6-54. Distribution of mass content  $u$  (%) in the interior of cellulose ( $R$  is the distance of the layer centers from the outer surface of the body, in mm) for various times of drying

A virtual variation of the individual similarity criteria shows (Figure 6-52) that for lower Fedorov numbers the distribution of  $\Theta$  at the initial stage of the process is more nonuniform and that at the same time the wave of the reduced dimensionless mass-transfer potential is formed closer to the surface of the material and has a larger amplitude. Also, the "dehydration" which the central layers of the material undergo during the initial moments will be greater (Figure 6-53). As the Fedorov number increases, the amplitude of the "moistening" wave decreases and for sufficiently large  $Fe$  this wave disappears.

As the criterion  $Lu$  of the coupling between the heat and mass transfer increases, the development of the mass fields becomes more intensive; the absolute value of the extremum first increases, then for  $Lu > 0.3$  it begins to decrease, and finally it degenerates. Consequently, a wave of increased mass content need not be observed in all materials. For materials with high  $Lu$  this wave may be absent. It is important to note that, from the thermodynamic point of view, heat and mass transfer are not equally probable for all values of  $Lu$  and  $\epsilon KoPn$ . For low  $Lu$ , values of  $\epsilon KoPn < 0.3$  are more probable, while for  $Lu \geq 0.3$ , we have  $Fe = \epsilon KoPn > 0.4$ . Under quasisteady conditions  $\Theta$  is a linear function of  $Lu$  and  $\epsilon KoPn$ .

The above-described properties of the distributions of mass-transfer potentials for constant mass-exchange rates have been confirmed experimentally by the interesting studies made by co-workers of Kazanskii (Lutsik and Oleinikov) /33, 34/. These investigators used gamma-ray spectroscopy to observe the unsteady hydrothermal field in a typical porous colloidal body (see Figure 6-48, b and c, and Figure 6-54).

The mass fields are affected only by one of the dimensionless numbers characterizing the surface heat and mass exchange (that is, by  $Ki_m$  and not by  $Bi_q$ ). As the Kirpichev mass-exchange number increases, the dimensionless mass-transfer potential increases linearly. If  $Fo$  is increased, then the rapidity of increase of  $\Theta$  as a function of  $Ki_m$  increases too (Figure 6-50). The average dimensionless mass-transfer potential is self-similar with respect to the quantity  $Bi_q$  (Figure 6-51), whereas the local potential is self-similar only for  $Fo > 1.0$ . The latter fact is also evident from equation (6-2-43); under quasisteady conditions  $\Theta$  is independent of  $Bi_q$ .

When the mass-exchange rate is a function of the surface mass-transfer potential, unsteady fields of transfer potential do not show extrema.

Therefore, let us analyze the process under these conditions from the point of view of the effects of individual process similarity criteria on the transfer.

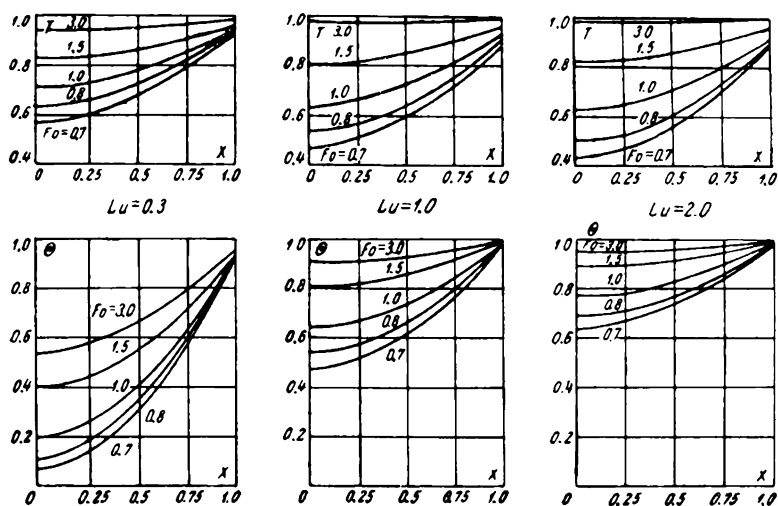


FIGURE 6-55. Distributions of heat-transfer and mass-transfer potentials inside a plate, for different values of  $Lu$

The most important effect which a dimensionless number has on the heat and mass transfer is that of  $Lu$ . Figure 6-55 indicates that for low values of  $Lu$  the heat-transfer potential (temperature) spreads considerably more

TABLE 6-53

Effects of  $Lu$  on the average heat-transfer and mass-transfer potentials and on their variation rates

$Fo$	$\bar{T}/\bar{\theta}$ for $Lu$ values of			$\frac{d\bar{T}/dFo}{d\bar{\theta}/dFo}$ for $Lu$ values of		
	0.5	1.0	2.0	0.5	1.0	2.0
0.4	1.44	0.908	1.61	—	—	—
0.6	1.41	0.946	1.35	1.23	1.10	0.63
0.8	1.35	0.965	1.22	0.92	1.09	0.60
1.0	1.29	0.976	1.15	0.79	1.08	0.59

rapidly than the mass-transfer potential. For  $Lu = 1$  heat and mass are transported at the same rate, while for  $Lu > 1$  the propagation of the potential  $\theta$  leads the development of the temperature field. However, perfect symmetry in the transfer properties at  $Lu = 1.0$  is not observed.

In order to compare the transfer characteristics let us consider what happens when  $Lu$  is equal to 0.5 and to 2.0, that is, when the temperature conductivity [thermal diffusivity] is, respectively, twice as great and half

as great as the mass-transfer potential conductivity. Table 6-53 and Figure 6-56 show that the difference amounts to  $\pm 12\%$  for the average potentials and as much as 200% for the ratio of their variation rates. Perfect equality of the heat-transfer and mass-transfer potentials is not observed at exactly

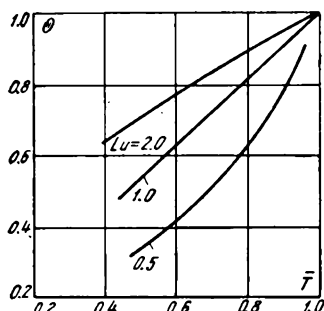


FIGURE 6-56. Symmetry of potentials  $\bar{T}$  and  $\bar{\theta}$  with respect to  $Lu = 1.0$

$Lu = 1.0$ , but occurs at  $Lu < 1.0$ . If  $Lu$  increases for a given value of  $Fo$ , the average temperature of the body drops and the rate of variation of  $\bar{\theta}$  and  $\bar{T}$  increases; the process of heat and mass transfer then takes place more uniformly (Figure 6-57).

Heat and mass transfer are influenced considerably by the surface heat-exchange and mass-exchange numbers ( $Bi_q$  and  $Bi_m$ ). For low values of  $Bi$ , the rates of variation of potentials  $T$  and  $\theta$  are negligible, and the potential gradients are small. As  $Bi_q$  and  $Bi_m$  increase, the heat and mass transfer intensifies, while the potential gradients increase. As a result, considerable stresses may appear in the material, so that conditions are created in

which buckling may appear, leading to the appearance of cracks and pores. The calculation results plotted in Figures 6-58 and 6-59 show that after stabilization of the process ( $Fo \geq 0.2$ ) the Biot heat-exchange number begins to have an effect only on the time development of the heat transfer ( $T$ ,  $dT/dFo$ ) and no longer affects the mass transfer ( $\theta$ ,  $d\theta/dFo$ ), whereas the Biot mass-exchange number begins to affect only the time development of the mass transfer. Consequently, a particular result is that the rate of variation of the average dimensionless potential is independent of the Biot heat-exchange number (Figure 6-60). At the same time the heating rate  $\frac{dT}{dFo}$  is a function of  $\bar{T}$  and  $Bi_q$ .

Under actual conditions the Biot heat-exchange and mass-exchange numbers are functions of one another. This interrelation is determined by the formulas describing the heat and mass balances. It is found that  $Bi_q$  varies with  $Bi_m$ , and that their numerical values are of approximately the same order. As  $Bi_m$  and  $Bi_q$  increase, the temperature rises considerably, the heating rate increases, and for lower  $\theta$  steady-state conditions are established. The experimental data of Lebedev and also Chizhskii confirm quite well the results of these analytical investigations.

The heat-transfer and mass-transfer criteria also include the Kossovich and Posnov numbers, as well as the phase-transition ratio  $\varepsilon$ . The properties of the first two are similar to those of the Biot number, but in contrast to the latter they describe internal processes of heat and mass transfer. The quantity  $Pn$  practically affects just the field of mass-transfer potential, while the Kossovich number practically affects just the temperature field (Figure 6-61). As the figure shows, increases in  $Ko$  and  $Pn$  are accompanied by decreases in the average and local heat-transfer and mass-transfer potentials, according to linear laws. This linearity is observed for all values of  $Fo$ . When  $Pn$  decreases, the intensification of the mass transfer is accompanied by an improvement in the temperature conditions, since the process takes place at lower temperatures of the material. Calculations



show that as the mass content decreases a sharp drop in  $P_n$  occurs, a result which has been confirmed experimentally.

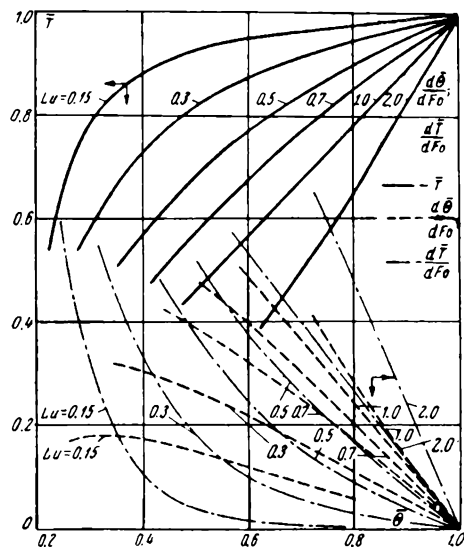


FIGURE 6-57. Coupling between the heat-transfer and mass-transfer potentials and between their rates of variation, for various values of  $Lu$

The above analysis implies that the surface mass exchange is governed by  $Bi_m$  and the internal mass exchange is governed by  $P_n$ . Consequently, the two dimensionless numbers  $Bi_m$  and  $P_n$  (or their ratio) determine completely the mass exchange in a body, and the ratio between  $d\bar{\theta}/dFo$  and  $\bar{\theta}$  is uniquely determined by the single compound dimensionless number  $Bi_m/P_n$ .

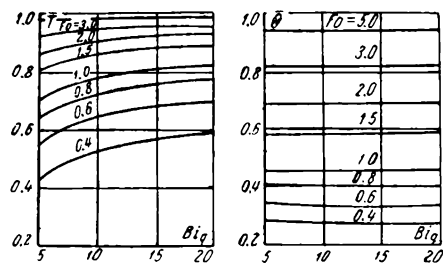


FIGURE 6-58. The heat-transfer and mass-transfer potentials as functions of  $Bi_q$

Actually, values of  $\bar{\theta}$  may be fitted quite well to a consecutive series of  $Bi_m/P_n$  values, as is evident from Table 6-54. Such a functional relationship

must apply equally well to different modes of evaporation of liquid from a material, since the mechanisms of heat and mass transfer are the same in all cases. The relation obtained between the dimensionless numbers has been completely confirmed by the experiments of Lebedev and Lisenkov /20/.

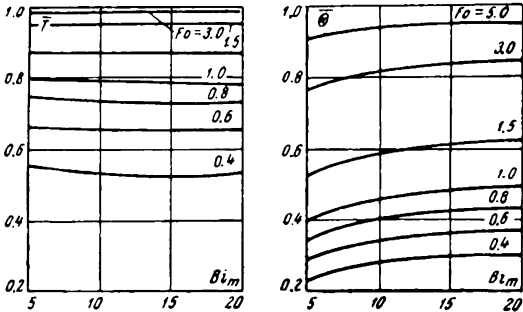


FIGURE 6-59. Heat-transfer and mass-transfer potentials as functions of  $Bi_m$

whose results are plotted in Figure 6-62. It is a very interesting fact that this relation between the dimensionless quantities is valid both for a constant mass-exchange rate and for a rate which is a function of the surface mass-transfer potential. Analogously, the rate of heating of a dispersed

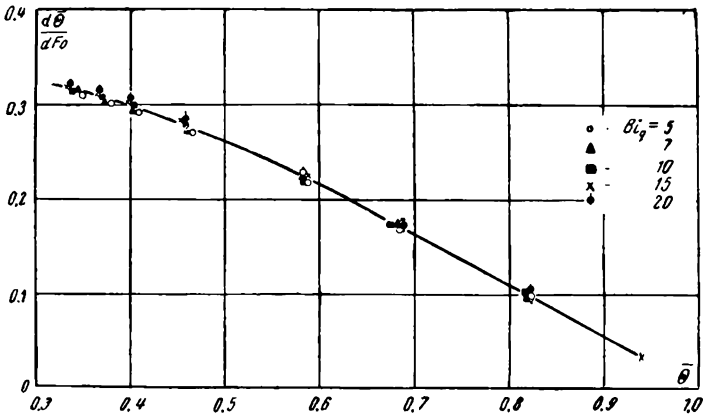


FIGURE 6-60. Rate of variation of dimensionless mass-transfer potential, for various values of  $Bi_q$

medium is also governed by  $Bi_q$  and  $Ko$ , and the relation between  $d\bar{T}/dFo$  and  $\bar{T}$  is uniquely determined by the compound dimensionless number  $Bi_q/Ko$  (Table 6-55).

The phase-transition ratio  $\epsilon$  has the same effect on the heat and mass transfer as the Kossovich number. The only difference lies in the intensity

of the influence exerted on the heat transfer: that of  $\epsilon$  is less than that of  $Ko$ . The curves describing the effect of  $\epsilon$  on potentials  $\bar{\theta}$  and  $\bar{T}$  are similar to those for  $Ko$ .

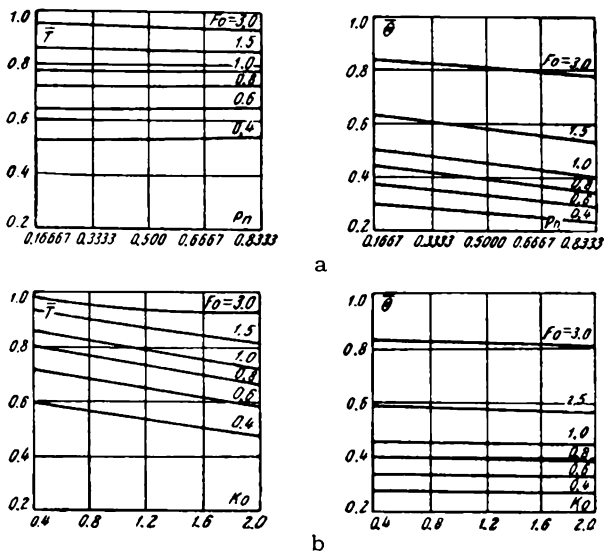


FIGURE 6-61. Heat-transfer and mass-transfer potentials as functions of the Posnov and Kossovich numbers

Let us now estimate what effect the phase-transition term in differential heat-transfer equation (4-1-2) has on the distributions of heat-transfer and mass-transfer potentials and on their variation rates. Analytically, this

TABLE 6-54  
Values of  $\bar{\theta}$  for various values of  $Bi_m/Pn$

$Bi_m/Pn$	$Po$								Separate values	
	0.4	0.5	0.6	0.7	0.8	1.0	1.5	2.0	$Bi_m$	$Pn$
10	0.225	0.254	0.283	0.312	0.340	0.395	0.518	0.618	5	0.5
12	0.250	0.277	0.305	0.333	0.361	0.417	0.544	0.646	10	0.8333
14	0.253	0.283	0.314	0.344	0.374	0.431	0.556	0.656	7	0.5
15	0.262	0.292	0.322	0.351	0.381	0.438	0.564	0.664	10	0.6667
20	0.275	0.307	0.339	0.370	0.401	0.459	0.584	0.683	10	0.5
30	0.291	0.326	0.358	0.391	0.422	0.480	0.607	0.704	10	0.3333
									15	0.5
40	0.304	0.338	0.371	0.404	0.435	0.494	0.620	0.717	20	0.5
60	0.305	0.341	0.377	0.412	0.444	0.505	0.630	0.724	10	0.1667

this can be done by comparing solutions of system of equations (4-1-2) and (4-1-3) with negative and positive  $\epsilon \frac{c_m}{c_q} \rho \frac{\partial \bar{\theta}}{\partial \tau}$ . The difference between the results for the two cases will be twice the quantity  $\epsilon \frac{c_m}{c_q} \rho \frac{\partial \bar{\theta}}{\partial \tau}$ . The calculation results are given in Table 6-56, which indicates that the heat source has

its greatest effect on the temperature field and has almost no effect on the field of mass-transfer potential or on the rate of heating of the body. The effect of the heat-source strength on the rate of variation of mass-transfer potential is also small, amounting to about 5 %.

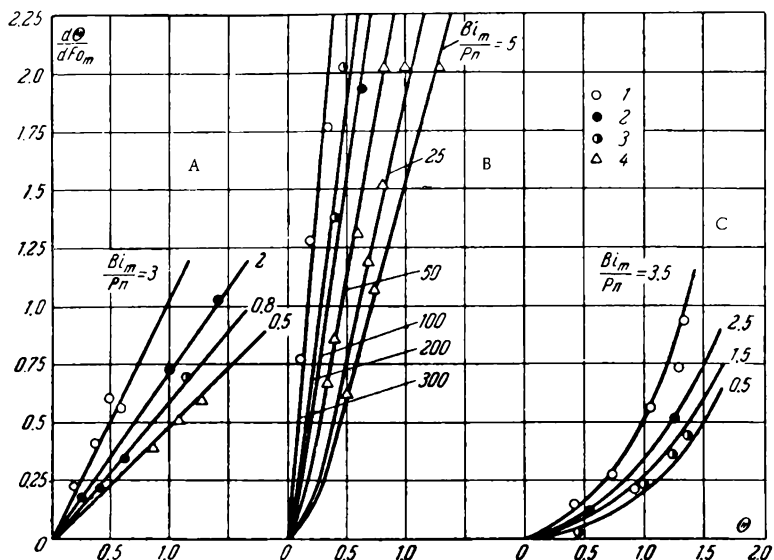


FIGURE 6-62. The function  $\frac{d\theta}{dFo} = f(\theta)$  for various values of the compound dimensionless number  $Bi_m/Pn$ : for sand (A), clay (B), and pine wood (C)

- 1) drying in a high-frequency field; 2) drying by infrared radiation; 3) drying by hot air; 4) contact drying

The dimensionless parameters describing the nonuniformity of the initial distribution of transfer potentials ( $W = \frac{t_s - t_{ce}}{t_c - t_s}$  and  $V = \frac{\theta_s - \theta_{ce}}{\theta_s - \theta_{ed}}$ ) affect the transfer of

TABLE 6-55  
Values of  $\bar{T}$  for various values of  $Bi_q/Ko$

$Bi_q/Ko$	$Fo$						Separate values	
	0.4	0.5	0.6	0.7	0.8	1.0	$Bi_q$	$Ko$
4.167	0.428	0.493	0.551	0.599	0.641	0.707	5	1.2
5.000	0.478	0.540	0.591	0.633	0.668	0.724	10	2.0
5.833	0.486	0.552	0.607	0.653	0.692	0.752	7	1.2
6.250	0.506	0.569	0.621	0.664	0.699	0.755	10	1.6
8.333	0.535	0.599	0.652	0.695	0.732	0.787	10	1.2
12.500	0.561	0.628	0.682	0.727	0.764	0.820	10	0.8
16.667	0.591	0.654	0.704	0.744	0.777	0.826	20	1.2
25.000	0.591	0.659	0.715	0.761	0.799	0.855	10	0.4

heat and matter only at the beginning of the process. Subsequently, during the stage of the ordered process, the rates of heat and mass transfer are independent of these parameters. To the accuracy usually necessary in

practice it is possible to neglect the effect of  $W$  on the mass characteristics for all values of  $Fo \geq 1.0$ , whereas the effect of parameter  $V$  on the distribution of  $\bar{T}$  and  $\bar{\theta}$  should still be taken into account up to values of  $\bar{\theta} = 0.7$  and  $Fo = 2.5$ .

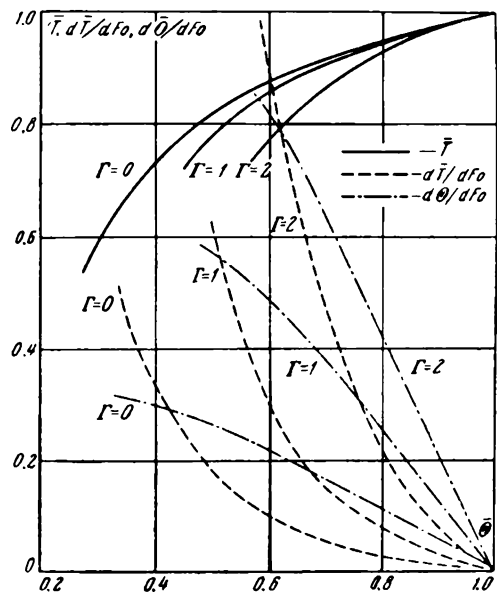


FIGURE 6-63. Effect of form factor  $\Gamma$  on heat and mass transfer

In conclusion, let us consider the effect which the shape of the body (as expressed by the form factor  $\Gamma$ ) has on the process of heat and mass transfer.

TABLE 6-56

Percent effect of heat source due to phase transitions on heat-transfer and mass-transfer potentials and on the rates of potential variation, as functions of various dimensionless quantities (mean arithmetic values)

Dimensionless number	$\bar{T}$	$\bar{\theta}$	$d\bar{T}/dFo$	$d\bar{\theta}/dFo$
Lu	20	4	2	10
$Bi_q$	15	4	5	9
$Bi_m$	20	4	8	9
Pn	21	3	6	11
Ko	21	3	6	12
Average	19.4	3.8	5.4	10.2

Figure 6-63 indicates that, as the form factor  $\Gamma$  increases, the average temperature of the body drops, for a given value of the dimensionless

mass-transfer potential. The rate of variation of mass-transfer potential throughout the whole process is considerably higher for a sphere than for a cylinder or for an infinite plate. The drop in heating rate with an increase in  $\Theta$  is also more marked for bodies with larger values of  $\Gamma$ . Thus, we may say that an increase in  $\Gamma$  accelerates considerably the mass exchange and reduces the average temperature of the material.

On the basis of the above analysis, it is possible to neglect some of the dimensionless parameters during calculations of unsteady potential fields, and in this way to simplify considerably the analytical model describing molecular heat and mass transfer. Thus, the simplified equations for the average transfer characteristics can be represented by the following functions of dimensionless numbers:

$$\bar{T} = \bar{T}(Fo, Lu, Bi_q/Ko, \epsilon, V, W);$$

$$\bar{\Theta} = \bar{\Theta}(Fo, Lu, Bi_m/Pn, V).$$

For constant initial potential distributions, or in the ordered-process regime for  $V=W=0$ , these functions simplify to

$$\bar{T} = \bar{T}(Fo, Lu, Bi_q/Ko, \epsilon); \quad (6-6-1)$$

$$\bar{\Theta} = \bar{\Theta}(Fo, Lu, Bi_m/Pn). \quad (6-6-2)$$

Graphoanalytical processings of the many calculations made for unsteady fields of temperature and mass-transfer potential have made it possible to obtain the analytical functional equations (6-6-1) and (6-6-2) in explicit forms. Thus, for an infinite plate these functions are

$$\bar{T} = 0.559 Fo^{0.30} Lu^{-0.073} (Bi_q/Ko)^{0.032} \epsilon^{-0.072};$$

$$\bar{\Theta} = 0.462 Fo^{0.67} Lu^{0.43} (Bi_m/Pn)^{0.11}.$$

For  $\bar{\Theta}$  this relation is valid for the following range of the dimensionless numbers entering into it:  $0.4 \leq Fo \leq 2.0$ ;  $10 \leq \frac{Bi_m}{Pn} \leq 40$ ; and  $0.15 \leq Lu \leq 1.0$ .

The maximum deviation from the exact analytical results amounts to 7%, with a mean deviation of 2.5%. The minimum deviation of the calculations is in the range of  $Lu$  from 0.2 to 0.65 and for  $Bi_m/Pn$  from 11 to 35 (with a maximum error less than 4%). The greatest deviation occurs for  $Bi_m/Pn = 10$  and  $Lu > 0.7$ . For  $\bar{T}$  the relation is valid in the following ranges:

$0.5 \leq Fo \leq 2.0$ ;  $0.15 \leq Lu \leq 1.0$ ;  $5 \leq \frac{Bi_q}{Ko} \leq 17$ ; and  $0.16 \leq \epsilon \leq 1.0$ . The maximum deviation is 9%, with an average deviation of 2.8%. The greatest error here is obtained for  $Lu = 1.0$  (5.81%) and for  $Fo = 0.5$ .

## Chapter VII

### HEAT AND MASS TRANSFER IN A MEDIUM WITH VARIABLE POTENTIALS

#### 7-1. Unsteady Fields of Transfer Potential in a Medium Whose Temperature Varies Linearly or Exponentially

In the foregoing we have considered heat and mass transfer in a medium with constant potentials. In most cases, however, real processes take place in a system whose temperatures and mass concentrations vary continuously with time. Thus, a body in contact with a heat carrier becomes heated, while the heat carrier, which has supplied heat to the body, cools off. This variation in the situation of the system and surroundings is particularly important during calculations of the kinetics of diffusive extraction (fractional distillation, flotation, etc.) and diffusive electrical processes in electrolytes, as well as during calculations of dissolution, semicoking, drying, etc. In recent years particularly, engineers have often been involved with problems in which the temperature of the external medium varies appreciably with time. A mathematical model of actual heat and mass transfer must thus represent the variation in the potentials of the external medium; this is done in the boundary conditions. Such a generalization of the conditions can at the same time make it possible, to some extent, to apply the system of differential heat-transfer and mass-transfer equations to transfer phenomena in a stationary or even a moving layer of matter.

Let us now consider solutions of system of equations (4-1-2) and (4-1-3):

$$\frac{\partial T}{\partial F_0} = \left[ \frac{\partial^2 T}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial T}{\partial X} \right] - \bullet K_0 \frac{\partial \Theta}{\partial F_0};$$

$$\frac{\partial \Theta}{\partial F_0} = \text{Lu} \left[ \frac{\partial^2 \Theta}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \Theta}{\partial X} \right] - \text{LuPn} \left[ \frac{\partial^2 T}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial T}{\partial X} \right],$$

for cases when the temperature of the medium varies linearly or exponentially. The linear case is introduced because of its simplicity, and the exponential case is treated because it is a good representation of the actual kinetics of the process.

The boundary conditions for a one-dimensional body, when the temperature of the medium drops linearly with time and when the mass exchange at the surface of the body is a function of the mass-transfer potential, are written as follows:

$$\frac{\partial T(1, F_0)}{\partial X} - \text{Bi}_q [1 - T(1, F_0) - \text{Pd} F_0] + (1 - \bullet) \text{Lu} K_0 \text{Bi}_m [1 - \Theta(1, F_0)] = 0; \quad (7-1-1)$$

$$-\frac{\partial \Theta(1, F_0)}{\partial X} + \text{Pn} \frac{\partial T(1, F_0)}{\partial X} + \text{Bi}_m [1 - \Theta(1, F_0)] = 0, \quad (7-1-2)$$

where  $Pd = \frac{bR^2}{a_q(t_{c0} - t_0)}$  is the Predvoditelev number, a quantity characterizing the variation rate of the temperature of the surroundings ( $b$  is the constant rate of the linear variation of the temperature of the surroundings and  $t_{c0}$  is the temperature at the initial moment). This definition for the Predvoditelev number is only a particular case of the general definition  $Pd = \left| \frac{dT_c}{dT_0} \right|_{\max}$ .

It should be noted that heat-exchange boundary condition (7-1-1) was derived for the case when the temperature of the surroundings drops linearly with time, namely for  $t_c(\tau) = t_{c0} - b\tau$  (which in dimensionless form is  $T_c = 1 - PdFo$ ). If the temperature rises with time, so that  $t_c = t_{c0} + b\tau = t_0 + b\tau$ , then in all the solutions the sign of  $Pd$  must be reversed.

The heat-exchange boundary condition when the temperature of the surroundings varies exponentially with time, namely when  $t_c = t_{c0} \exp(-\kappa\tau)$ , is

$$\begin{aligned} \frac{\partial T(1, Fo)}{\partial X} - Bi_q [1 - \vartheta(1 - \exp[-Pd'Fo]) - T(1, Fo)] + \\ + (1 - \varepsilon) Ko Lu Bi_m [1 - \theta(1, Fo)] = 0. \end{aligned} \quad (7-1-3)$$

Here  $Pd' = \frac{\kappa R^2}{a_q}$  is the Predvoditelev number for the given case of temperature variation, and  $\vartheta = \frac{t_{c0}}{t_{c0} - t_0}$  is the dimensionless temperature parameter.

Solutions of system of heat-transfer and mass-transfer equations (4-1-2) and (4-1-3) with boundary conditions (7-1-1) and (7-1-2) or (7-1-3) and (7-1-2), for constant or parabolic initial conditions, may be found by means of Laplace integral transformations. The methods used to solve this type of problem are no different from those considered in Chapters V and VI. Therefore, we will give here only the final results in generalized form, a particular case of which is represented by the solutions already considered in §6-3. The solutions are

$$T(X, Fo) = 1 - \Pi_T - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \Phi_r(v_i \mu_n X) \exp(-\mu_n^2 Fo); \quad (7-1-4)$$

$$\theta(X, Fo) = 1 + \Pi_{\theta} + \frac{1}{\varepsilon Ko} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2) \Phi_r(v_i \mu_n X) \exp(-\mu_n^2 Fo); \quad (7-1-5)$$

and

$$\bar{T} = 1 - \bar{\Pi}_T - \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} \exp(-\mu_n^2 Fo); \quad (7-1-6)$$

$$\bar{\theta} = 1 + \bar{\Pi}_{\theta} + \frac{1}{\varepsilon Ko} \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} (1 - v_i^2) \exp(-\mu_n^2 Fo), \quad (7-1-7)$$

where

$$C_{n1} = 2 \frac{G_1 P_{n2} + G_2 Q_{n2}}{\mu_n \psi_n}; \quad C_{n2} = -2 \frac{G_1 P_{n1} + G_2 Q_{n1}}{\mu_n \psi_n};$$

$$D_{ni} = (\Gamma + 1) C_{ni} \Omega_r;$$

$$\psi_n = v_1 A_{n1} P_{n2} + v_2 B_{n2} Q_{n1} - v_2 A_{n2} P_{n1} - v_1 B_{n1} Q_{n2};$$

$$v_i^2 = \frac{1}{2} \left[ \left( 1 + \varepsilon Ko Pn + \frac{1}{Lu} \right) + (-1)^i \sqrt{\left( 1 + \varepsilon Ko Pn + \frac{1}{Lu} \right)^2 - \frac{4}{Lu}} \right];$$



$$G_1 = 1 - \varepsilon \text{Ko} K_1 - 2W \left( \frac{1}{\text{Bl}_q} - \frac{1 + \Gamma}{\mu_n^2} \right) +$$

$$+ 2(\Gamma + 1) \varepsilon \text{Ko} (\text{Pn} W - V) \frac{(1 + K_1) Lu}{\mu_n^2} + M;$$

$$G_2 = \varepsilon \text{Ko} + 2\varepsilon \text{Ko} (\text{Pn} W - V) \left[ \frac{1}{\text{Bl}_m} - \frac{(\Gamma + 1) Lu}{\mu_n^2} \right],$$

and  $\mu_n$  are the roots of the characteristic equation

$$P_{n1} Q_{n2} - P_{n2} Q_{n1} = 0. \quad (7-1-8)$$

For an infinite plate:

$$\Gamma = 0; \quad \Phi_0(\mathbf{v}_i \mu_n X) = \cos \mathbf{v}_i \mu_n X; \quad \Omega_0 = \frac{\sin \mathbf{v}_i \mu_n}{\mathbf{v}_i \mu_n};$$

$$A_{ni} = \left[ 1 + \frac{1}{\text{Bl}_q} + K_1 (1 - \mathbf{v}_i^2) \right] \sin \mathbf{v}_i \mu_n + \frac{1}{\text{Bl}_q} \cos \mathbf{v}_i \mu_n;$$

$$B_{ni} = (1 - \mathbf{v}_i^2) \sin \mathbf{v}_i \mu_n + \frac{(1 - \mathbf{v}_i^2) + \varepsilon \text{Ko} \text{Pn}}{\text{Bl}_m} (\sin \mathbf{v}_i \mu_n + \mathbf{v}_i \mu_n \cos \mathbf{v}_i \mu_n);$$

$$Q_{ni} = [1 + K_1 (1 - \mathbf{v}_i^2)] \cos \mathbf{v}_i \mu_n - \frac{1}{\text{Bl}_q} \mathbf{v}_i \mu_n \sin \mathbf{v}_i \mu_n;$$

$$P_{ni} = (1 - \mathbf{v}_i^2) \cos \mathbf{v}_i \mu_n - \frac{(1 - \mathbf{v}_i^2) + \varepsilon \text{Ko} \text{Pn}}{\text{Bl}_m} \mathbf{v}_i \mu_n \sin \mathbf{v}_i \mu_n.$$

For an infinite cylinder:

$$\Gamma = 1; \quad \Phi_1(\mathbf{v}_i \mu_n X) = J_0(\mathbf{v}_i \mu_n X); \quad \Omega_1 = \frac{J_1(\mathbf{v}_i \mu_n)}{\mathbf{v}_i \mu_n};$$

$$A_{ni} = [1 + K_1 (1 - \mathbf{v}_i^2)] J_1(\mathbf{v}_i \mu_n) + \frac{1}{\text{Bl}_q} \mathbf{v}_i \mu_n J_0(\mathbf{v}_i \mu_n);$$

$$B_{ni} = (1 - \mathbf{v}_i^2) J_1(\mathbf{v}_i \mu_n) + \frac{(1 - \mathbf{v}_i^2) + \varepsilon \text{Ko} \text{Pn}}{\text{Bl}_m} \mathbf{v}_i \mu_n J_0(\mathbf{v}_i \mu_n);$$

$$Q_{ni} = [1 + K_1 (1 - \mathbf{v}_i^2)] J_0(\mathbf{v}_i \mu_n) - \frac{1}{\text{Bl}_q} \mathbf{v}_i \mu_n J_1(\mathbf{v}_i \mu_n);$$

$$P_{ni} = (1 - \mathbf{v}_i^2) J_0(\mathbf{v}_i \mu_n) - \frac{(1 - \mathbf{v}_i^2) + \varepsilon \text{Ko} \text{Pn}}{\text{Bl}_m} \mathbf{v}_i \mu_n J_1(\mathbf{v}_i \mu_n).$$

For a sphere:

$$\Gamma = 2; \quad \Phi_2(\mathbf{v}_i \mu_n X) = -\frac{\sin \mathbf{v}_i \mu_n X}{X}; \quad \Omega_2 = \frac{\sin \mathbf{v}_i \mu_n - \mathbf{v}_i \mu_n \cos \mathbf{v}_i \mu_n}{\mathbf{v}_i^2 \mu_n^2};$$

$$A_{ni} = [1 + K_1 (1 - \mathbf{v}_i^2)] \cos \mathbf{v}_i \mu_n - \frac{1}{\text{Bl}_q} \mathbf{v}_i \mu_n \sin \mathbf{v}_i \mu_n;$$

$$B_{ni} = (1 - \mathbf{v}_i^2) \cos \mathbf{v}_i \mu_n - \frac{(1 - \mathbf{v}_i^2) + \varepsilon \text{Ko} \text{Pn}}{\text{Bl}_m} \mathbf{v}_i \mu_n \sin \mathbf{v}_i \mu_n;$$

$$Q_{ni} = \left[ 1 - \frac{1}{\text{Bl}_q} + K_1 (1 - \mathbf{v}_i^2) \right] \sin \mathbf{v}_i \mu_n + \frac{1}{\text{Bl}_q} \mathbf{v}_i \mu_n \cos \mathbf{v}_i \mu_n;$$

$$P_{ni} = (1 - \mathbf{v}_i^2) \sin \mathbf{v}_i \mu_n - \frac{(1 - \mathbf{v}_i^2) + \varepsilon \text{Ko} \text{Pn}}{\text{Bl}_m} (\sin \mathbf{v}_i \mu_n - \mathbf{v}_i \mu_n \cos \mathbf{v}_i \mu_n).$$

The law describing the temperature variation in the surroundings is:

for  $t_e = \text{const}$

$$\Pi_T = 0; \quad \Pi_\theta = 0; \quad \bar{\Pi}_T = 0; \quad \bar{\Pi}_\theta = 0; \quad M = 0;$$

$$\begin{aligned} \text{for } t_0 = t_{c_0} - b\tau \\ \Pi_r = \text{Pd} \left[ \text{Fo} - \frac{1}{2(\Gamma+1)} \left( 1 - X^2 + \frac{2}{\text{Bi}_q} \right) \right]; \\ \Pi_0 = \frac{1}{2(\Gamma+1)} \text{Pd Pn} (1 - X^2); \\ \bar{\Pi}_r = \text{Pd} \left[ \text{Fo} - \frac{1}{(\Gamma+1)(\Gamma+3)} \left( 1 + \frac{\Gamma+3}{\text{Bi}_q} \right) \right]; \\ \bar{\Pi}_0 = \frac{1}{(\Gamma+1)(\Gamma+3)} \text{Pd Pn}; \quad M = \frac{\text{Pd}}{\mu_n^2}; \end{aligned}$$

$$\text{for } t_0 = t_{c_0} \exp(-\kappa\tau)$$

$$\begin{aligned} \Pi_r = \vartheta \left[ 1 + \sum_{i=1}^2 \bar{C}_i \Phi_r(\nu_i \sqrt{\text{Pd}' X}) \exp(-\text{Pd}' \text{Fo}) \right]; \\ \Pi_0 = \frac{\vartheta}{\varepsilon \text{Ko}} \sum_{i=1}^2 \bar{C}_i (1 - \nu_i^2) \Phi_r(\nu_i \sqrt{\text{Pd}' X}) \exp(-\text{Pd}' \text{Fo}); \\ \bar{\Pi}_r = \vartheta \left[ 1 + \sum_{i=1}^2 \bar{D}_i \exp(-\text{Pd}' \text{Fo}) \right]; \\ \bar{\Pi}_0 = \frac{\vartheta}{\varepsilon \text{Ko}} \sum_{i=1}^2 \bar{D}_i (1 - \nu_i^2) \exp(-\text{Pd}' \text{Fo}); \quad M = \vartheta \frac{\text{Pd}'}{\mu_n^2 - \text{Pd}'}, \end{aligned}$$

where

$$\bar{C}_1 = \frac{\bar{P}_2}{\bar{P}_1 \bar{Q}_1 - \bar{P}_2 \bar{Q}_1}; \quad \bar{C}_2 = -\frac{\bar{P}_1}{\bar{P}_1 \bar{Q}_1 - \bar{P}_2 \bar{Q}_1}; \quad \bar{D}_i = (\Gamma+1) \bar{C}_i \bar{Q}_r.$$

The quantities  $\bar{Q}_i$ ,  $\bar{P}_i$ ,  $\Phi_r(\nu_i \sqrt{\text{Pd}' X})$ , and  $\bar{Q}_r$  for the respective body shapes have the same definitions as  $Q_i$ ,  $P_i$ ,  $\Phi_r(\nu_i \mu_n X)$ , and  $Q_r$  except that  $\mu_n$  is replaced by  $\sqrt{\text{Pd}'}$ .

For constant initial conditions  $W=V=0$ , and

$$T = \frac{t - t_0}{t_{c_0} - t_0}; \quad \theta = \frac{\theta_0 - \theta}{\theta_0 - \theta_{c_0}}; \quad \vartheta = \frac{t_{c_0}}{t_{c_0} - t_0}.$$

An analysis of solutions (7-1-4) through (7-1-7) shows that when the temperature variation in the environment is linear, the values of Fo above which the calculations may be simplified (since only two terms of the infinite series need be retained) remain approximately the same (up to  $\text{Pd} \approx 0.25$ ) as those when the temperature of the surroundings is constant (see Table 7-1). As Pd is raised above 0.25, the "initial" values of Fo drop gradually. If greater values of the dimensionless time are taken, the effect of the infinite series on the values of the dimensionless potentials also decreases. For example, in the equation involving the dimensionless temperature (for  $\text{Pd} = 0.25$ ;  $\text{Bi}_q = 10$ ;  $\text{Bi}_m = 10$ ,  $\varepsilon = 0.5$ ,  $\text{Ko} = 1.2$ ;  $\text{Pn} = 0.5$ ;  $X = 0$ ) the infinite sum compares with the free term as follows:

for  $\text{Fo} = 5.0$ , it is 23.7% of it; for  $\text{Fo} = 7.0$ , it is 1.31% of it; for  $\text{Fo} = 10$ , it is 0.11% of it.

Beginning with the value  $\text{Fo} = \text{Fo}_{\text{quasist}}$ , the dimensionless potentials are determined exclusively by the free term. The temperature at any point on a plate then becomes a linear function of the time and the distribution of temperature over the thickness of the material becomes parabolic. Consequently, above  $\text{Fo} = \text{Fo}_{\text{quasist}}$  the same spatial temperature distribution is always observed, and so this range is called quasisteady. By this time all

redistribution of mass-transfer potential within the body has stopped and a steady-state parabolic distribution of the potential  $\Theta$  has been established. For higher  $Pd$ , quasisteady conditions of the heating (or cooling) of a body are established at lower values of  $Fo$ . A similar situation is observed for other body shapes.

TABLE 7-1

Values of  $Fo$  above which the calculations of heat and mass transfer may be simplified; infinite plate and  $t_c = t_{c0} - b\tau$

Parameter		$Fo$	
		$\Delta$  , %	
		1.0	0.5
$1 - T$	$x/R = 0$	0.73	0.88
	$x/R = 1$	0.93	1.09
$1 - \Theta$	$x/R = 0$	0.79	0.95
	$x/R = 1$	1.04	
$1 - \bar{T}$		0.12	0.17
$1 - \bar{\Theta}$		0.14	0.30
$d\bar{T}/dFo$		0.26	0.35
$d\bar{\Theta}/dFo$		1.04	

For the exponential case, a quasisteady transfer potential distribution is also established above some value of the Fourier number. However, in contrast to the linear problem, this distribution is more complicated and approaches some exponential curve asymptotically.

Thus, for  $Fo \geq Fo_{\text{quasist}}$  equations (7-1-4) through (7-1-7) simplify to

$$T(X, Fo) = 1 - \Pi_r;$$

$$\Theta(X, Fo) = 1 + \Pi_\theta;$$

$$\bar{T}(Fo) = 1 - \bar{\Pi}_r;$$

$$\bar{\Theta}(Fo) = 1 + \bar{\Pi}_\theta.$$

To facilitate calculations of heat and mass transfer in a medium in which the temperature drop is linear,  $t_c = t_{c0} - b\tau$ , Tables 7-2 through 7-5 give the numerical values of coefficients  $C_{ni}$ ,  $D_{Tn}$ , and  $D_{\Theta n}$  as functions of the Predvoditelev number. Values of the characteristic-equation roots for the respective body shapes are given in the tables and graphs of Chapter VI.

On the basis of the above generalized solutions it is possible to derive several solutions for particular cases of heat and mass transfer.

Let us first give the solution for the case when the temperature of the medium varies linearly.

1. Uncoupled transfer of heat and mass ( $Pn = 0$ ;  $Ko = 0$ ;  $e = 0$ ):

$$T = 1 - Pd \left[ Fo - \frac{1}{2(\Gamma + 1)} \left( 1 - X^2 + \frac{2}{Bi_q} \right) \right] - \sum_{n=1}^{\infty} \left( 1 + \frac{Pd}{\mu_n^2} \right) \times \\ \times A_n \Phi_r(\mu_n X) \exp(-\mu_n^2 Fo); \quad (7-1-9)$$

TABLE 7-2

Coefficients  $C_{n1}$ ,  $D_{Tn}$ , and  $D_{\theta n}$  as functions of  $Pd = \frac{bR^2}{a_q(t_{c0} - t_0)}$ , for an infinite plate  
( $Lu = 0.3$ ;  $\epsilon = 0.5$ ;  $Ko = 1.2$ ;  $Pn = 0.5$ ;  $Bl_m = 10$ ;  $Bl_q = 10$ )

Pd	$C_{n1}$		$C_{n2}$		$D_{Tn}$		$D_{\theta n}$	
	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
0	0.04880	0.9378	0.3080	0.01452	0.2570	0.6500	-0.9616	0.1137
0.1	0.04976	0.9903	0.3152	0.01532	0.2628	0.6867	-0.9842	0.1202
0.2	0.05073	1.044	0.3225	0.01612	0.2687	0.7235	-1.007	0.1265
0.25	0.05125	1.070	0.3261	0.01652	0.2716	0.7416	-1.018	0.1297
0.3	0.05169	1.097	0.3297	0.01692	0.2745	0.7603	-1.029	0.1330
0.4	0.05266	1.150	0.3369	0.01772	0.2804	0.7971	-1.052	0.1394
0.5	0.05362	1.203	0.3441	0.01852	0.2862	0.8338	-1.075	0.1458
0.6	0.05458	1.256	0.3514	0.01932	0.2921	0.8706	-1.098	0.1523
0.7	0.05555	1.309	0.3586	0.02012	0.2980	0.9072	-1.120	0.1587
0.75	0.05603	1.355	0.3622	0.02052	0.3009	0.9252	-1.132	0.1618
0.8	0.05651	1.362	0.3658	0.02092	0.3039	0.9440	-1.143	0.1650
0.9	0.05748	1.414	0.3730	0.02172	0.3097	0.9800	-1.165	0.1714
1.0	0.05844	1.467	0.3803	0.02252	0.3156	1.031	-1.88	0.1778

TABLE 7-3

Coefficients  $C_{n1}$ ,  $D_{Tn}$ , and  $D_{\theta n}$  as functions of  $Pd$ , for an infinite plate ( $Lu = 1.0$ ;  $\epsilon = 0.5$ ;  
 $Ko = 1.2$ ;  $Pn = 0.5$ ;  $Bl_m = 10$ ;  $Bl_q = 10$ )

Pd	$C_{n1}$		$C_{n2}$		$D_{Tn}$		$D_{\theta n}$	
	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
0	0.1399	0.03438	1.135	0.003402	0.9009	0.02439	-0.8436	0.01538
0.25	0.1498	0.08778	1.222	0.007787	0.9692	0.06206	-0.909	0.03962
0.50	0.1596	0.1412	1.309	0.01217	1.037	0.09975	-0.974	0.06386
0.75	0.1695	0.1946	1.396	0.01656	1.106	0.1374	-1.039	0.08806
1.00	0.1793	0.2480	1.483	0.02094	1.174	0.1751	-1.104	0.1123

TABLE 7-4

Coefficients  $C_{n1}$ ,  $D_{Tn}$ , and  $D_{\theta n}$  as functions of  $Pd$ , for an infinite plate ( $Lu = 0.3$ ;  $\epsilon = 0.5$ ;  
 $Ko = 1.2$ ;  $Pn = 0.1667$ ;  $Bl_m = 10$ ;  $Bl_q = 10$ )

Pd	$C_{n1}$		$C_{n2}$		$D_{Tn}$		$D_{\theta n}$	
	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
0	0.04618	0.9158	0.3188	0.005276	0.2620	0.6353	-0.9045	0.03952
0.25	0.04536	1.062	0.3258	0.006395	0.2662	0.7368	-0.9248	0.04563
0.50	0.04455	1.207	0.3327	0.006914	0.2702	0.8372	-0.9446	0.05209
0.75	0.04373	1.353	0.3397	0.007733	0.2743	0.9386	-0.9645	0.05839
1.00	0.04292	1.499	0.3466	0.008552	0.2783	1.040	-0.9843	0.06471

TABLE 7-5

Coefficients  $C_{n1}$ ,  $D_{Tn}$ , and  $D_{\theta n}$  as functions of the Biot number; temperature of medium drops linearly (infinite plate)

$Bl_q$	$C_{n1}$		$C_{n2}$		$D_{Tn}$		$D_{\theta n}$	
	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
5	0.1643	1.010	0.3295	0.00990	0.3789	0.7390	-1.013	0.1278
7	0.09726	1.054	0.3274	0.01266	0.3152	0.7500	-1.016	0.1303
10	0.05125	1.070	0.3261	0.01652	0.2716	0.7416	-1.018	0.1297
15	0.01597	1.079	0.3251	0.01773	0.2383	0.7332	-1.019	0.1299
20	-0.003319	1.088	0.3243	0.01860	0.2197	0.7314	-1.019	0.1305

$$\bar{T} = 1 - \text{Pd} \left[ \text{Fo} - \frac{1}{(\Gamma+1)(\Gamma+3)} \left( 1 + \frac{\Gamma+3}{\text{Bi}_q} \right) \right] - \sum_{n=1}^{\infty} \left( 1 + \frac{\text{Pd}}{\mu_n^2} \right) D_n \exp(-\mu_n^2 \text{Fo}), \quad (7-1-10)$$

where for an infinite plate, an infinite cylinder, and a sphere, respectively, we have:

$$\Phi_0 = \cos \mu_n X; \quad \Phi_1 = J_0(\mu_n X); \quad \Phi_2 = \frac{\sin \mu_n X}{\mu_n X};$$

and where coefficients  $A_n$  and  $D_n$  are defined by equations (6-5-5) and (6-5-5'), (6-5-7) and (6-5-7'), (6-5-9) and (6-5-9'), and the characteristic roots obey equations (6-5-4), (6-5-6), and (6-5-8).

2. If the temperature of the external medium increases linearly with time,  $t_c(\tau) = t_0 + b\tau$ , equations (7-1-9) and (7-1-10) can be rewritten as

$$T^* = \frac{t - t_0}{t_c(\tau) - t_0} = 1 - \frac{1}{2(\Gamma+1)\text{Fo}} \left( 1 - X^2 + \frac{2}{\text{Bi}_q} \right) + \sum_{n=1}^{\infty} \frac{A_n}{\mu_n^2 \text{Fo}} \Phi_n(\mu_n X) \exp(-\mu_n^2 \text{Fo}); \quad (7-1-11)$$

$$\bar{T}^* = \frac{\bar{t} - t_0}{t_c(\tau) - t_0} = 1 - \frac{1}{(\Gamma+1)(\Gamma+3)\text{Fo}} \left( 1 + \frac{\Gamma+3}{\text{Bi}_q} \right) + \sum_{n=1}^{\infty} \frac{D_n}{\mu_n^2 \text{Fo}} \exp(-\mu_n^2 \text{Fo}). \quad (7-1-12)$$

It is important to note that the quantity  $\bar{T}^*$  defined by (7-1-12) is at the same time the relative heat content of the body. This is evident if we recall that the specific heat consumption may be defined as

$$Q = c_q \gamma [\bar{t}(\tau) - t_0],$$

while the possible heat intake is

$$Q_0 = c_q \gamma [t_c(\tau) - t_0],$$

so that

$$\frac{Q}{Q_0} = \frac{\bar{t}(\tau) - t_0}{t_c(\tau) - t_0} = \bar{T}^*.$$

In order to determine the dimensionless temperatures at the surface ( $T^*_s$ ) and at the center ( $T^*_{ce}$ ) of a plate, cylinder, or sphere, as well as the relative heat contents  $\bar{T}^*$  for these bodies, Kutateladze and Vinnikov /1/ constructed nomograms. These graphs, calculated according to (7-1-11) and (7-1-12), are given in Figures 7-1 through 7-9.

3. If we set  $\text{Bi}_q = \infty$  in the above solutions, then the temperature at the body surface instantaneously becomes equal to the temperature of the surroundings but then varies according to the linear law  $T(1, \text{Fo}) = 1 - \frac{\text{Pd}}{2} \text{Fo}$ , that is, we obtain boundary conditions of the first kind. For  $\text{Bi}_q = \infty$  equations (7-1-9) and (7-1-10) become

$$T = 1 - \text{Pd} \left[ \text{Fo} - \frac{1}{2(\Gamma+1)} (1 - X^2) \right] - \sum_{n=1}^{\infty} \left( 1 + \frac{\text{Pd}}{\mu_n^2} \right) A_n \Phi_n(\mu_n X) \exp(-\mu_n^2 \text{Fo}); \quad (7-1-13)$$

$$\bar{T} = 1 - \text{Pd} \left[ \text{Fo} - \frac{1}{(\Gamma+1)(\Gamma+3)} \right] - \sum_{n=1}^{\infty} \left( 1 + \frac{\text{Pd}}{\mu_n^2} \right) D_n \exp(-\mu_n^2 \text{Fo}). \quad (7-1-14)$$

Here, for a plate

$$\mu_n = (2n-1) \frac{\pi}{2}; \quad A_n = (-1)^{n+1} \frac{4}{(2n-1)\pi}; \quad D_n = \frac{8}{(2n-1)^2 \pi^2};$$

for a cylinder,  $\mu_n$  are the roots of the equation  $J_0(\mu) = 0$  and the coefficients are  $A_n = \frac{2}{\mu_n J_1(\mu_n)}$  and  $D_n = \frac{4}{\mu_n^2}$ ;

for a sphere,

$$\mu_n = n\pi; \quad A_n = 2(-1)^{n+1}; \quad D_n = \frac{6}{\mu_n^2}.$$

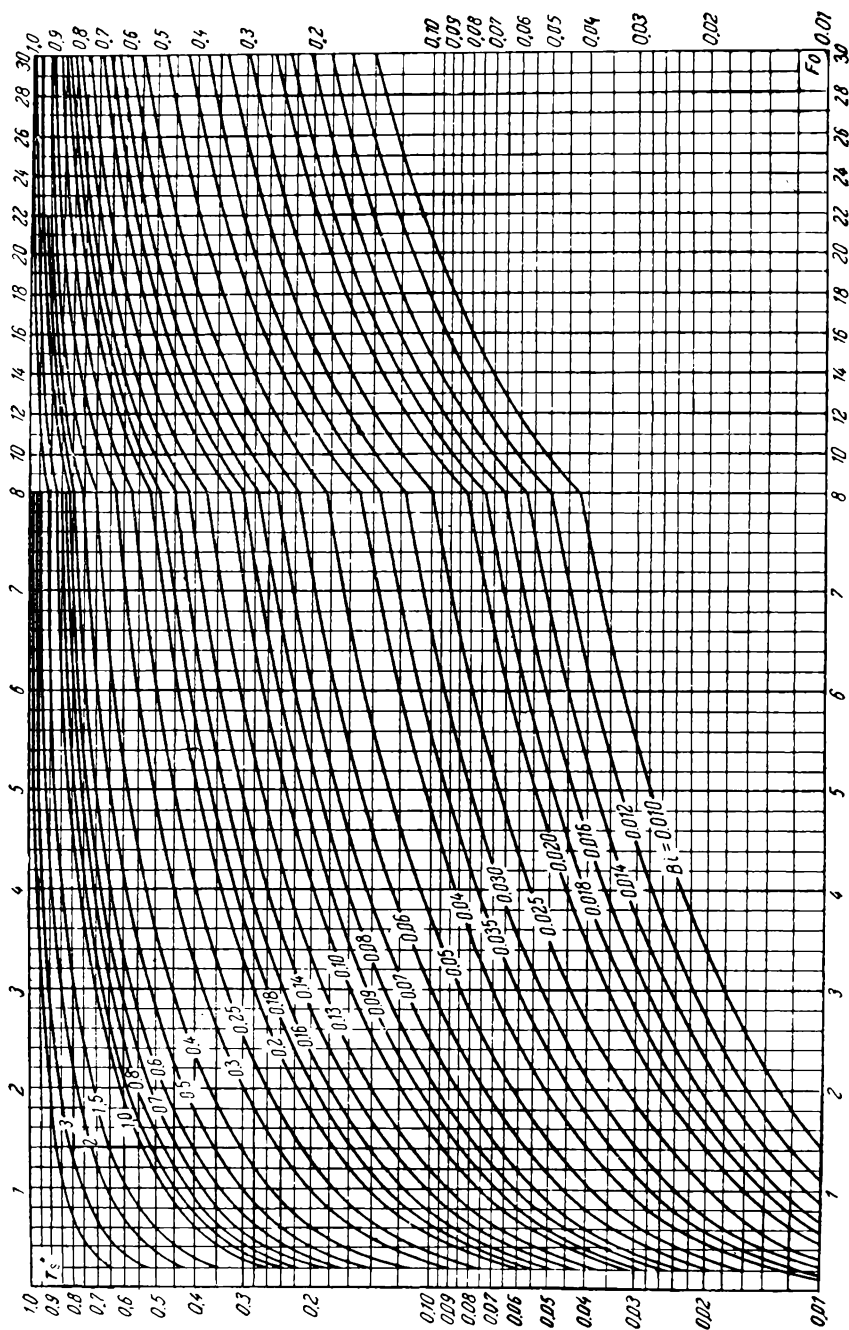


FIGURE 7-1. Dimensionless temperature  $T^*$  at surface of plate, for linear variation of temperature of surroundings

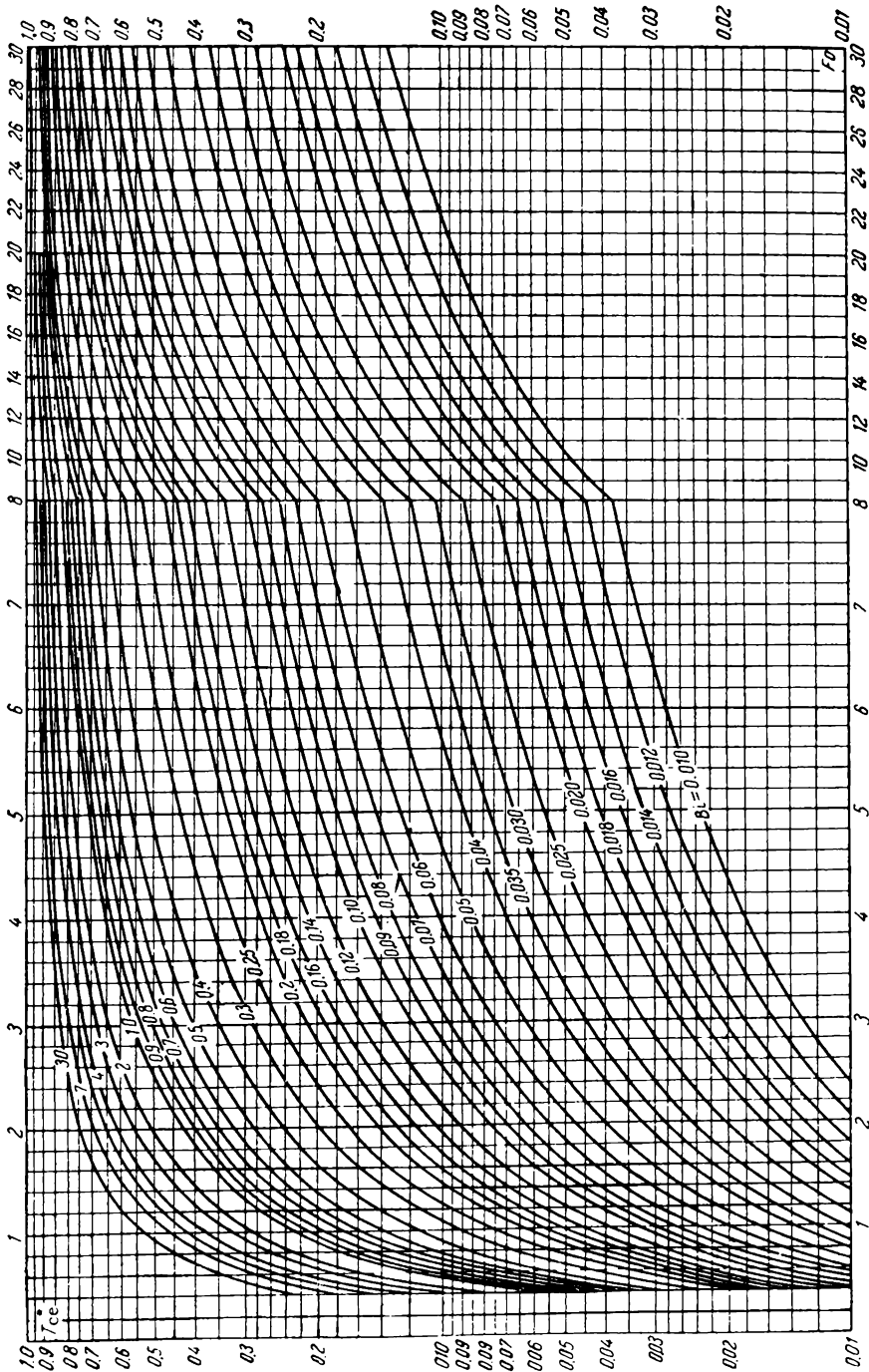


FIGURE 7-2. Dimensionless temperature  $T^*_{ceat}$  of plate, for linear variation of temperature of surroundings

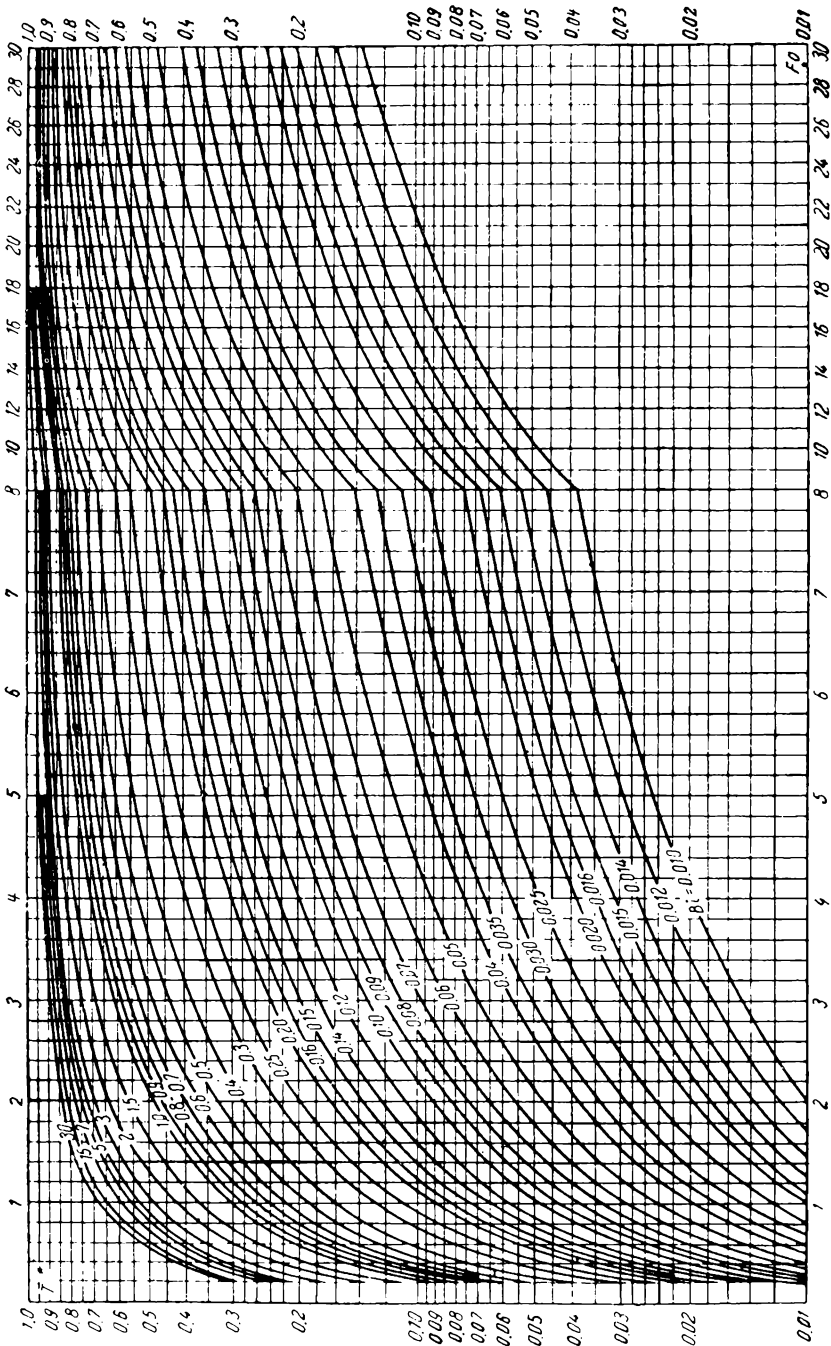


FIGURE 7-3. Relative heat content  $T^*$  of infinite plate, for linear variation of temperature of surroundings



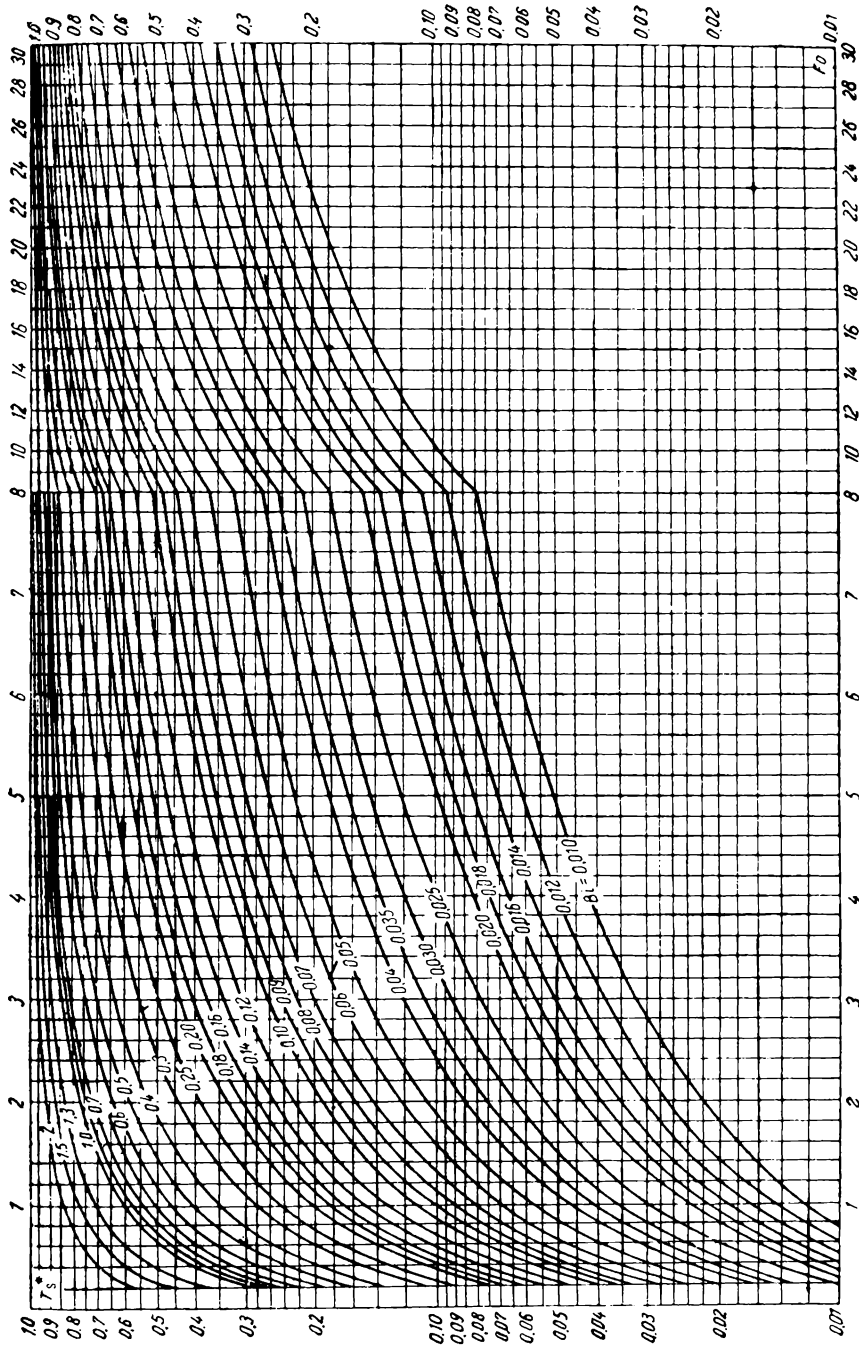


FIGURE 7-4. Dimensionless temperature  $T^*$  at surface of cylinder, for linear variation of temperature of surroundings

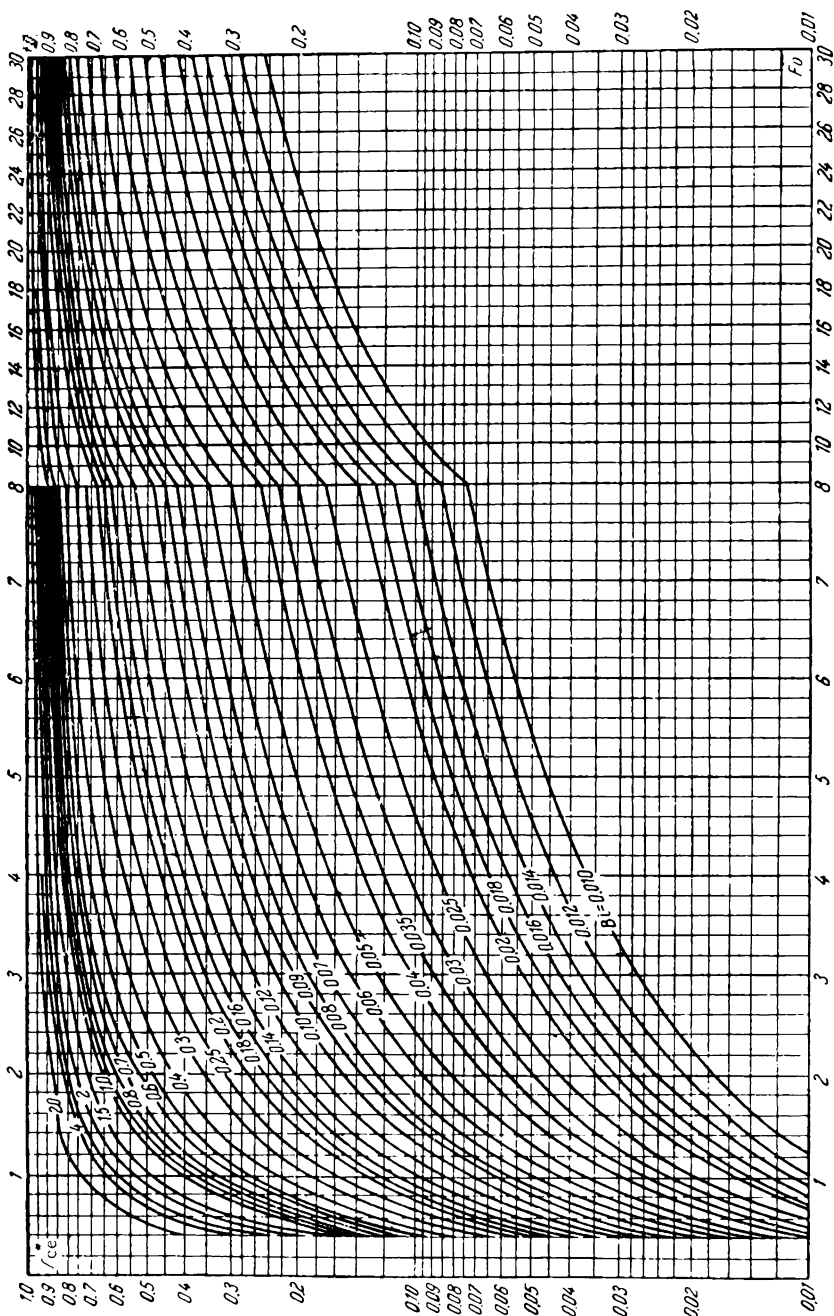


FIGURE 7-5. Dimensionless temperature  $T^*_{ce}$  on axis of cylinder, for linear variation of temperature of surroundings

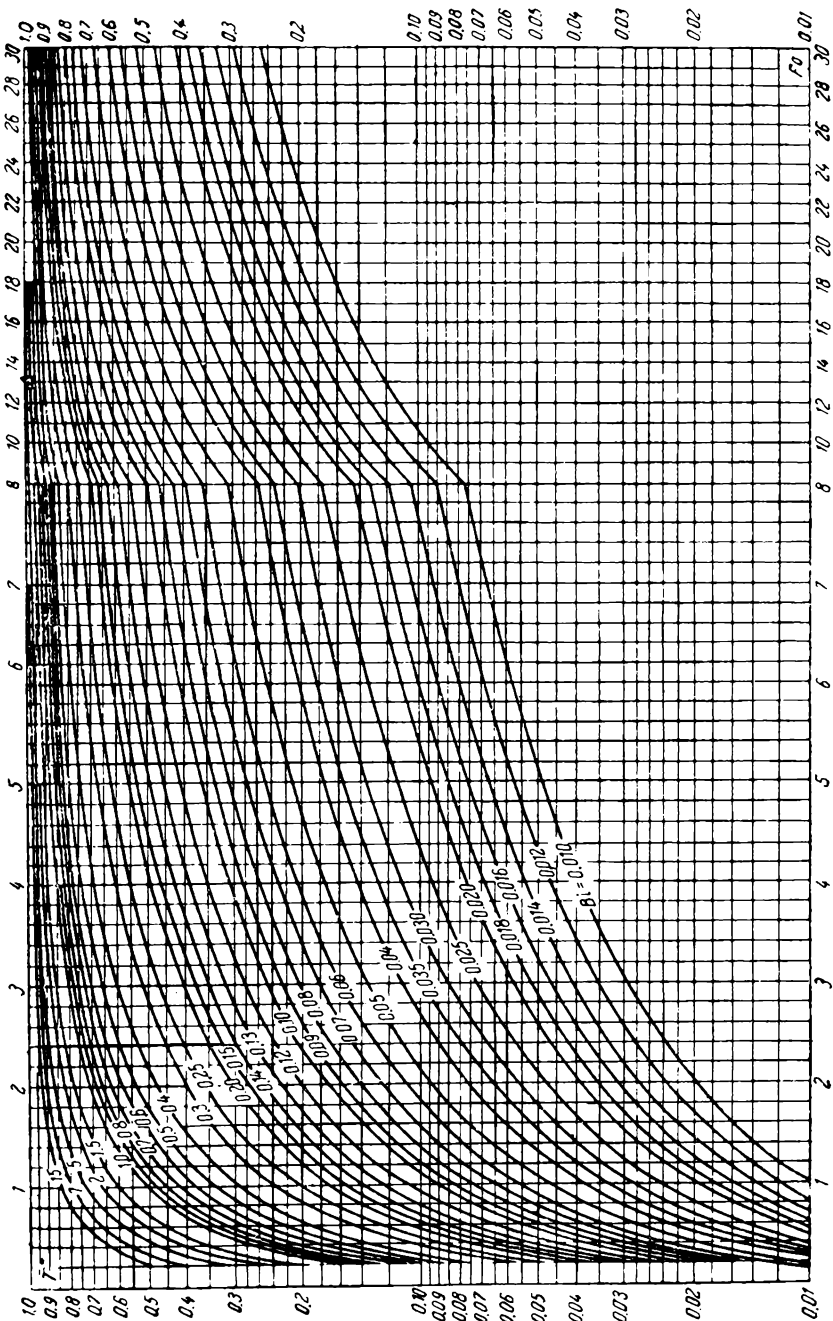


FIGURE 7-6. Relative heat content  $T$  of infinite cylinder, for linear variation of temperature of surroundings

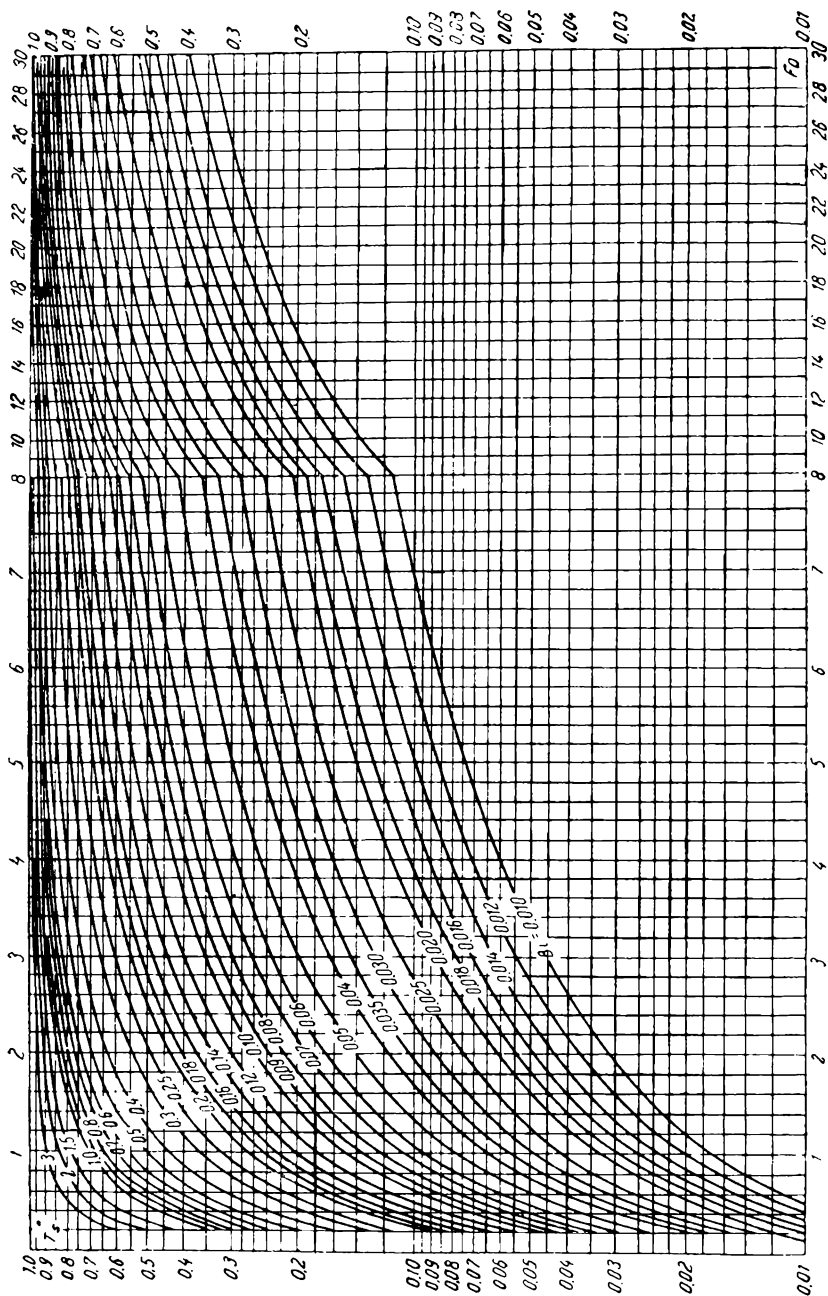


FIGURE 7-7. Dimensionless temperature  $T^*$  at surface of sphere, for linear variation of temperature of surroundings

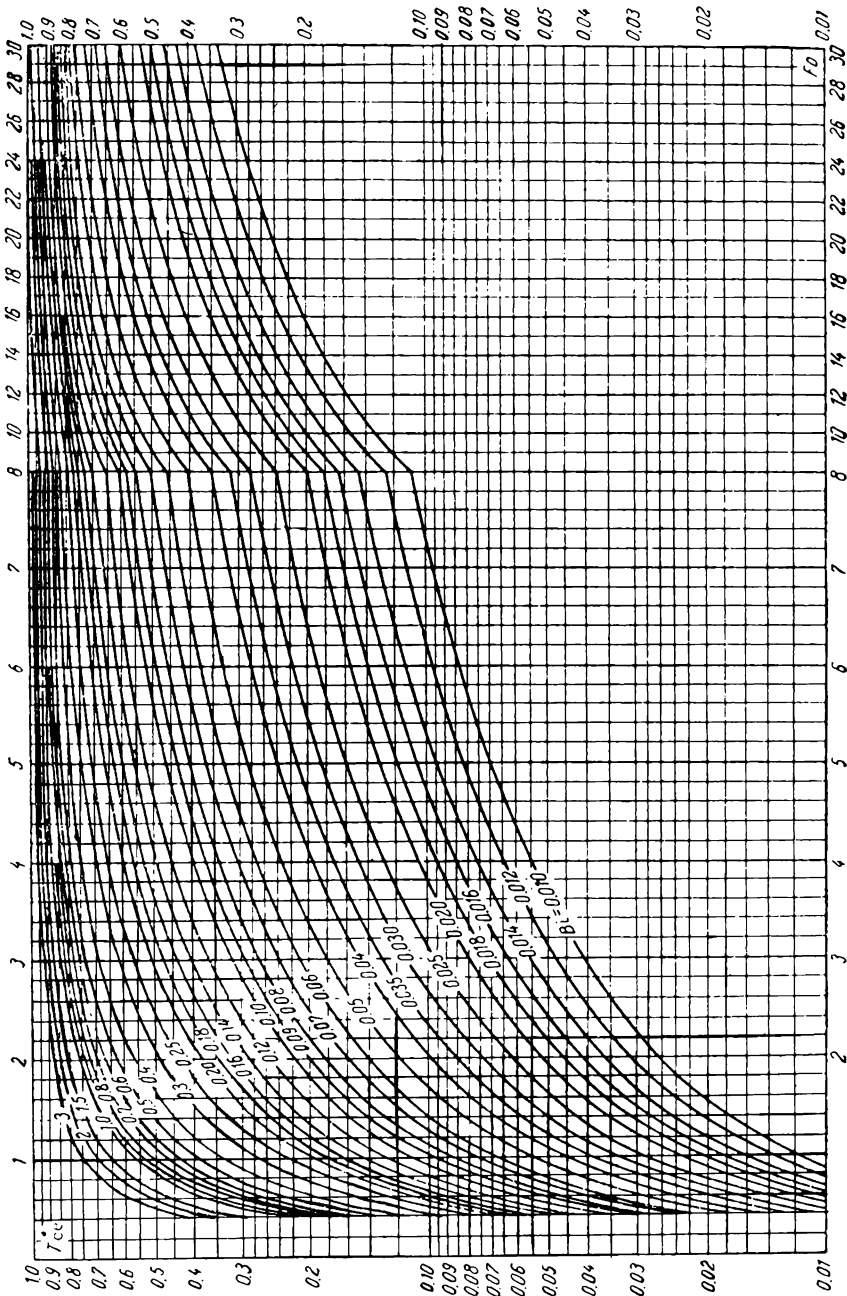


FIGURE 7-8. Dimensionless temperature  $T^*$  at center of sphere, for linear variation of temperature of surroundings

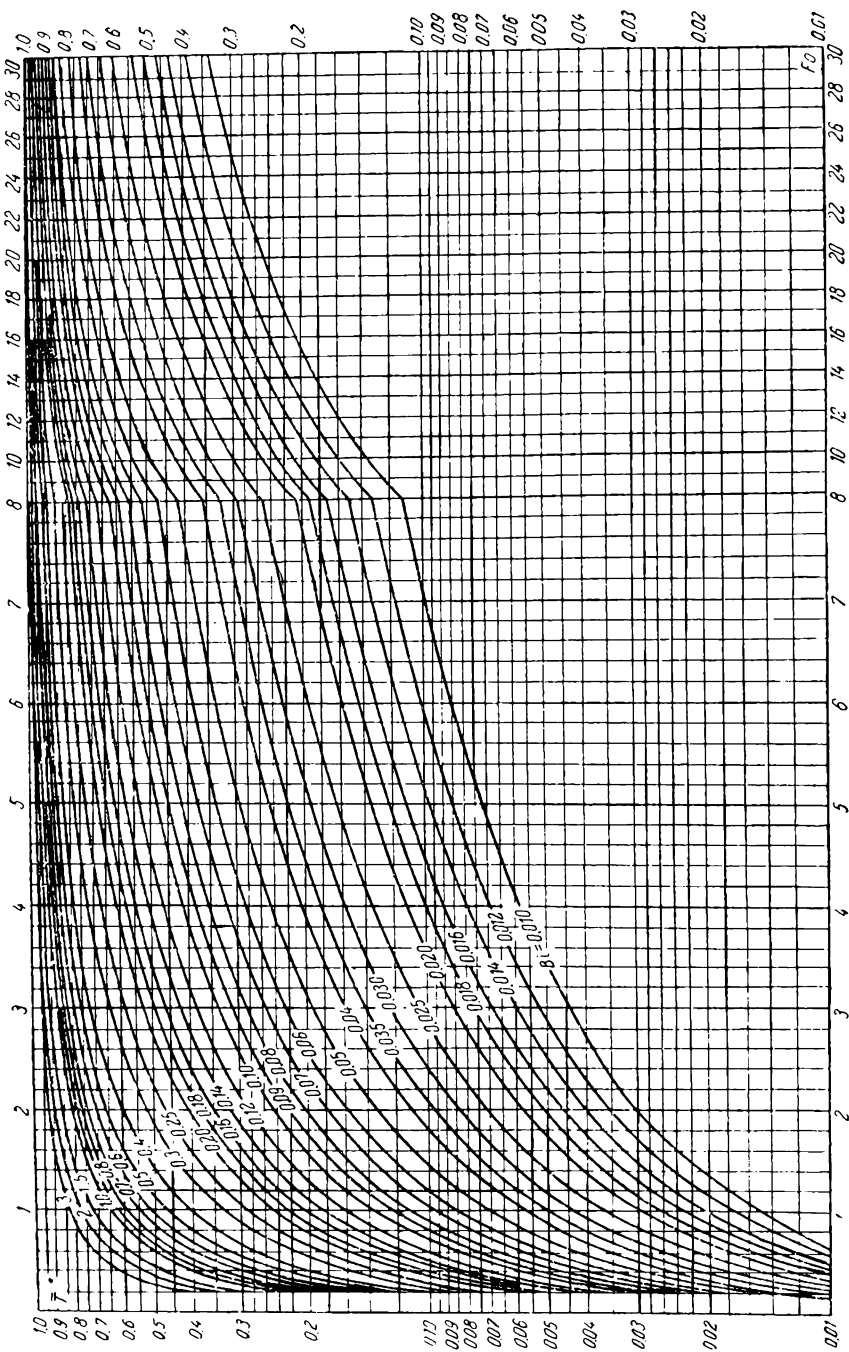


FIGURE 7-9. Relative heat content  $T^*$  of sphere, for linear variation of temperature of surroundings

It should be noted that the above solutions were derived for "pure" heat conduction. However, they may also be used for calculations of mass conduction which is not complicated by heat transfer (in particular, to calculate the kinetics of a diffusion process). In this case, instead of the Biot heat-exchange number the Biot mass-exchange number  $Bi_m$  is introduced, and instead of  $Fo$  the dimensionless number  $Fo_m = Lu Fo$  is introduced.

Let us now consider some particular problems in which the temperature of the surroundings varies exponentially.

1. Uncoupled transfer of heat and mass ( $Pn = 0$ ;  $Ko = 0$ ;  $e = 0$ ):

$$T = 1 - \theta [1 - G_T(\sqrt{Pd'}X) \exp(-Pd'Fo)] - \sum_{n=1}^{\infty} \frac{\mu_n^2 - (1 - \theta)Pd'}{\mu_n^2 - Pd'} A_n \Phi_T(\mu_n X) \exp(-\mu_n^2 Fo); \quad (7-1-15)$$

$$\bar{T} = 1 - \theta [1 - \bar{G}_T(\sqrt{Pd'}) \exp(-Pd'Fo)] - \sum_{n=1}^{\infty} \frac{\mu_n^2 - (1 - \theta)Pd'}{\mu_n^2 - Pd'} D_n \exp(-\mu_n^2 Fo); \quad (7-1-16)$$

where  $\mu_n$ ,  $A_n$ ,  $D_n$ , and  $\Phi_T(\mu_n X)$  are defined by the corresponding relations for the analogous linear problem, considered above. In addition, for an infinite plate

$$G_0 = \frac{\cos \sqrt{Pd'}X}{\cos \sqrt{Pd'} - \frac{1}{Bi_q} \sqrt{Pd'} \sin \sqrt{Pd'}}; \quad \bar{G}_0 = \frac{1}{\sqrt{Pd'} (\cot \sqrt{Pd'} - \frac{1}{Bi_q} \sqrt{Pd'})};$$

for an infinite cylinder

$$G_1 = \frac{J_0(\sqrt{Pd'}X)}{J_0(\sqrt{Pd'}) - \frac{1}{Bi_q} \sqrt{Pd'} J_1(\sqrt{Pd'})}; \quad \bar{G}_1 = \frac{2J_1(\sqrt{Pd'})}{\sqrt{Pd'} J_0(\sqrt{Pd'}) - \frac{1}{Bi_q} Pd' J_1(\sqrt{Pd'})};$$

and for a sphere

$$G_2 = \frac{Bi_q \frac{\sin \sqrt{Pd'}X}{X}}{(Bi_q - 1) \sin \sqrt{Pd'} + \sqrt{Pd'} \cos \sqrt{Pd'}}; \quad \bar{G}_2 = \frac{3 Bi_q (\tan \sqrt{Pd'} - \sqrt{Pd'})}{Pd' [(Bi_q - 1) \tan \sqrt{Pd'} - \sqrt{Pd'}]}.$$

2. If the temperature of the surroundings rises according to the law  $t_0 = t_m - (t_m - t_0) \exp(-\pi\tau)$ , where  $t_m$  is the maximum temperature, at  $\tau = \infty$ , then (7-1-15) and (7-1-16) become

$$\frac{t - t_0}{t_m - t_0} = 1 - G_T(\sqrt{Pd'}X) \exp(-Pd'Fo) - \sum_{n=1}^{\infty} \frac{A_n}{1 - \mu_n^2 / Pd'} \Phi_T(\mu_n X) \exp(-\mu_n^2 Fo); \quad (7-1-17)$$

$$\frac{\bar{t} - t_0}{t_m - t_0} = 1 - \bar{G}_T(\sqrt{Pd'}) \exp(-Pd'Fo) - \sum_{n=1}^{\infty} \frac{D_n}{1 - \mu_n^2 / Pd'} \exp(-\mu_n^2 Fo); \quad (7-1-18)$$

3. If we set  $Bi_q = \infty$  in (7-1-17) and (7-1-18), then we obtain solutions for the case when the surface temperature is an exponential function of time. The form of the solution is the same as solutions (7-1-17) and (7-1-18), but  $\mu_n$ ,  $A_n$ , and  $D_n$  are here defined by the corresponding relations for the analogous linear problem, while  $G_T$  and  $\bar{G}_T$  are:

for a plate

$$G_0 = \frac{\cos \sqrt{Pd'}X}{\cos \sqrt{Pd'}}; \quad \bar{G}_0 = \frac{\tan \sqrt{Pd'}}{\sqrt{Pd'}};$$

for a cylinder

$$G_1 = \frac{J_0(\sqrt{Pd'}X)}{J_0(\sqrt{Pd'})}; \quad \bar{G}_1 = \frac{2J_1(\sqrt{Pd'})}{\sqrt{Pd'} J_0(\sqrt{Pd'})};$$

and for a sphere

$$G_2 = \frac{\sin \sqrt{Pd'}X}{X \sin \sqrt{Pd'}}; \quad \bar{G}_2 = \frac{3 (\tan \sqrt{Pd'} - \sqrt{Pd'})}{Pd' \tan \sqrt{Pd'}}.$$

The variation with time of the average dimensionless temperature of the body,  $\bar{T} = \frac{t - t_o}{t_m - t_o}$ , for various values of  $Pd'$  and for  $Bi_q = \infty$ , is shown in Figures 7-10 through 7-12.

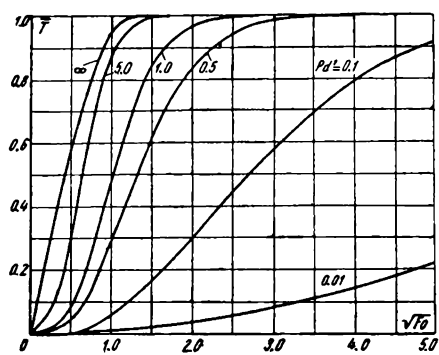


FIGURE 7-10. Time variation of average dimensionless plate temperature, for various values of  $Pd'$ . At the plate

$$\text{surface } T_s = \frac{t_m - t_s}{t_m - t_o} = \exp(-Pd' Fo)$$

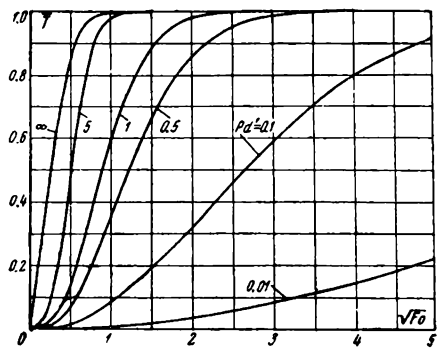


FIGURE 7-11. Time variation of average dimensionless cylinder temperature, for various values of  $Pd'$ . At the

$$\text{cylinder surface } T_s = \frac{t_m - t_s}{t_m - t_o} = \exp(-Pd' Fo)$$

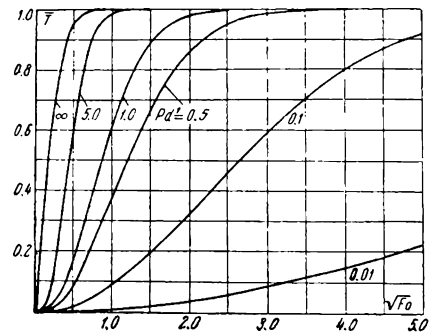


FIGURE 7-12. Time variation of average dimensionless sphere temperature, for various values of  $Pd'$ . At sphere

$$\text{surface } T_s = \frac{t_m - t_s}{t_m - t_o} = \exp(-Pd' Fo)$$



## 7-2. Heat and Mass Transfer in a Medium with Variable Temperature

If the temperature of the medium surrounding a body varies with time, then the Predvoditelev number (Pd), characterizing the rate of this variation, appears in the analytical solution. Let us now evaluate the effect of the dimensionless number Pd on the heat and mass transfer when the temperature of the heat carrier drops linearly with time:  $t_c = t_{co} - b\tau$ .

As Fo increases, the dimensionless temperature first rises, then reaches a maximum ( $T_{\max}$ ), and finally begins to fall (Figure 7-13). The

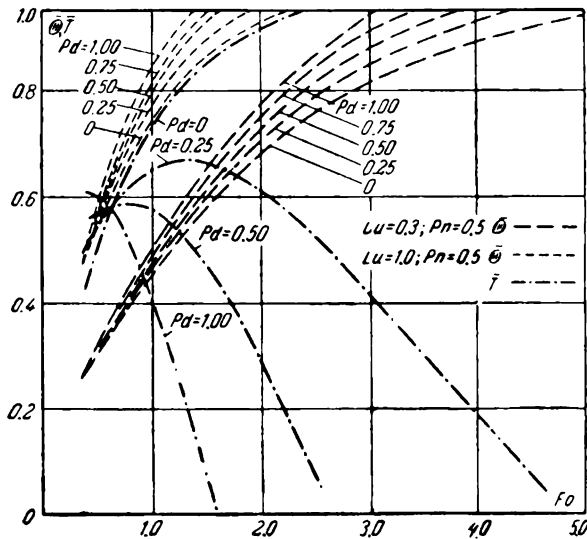


FIGURE 7-13. Dimensionless temperature  $\bar{T}$  and dimensionless mass-transfer potential  $\bar{\theta}$  as functions of Fourier number, for various values of Pd and for Pn = 0.5

greater the value of Pd, the earlier (for low Fo and  $\theta$ ) the temperature of the material will attain a maximum. For all  $Pd < 0.7$  the absolute value of the temperature maximum decreases with increasing Pd. Thus for

$$Pd=0, \quad Fo_{\max}=2.3 \text{ and } \bar{T}_{\max}=1.0;$$

$$Pd=0.25, \quad Fo_{\max}=1.3 \text{ and } \bar{T}_{\max}=0.64;$$

$$Pd=0.50, \quad Fo_{\max}=0.8 \text{ and } \bar{T}_{\max}=0.59$$

[where  $Fo_{\max}$  is the value of Fo at maximum temperature]. For increasing Lu and decreasing Pn, the value of the dimensionless mass-transfer potential to which the temperature maximum corresponds increases. Beyond  $T_{\max}$ , a sharper temperature drop is observed for higher values of Pd (Figure 7-14).

It is interesting to consider the nature of the temperature variations at the center and surface of the body. Up to  $\bar{T}=T_{\max}$ , the temperature at the surface rises considerably more rapidly than that at the center of the material.

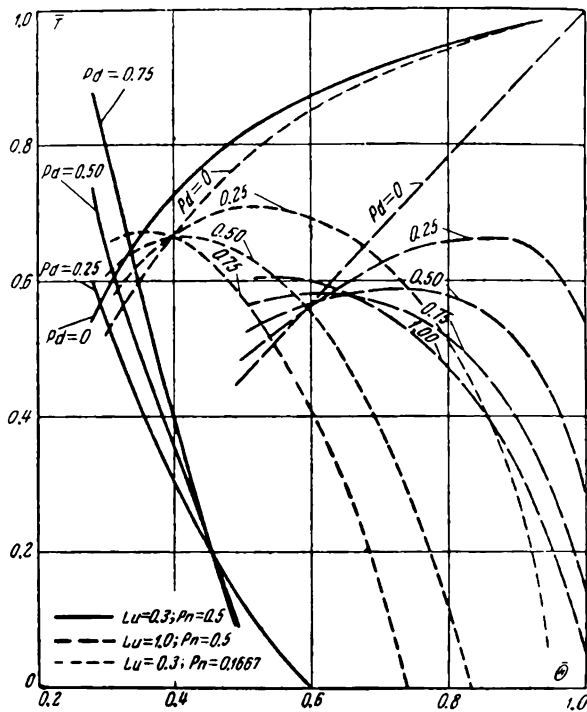


FIGURE 7-14. Development of the  $\bar{T} = f(\bar{\theta})$  curves as  $P_d$  varies, for different values of  $Lu$  and  $Pn$

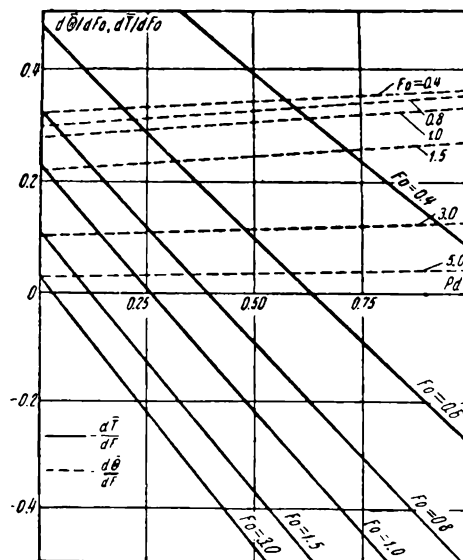


FIGURE 7-15. Rates of variation of dimensionless transfer potentials as functions of  $P_d$

Once  $T_{\max}$  has been reached, however, the surface layer of the body begins to become cooled, since the temperature of the surroundings is then lower than the temperature of the body. On the other hand, heating continues at the center, since  $t_s > t_{ce}$ . As soon as  $t_{ce} \approx t_s$ , all the layers of the body begin to cool off. The moment at which the maximum temperature occurs, as well as its absolute value, must be known for the proper control of various technological processes. In order to prevent condensation of the heat carrier during certain transfer processes, it is advisable to stop the development of the process at the moment  $Fo_{\max}$ .

As  $Pd$  increases, the rate of heating of the material decreases linearly (Figure 7-5). The variation in mass-transfer potential and in its rate as the Predvoditelev number varies is negligible; they both increase with the increases in  $Pd$  and  $Fo$ . Thus, for  $Fo = 0.4$  the increase in the rate of variation  $\left(\frac{d\bar{\theta}}{dFo}\right)$  of mass-transfer potential from  $Pd = 0.25$  to  $Pd = 1.0$  is 8.5% and the variation in  $\bar{\theta}$  is 3.5%. Accordingly, for  $Fo = 0.8$  the rises are 11.8% and 5.3%. As  $Lu$  increases, this same dependence of  $\bar{\theta}$  on  $Pd$  is maintained, but is observed for higher values of  $\bar{\theta}$ .

### 7-3. Heat and Mass Transfer with Periodic Variation of Temperature of Surroundings

Periodic processes are among the most interesting of all unsteady-state transfer phenomena, particularly periodic heat-transfer processes. The latter may be observed, for example, in internal-combustion engines, in cyclic regenerators, and in various shielding devices. Such problems are important in the theory of automatic control and regulation, in meteorology and agrophysics, in the determination of diffusion coefficients, etc. Let us now consider some problems in which either the temperature of the surroundings or that of the body surface varies according to a simple harmonic law.

#### a) Transfer of heat and matter in a semi-infinite medium

Let us examine heat and mass transfer in soils. The experiments of Porkhaev /2/, Onchukov /3/, and others have shown that the transfers of heat and moisture in soils are coupled and that they obey basic laws which are the same as those considered by us for other media. Variations in the transfer potentials thus take place approximately as simple harmonic oscillations. The problems in this subsection will be solved in terms of dimensional quantities.

We have the system of differential equations:

$$\frac{\partial t(x, \tau)}{\partial \tau} = a_q \frac{\partial^2 t(x, \tau)}{\partial x^2} + \epsilon p \frac{c_m}{c_q} \frac{\partial \theta}{\partial \tau}; \quad (7-3-1)$$

$$\frac{\partial \theta(x, \tau)}{\partial \tau} = a_m \frac{\partial^2 \theta(x, \tau)}{\partial x^2}. \quad (7-3-2)$$

Let us assume that the heat transfer is subject to boundary conditions of

the first kind:

$$t(0, \tau) = t_m \sin \omega \tau, \quad (7-3-3)$$

where  $t_m$  is the amplitude of the temperature oscillations at the surface of the medium, and  $\omega = 2\pi\nu$  is the angular frequency ( $\frac{1}{\nu}$  being the oscillation period). The boundary conditions for equation (7-3-2) may be written as

$$\lambda_m \frac{\partial \theta(0, \tau)}{\partial x} + \alpha_m [\theta_0 - \theta(0, \tau)] = 0, \quad (7-3-4)$$

where  $\theta_0$  is the transfer potential for vapor in the air over the soil. In addition, we assume that

$$t(x, 0) = 0; \quad \theta(x, 0) = \theta_0 = \text{const}; \quad (7-3-5)$$

$$\frac{\partial t(\infty, \tau)}{\partial x} = 0; \quad \frac{\partial \theta(\infty, \tau)}{\partial x} = 0. \quad (7-3-6)$$

The transform solution of equation (7-3-2) subject to conditions (7-3-5) and (7-3-6) is known to be

$$\theta_L = \frac{\theta_0}{s} - \frac{\theta_0 - \theta_e}{s \left(1 + \frac{\lambda_m}{\alpha_m} \sqrt{\frac{s}{a_m}}\right)} \exp\left(-\sqrt{\frac{s}{a_m}} x\right). \quad (7-3-7)$$

If we now apply a Laplace transformation to equation (7-3-1), taking into account solution (7-3-7) and conditions (7-3-6) and (7-3-3), then we obtain

$$\begin{aligned} t_L = & \frac{\omega t_m}{s^2 + \omega^2} \exp\left(-\sqrt{\frac{s}{a_q}} x\right) + \frac{A \exp\left(-\sqrt{\frac{s}{a_q}} x\right)}{\frac{s}{a_m} \left(1 + \frac{\lambda_m}{\alpha_m} \sqrt{\frac{s}{a_m}}\right)} - \\ & - \frac{A \exp\left(-\sqrt{\frac{s}{a_m}} x\right)}{\frac{s}{a_m} \left(1 + \frac{\lambda_m}{\alpha_m} \sqrt{\frac{s}{a_m}}\right)}, \end{aligned} \quad (7-3-8)$$

where

$$A = \varepsilon \frac{\rho_{cm} \theta_0 - \theta_0}{c_\sigma 1 - Lu}.$$

The inverse transform of the first term is found using the expansion theorem, while those of the last two terms are taken from transform tables. As a result, the real part of the solution has the form

$$\begin{aligned} t(x, \tau) = & t_m \exp\left(-x \sqrt{\frac{\omega}{2a_q}}\right) \sin\left(\omega \tau - x \sqrt{\frac{\omega}{2a_q}}\right) + \\ & + A \frac{a_m}{a_q} \left\{ \text{erfc} \frac{x}{2 \sqrt{a_q \tau}} - \exp\left[\frac{a_m}{\lambda_m} \sqrt{\frac{a_m}{a_q}} x + \left(\frac{a_m}{\lambda_m}\right)^2 a_m \tau\right] \times \right. \\ & \times \text{erfc}\left(\frac{a_m}{\lambda_m} \sqrt{a_m \tau} + \frac{x}{2 \sqrt{a_q \tau}}\right) \Big\} - A \left\{ \text{erfc} \frac{x}{2 \sqrt{a_m \tau}} - \exp\left[\frac{a_m}{\lambda_m} x + \left(\frac{a_m}{\lambda_m}\right)^2 a_m \tau\right] \times \right. \\ & \times \text{erfc}\left(\frac{a_m}{\lambda_m} \sqrt{a_m \tau} + \frac{x}{2 \sqrt{a_m \tau}}\right) \Big\} \end{aligned}$$

or in terms of dimensionless quantities

$$\begin{aligned} \frac{t(x, \tau)}{t_m} = & \exp(-\sqrt{\text{Pd}_x/2}) \sin(\text{Pd}_x \text{Fo}_x - \sqrt{\text{Pd}_x/2}) + \\ & + \frac{*K_o \text{Lu}}{1 - \text{Lu}} \left\{ \text{erfc} \frac{1}{2\sqrt{\text{Fo}_x}} - \exp[(\text{Bi}_m)_x \sqrt{\text{Lu}} + (\text{Bi}_m)_x^2 \text{Lu} \text{Fo}_x] \times \right. \\ & \times \text{erfc} \left[ (\text{Bi}_m)_x \sqrt{\text{Lu} \text{Fo}_x} + \frac{1}{2\sqrt{\text{Fo}_x}} \right] \left. \right\} - \frac{*K_o}{1 - \text{Lu}} \left\{ \text{erfc} \frac{1}{2\sqrt{\text{Lu} \text{Fo}_x}} - \right. \\ & \left. - \text{erfc}[(\text{Bi}_m)_x + (\text{Bi}_m)_x^2 \text{Lu} \text{Fo}_x] \text{erfc} \left[ (\text{Bi}_m)_x \sqrt{\text{Lu} \text{Fo}_x} + \frac{1}{2\sqrt{\text{Lu} \text{Fo}_x}} \right] \right\}. \end{aligned} \quad (7-3-9)$$

In this equation  $\text{Fo}_x = \frac{a_q \tau}{x^2}$  is the Fourier number for the coordinate  $x$ ,  $(\text{Bi}_m)_x = \frac{a_m}{\lambda_m} x$  is the Biot mass-exchange number for the coordinate  $x$ , and  $\text{Pd}_x = \frac{\omega}{a_q} x^2 = \frac{2\pi\nu}{a_q} x^2$  is the Predvoditelev number for the coordinate  $x$ .

In dimensionless form, the inverse transform of (7-3-7) is

$$\begin{aligned} \frac{\theta_s - \theta(x, \tau)}{\theta_s - \theta_c} = & \text{erfc} \frac{1}{2\sqrt{\text{Lu} \text{Fo}_x}} - \exp[(\text{Bi}_m)_x + (\text{Bi}_m)_x^2 \text{Lu} \text{Fo}_x] \times \\ & \times \text{erfc} \left[ \frac{1}{2\sqrt{\text{Lu} \text{Fo}_x}} + (\text{Bi}_m)_x \sqrt{\text{Lu} \text{Fo}_x} \right]. \end{aligned} \quad (7-3-10)$$

Let us now analyze solutions (7-3-9) and (7-3-10). At the beginning of the process the transfer-potential fields will be affected by their initial distributions, the effects of which then rapidly decrease. As the argument  $u$

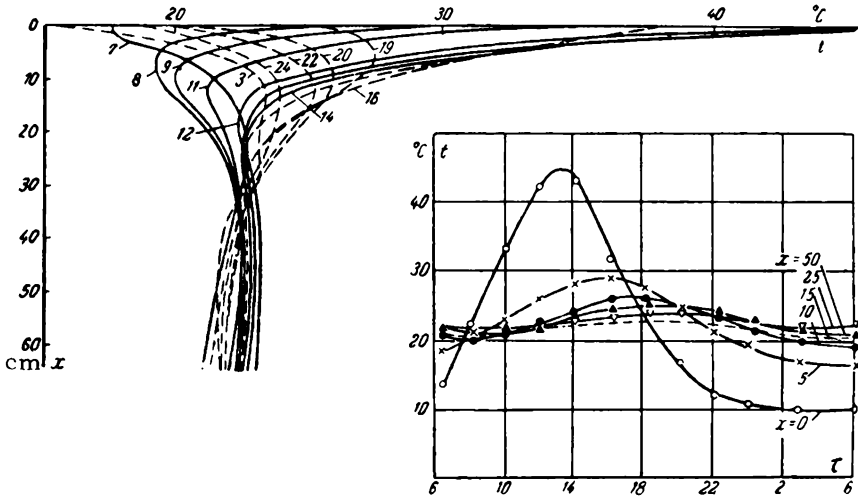


FIGURE 7-16. Temperature distribution throughout the soil interior at various times in a 24-hour period

of the [error] function  $\text{erfc } u$  increases, this function decreases rapidly until it becomes practically zero at  $u = 2.8$ . Consequently, it follows from equation (7-3-10) that a wave of dimensionless mass-transfer potential is rapidly attenuated as it penetrates more deeply into the soil, and as a result the moisture transfer which may occur due to the daily temperature

oscillations at the surface will propagate to only a relatively small depth. The time variation in dimensionless mass-transfer potential at the surface of the medium is

$$\theta(0, Fo) = 1 - \exp[(Bi_m)_x^2 Lu Fo_x] \operatorname{erfc}[(Bi_m)_x \sqrt{Lu Fo_x}].$$

The experiments of Onchukov /3/ confirm this result. If we set  $(Bi_m)_x = \infty$  in equation (7-3-10), then the solution is simplified considerably to

$$\theta = \operatorname{erfc} \frac{1}{2 \sqrt{Lu Fo_x}}.$$

The thermal-wave equation (7-3-9) represents the overall effect of heat and mass transfer. The latter creates a very complicated temperature distribution in the surface layer (Figure 7-16). As time goes by, the temperature wave becomes attenuated and the temperature field becomes more and more steady-state periodical, satisfying the equation

$$t(x, \tau) = t_m \exp\left(-x \sqrt{\frac{\omega}{2a_q}}\right) \sin\left(\omega \tau - x \sqrt{\frac{\omega}{2a_q}}\right) \quad (7-3-11)$$

or

$$\frac{t}{t_m} = \exp(-\sqrt{Pd_x/2}) \sin(Pd_x Fo_x - \sqrt{Pd_x/2}). \quad (7-3-11')$$

Equations (7-3-11) and (7-3-11') indicate that under these conditions the oscillation amplitude decreases exponentially with depth:  $A(x) = t_m \times \exp\left(-x \sqrt{\frac{\omega}{2a_q}}\right)$ . The temperature oscillations in the soil have a phase shift, and the time lags ( $\Delta \tau_{lag}$ ) of the temperature extrema, relative to the corresponding times at the surface, are proportional to the depth

$$\Delta \tau_{lag} = \sqrt{\frac{1}{2a_q \omega}} x.$$

For a given depth the time lag is directly proportional to the square root of the temperature-oscillation period  $\left(\frac{1}{\nu}\right)$ . The penetration depth of temperature into the soil depends on the temperature-oscillation period at the surface: the shorter this period, the smaller is the penetration depth for temperature.

If we set  $(Bi_m)_x = \infty$ , then solution (7-3-9) simplifies to

$$\begin{aligned} \frac{t(x, \tau)}{t_m} = & \exp(-\sqrt{Pd_x/2}) \sin(Pd_x Fo_x - \sqrt{Pd_x/2}) + \\ & + \frac{sKo}{1 - Lu} \left[ Lu \operatorname{erfc} \frac{1}{2 \sqrt{Fo_x}} - \operatorname{erfc} \frac{1}{2 \sqrt{Lu Fo_x}} \right], \end{aligned}$$

indicating that the steady-state temperature distribution involves the difference of two probability functions, which depend on the homochronicity numbers for the variations of the thermal and mass processes and also on the inertial properties of the medium and the rate of phase transitions.

On the basis of known properties of the propagation of temperature oscillations through soils, and taking into account the experimental data, this propagation may be considered using just the heat-conduction equation alone, where the effects of the mass transfer and phase transitions are taken into account by means of a source term which varies according to some harmonic law. This problem has been solved by Porkhaev and Smirnov /4/.

Let us consider the solution of the heat-conduction equation

$$\frac{\partial t(x, \tau)}{\partial \tau} = a_q \frac{\partial^2 t(x, \tau)}{\partial x^2} + \frac{w_0 \rho}{c_q \gamma} \cos \omega_1 \tau$$

subject to the limiting conditions

$$t(x, 0) = 0;$$

$$t(0, \tau) = t_m \cos \omega \tau;$$

$$t(\infty, \tau) = 0;$$

where  $w_0$  is the specific strength of the heat source. The solution of this problem is

$$\begin{aligned} t(x, \tau) = & t_m \exp\left(-x \sqrt{\frac{\omega}{2a_q}}\right) \cos\left(\omega \tau - x \sqrt{\frac{\omega}{2a_q}}\right) - \\ & - \frac{w_0 \rho}{\omega_1 c_q \gamma} \exp\left(-x \sqrt{\frac{\omega_1}{2a_q}}\right) \sin\left(\omega_1 \tau - x \sqrt{\frac{\omega_1}{2a_q}}\right) + \frac{w_0 \rho}{\omega_1 c_q \gamma} \sin \omega_1 \tau - \\ & - \frac{1}{\pi} \int_0^\infty e^{-u\tau} \sin x \sqrt{\frac{u}{a_q}} \left[ \frac{t_m u}{u^2 + \omega} + \frac{w_0 \rho}{c_q \gamma} \cdot \frac{1}{u^2 + \omega_1} \right] du. \end{aligned} \quad (7-3-12)$$

If we consider only real solutions, then the integral in (7-3-12) disappears. In the absence of a source ( $w_0 = 0$ ) the following classical solution is obtained /5/:

$$\begin{aligned} t(x, \tau) = & t_m \exp\left(-x \sqrt{\frac{\omega}{2a_q}}\right) \cos\left(\omega \tau - x \sqrt{\frac{\omega}{2a_q}}\right) - \\ & - \frac{t_m}{\pi} \int_0^\infty e^{-u\tau} \sin x \sqrt{\frac{u}{a_q}} \cdot \frac{u}{u^2 + \omega} du. \end{aligned} \quad (7-3-13)$$

For the case of a sinusoidal initial temperature distribution at the surface of the medium, equation (7-3-13) becomes

$$\begin{aligned} t(x, \tau) = & t_m \exp\left(-x \sqrt{\frac{\omega}{2a_q}}\right) \sin\left(\omega \tau - x \sqrt{\frac{\omega}{2a_q}}\right) + \\ & + \frac{\omega}{\pi} \int_0^\infty e^{-u\tau} \sin x \sqrt{\frac{u}{a_q}} \cdot \frac{du}{u^2 + \omega^2}. \end{aligned}$$

Some other particular solutions for the temperature-wave propagation are given by Carslaw and Jaeger /6/.

#### b) Thermal waves in bodies of regular geometrical shape.

Boundary conditions of the third and first kinds

Let us now consider the propagation of a thermal wave in a body of regular geometrical shape. The one-dimensional heat-conduction equation

$$\frac{\partial T}{\partial Fo} = \frac{\partial^2 T}{\partial X^2} + \frac{F}{X} \cdot \frac{\partial T}{\partial X} \quad (7-3-14)$$

will be solved with the following limiting conditions:

$$T(X, 0) = 0; \quad (7-3-15)$$

$$\frac{\partial T(0, Fo)}{\partial X} = 0; \quad (7-3-16)$$

$$-\frac{\partial T(1, Fo)}{\partial X} + Bi_q [T_o(Fo) - T(1, Fo)] = 0, \quad (7-3-17)$$

where  $T = \frac{t}{t_m}$  ( $t_m$  being the maximum temperature of the medium, namely the amplitude of oscillation of the temperature of the medium);  $T_c(Fo) = \cos Pd Fo$  (that is,  $t_c(\tau) = t_m \cos \omega \tau$ ), and  $Pd = \frac{\omega}{\alpha_q} R^2$  is the Predvoditelev number for a harmonic variation of the temperature of the medium.

Let us treat in detail the solution for an infinite plate. For the other body shapes only the final results will be stated.

The transform solution of equation (7-3-14), subject to conditions (7-3-15) and (7-3-16), is

$$T_L(X, s) = B \operatorname{ch} \sqrt{s} X. \quad (7-3-18)$$

As usual, the constant  $B$  is determined from the boundary condition, which in terms of transforms is

$$-T'_L(1, s) + Bi_q \left[ \frac{s}{s^2 + Pd^2} - T_L(1, s) \right] = 0.$$

From this, we obtain

$$B = \frac{s}{(s^2 + Pd^2) \left( \operatorname{ch} \sqrt{s} + \frac{1}{Bi_q} \sqrt{s} \operatorname{sh} \sqrt{s} \right)}. \quad (7-3-19)$$

Solution (7-3-18), taking into account (7-3-19), represents the ratio of two generalized polynomials, which satisfies the conditions for use of the expansion theorem. The roots of the denominator in this case are: 1)  $s_1 = iPd$  and  $s_2 = -iPd$ ; 2)  $s_n = -\mu_n^2$ . The latter relation gives an infinite number of roots, which are determined from the usual characteristic equation. After carrying out the necessary calculations and transformations, we obtain the final solution as

$$T = \frac{t}{t_m} = \frac{1}{2} [N_i \exp(i Pd Fo) + N_{-i} \exp(-i Pd Fo)] - \sum_{n=1}^{\infty} \frac{\mu_n^4}{\mu_n^4 + Pd^2} A_n \cos \mu_n X \exp(-\mu_n^2 Fo), \quad (7-3-20)$$

where

$$N_i = \frac{\operatorname{ch} \sqrt{iPd} X}{\operatorname{ch} \sqrt{iPd} + \frac{1}{Bi_q} \sqrt{iPd} \operatorname{sh} \sqrt{iPd}};$$

$$N_{-i} = \frac{\operatorname{ch} \sqrt{-iPd} X}{\operatorname{ch} \sqrt{-iPd} + \frac{1}{Bi_q} \sqrt{-iPd} \operatorname{sh} \sqrt{-iPd}};$$

and where  $A_n$  and  $\mu_n$  are the coefficients and characteristic-equation roots, determined from the familiar relations describing uncoupled transfer in an infinite plate.

As time goes by, the sum in (7-3-20) decreases, so that above some value ( $Fo > Fo_1$ ) a quasisteady state is established, and all series terms except the first two can be neglected. For the quasisteady state we obtain

$$T = \frac{1}{2} [N_i \exp(i Pd Fo) + N_{-i} \exp(-i Pd Fo)] = \frac{1}{2} [(N_i + N_{-i}) \cos Pd Fo + i(N_i - N_{-i}) \sin Pd Fo] = (N_i N_{-i})^{1/2} \cos \left[ Pd Fo - \arctan \left( i \frac{N_i - N_{-i}}{N_i + N_{-i}} \right) \right]. \quad (7-3-20')$$



With boundary conditions of the first kind ( $Bi_f = \infty$ ), this simplifies to

$$T(\pm 1, Fo) = \cos PdFo.$$

In this case

$$N_i = \frac{\text{ch} \sqrt{i Pd} X}{\text{ch} \sqrt{i Pd}} \quad \text{and} \quad N_{-i} = \frac{\text{ch} \sqrt{-i Pd} X}{\text{ch} \sqrt{-i Pd}}.$$

It follows from (7-3-20') that the temperature at any point of the plate performs a simple harmonic oscillation with the same frequency as, but with a phase lag of  $\arctan \left( i \frac{N_i - N_{-i}}{N_i + N_{-i}} \right)$  relative to, the temperature oscillation in the medium, since  $T_c(Fo)$  (the Predvoditelev number) is directly proportional to the frequency.

The amplitude of the temperature oscillations at any point of the plate decreases with depth. The maximum amplitude is therefore observed at the surface of the plate ( $X=1$ ), but it is smaller than the amplitude of oscillation of the surrounding temperature by a factor of  $(N_i N_{-i})^{1/2}$ .

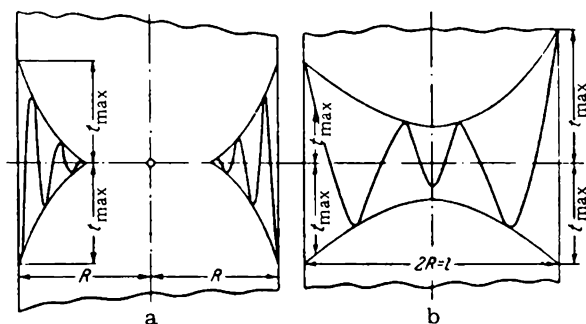


FIGURE 7-17. Penetration of temperature waves into an infinite plate:  
a) of considerable depth; b) of average depth

If the plate is very thick or if the temperature variations take place very rapidly, then the thermal waves propagating into the plate from its two surfaces will be attenuated completely before they reach the center (Figure 7-17, a). In this case each of the two symmetrical halves of the plate behaves like a body of infinite thickness (half-space). This is one limiting case.

In the opposite case (a very thin plate or an extremely slow temperature variation) it may be assumed that the entire plate thickness is subjected to temperature oscillations, without any decrease in amplitude or time lag with respect to the variations at the surface. Under these conditions the temperature throughout the whole thickness of the plate is the same (is independent of  $X$ ).

Intermediate to these two limiting cases, a very thick and a very thin plate, it is possible to consider all the real variants of the problem, especially for shielding devices. As we gradually reduce the thickness of the plate, we obtain the case when the oscillations propagating from both sides meet in the middle of the plate and begin to interfere with one another. The instantaneous temperature propagation in this case is that of Figure 7-17, b.

Coefficients  $N$  and  $N_{-i}$  can be calculated using the relations

$$\sqrt{i} = \pm \sqrt{\frac{1}{2}}(1+i) \text{ and } \sqrt{-i} = \pm \sqrt{\frac{1}{2}}(1-i)$$

(in the given problem the  $\pm$  signs may be omitted, since  $\operatorname{ch} z$  is an even function, whereas the odd function  $\operatorname{sh} z$  is preceded by the factor  $z$ , giving the same result) and familiar relations such as

$$\begin{aligned} \operatorname{ch} \sqrt{i} \operatorname{Pd} X &= \operatorname{ch} \sqrt{\operatorname{Pd}/2} (1+i) X = \operatorname{ch} \sqrt{\operatorname{Pd}/2} X \cos \sqrt{\operatorname{Pd}/2} X + \\ &+ i \operatorname{sh} \sqrt{\operatorname{Pd}/2} X \cdot \sin \sqrt{\operatorname{Pd}/2} X, \end{aligned}$$

that is,

$$\operatorname{ch} z \sqrt{\pm i} = A_0 \pm i B_0; \operatorname{sh} z \sqrt{\pm i} = A_s \pm i B_s.$$

To simplify calculations, some values of coefficients  $A_0$ ,  $B_0$ ,  $A_s$ , and  $B_s$  are listed in Table 7-6.

The average dimensionless temperature over the volume is

$$\begin{aligned} \bar{T} &= \frac{1}{2} [\bar{N}_i \exp(i \operatorname{Pd} \operatorname{Fo}) + \bar{N}_{-i} \exp(-i \operatorname{Pd} \operatorname{Fo})] - \\ &- \sum_{n=1}^{\infty} \frac{\mu_n^4}{\mu_n^4 + \operatorname{Pd}^2} D_n \exp(-\mu_n^2 \operatorname{Fo}), \end{aligned} \quad (7-3-21)$$

where  $D_n$  are constant coefficients defined by (6-5-5') in § 6-5, and coefficients  $\bar{N}_i$  and  $\bar{N}_{-i}$  are

$$\begin{aligned} \bar{N}_i &= \frac{1}{\left[ \sqrt{i} \operatorname{Pd} \coth \sqrt{i} \operatorname{Pd} + \frac{1}{\operatorname{Bi}_q} i \operatorname{Pd} \right]}; \\ \bar{N}_{-i} &= \frac{1}{\sqrt{-i} \operatorname{Pd} \coth \sqrt{-i} \operatorname{Pd} - \frac{1}{\operatorname{Bi}_q} i \operatorname{Pd}}. \end{aligned}$$

The generalized solutions (7-3-20) and (7-3-21) for a plate, cylinder, and sphere are

$$\begin{aligned} T &= \frac{1}{2} [N_i \exp(i \operatorname{Pd} \operatorname{Fo}) + N_{-i} \exp(-i \operatorname{Pd} \operatorname{Fo})] - \\ &- \sum_{n=1}^{\infty} \frac{\mu_n^4}{\mu_n^4 + \operatorname{Pd}^2} A_{nr} \Phi_r(\mu_n X) \exp(-\mu_n^2 \operatorname{Fo}); \\ \bar{T} &= \frac{1}{2} [\bar{N}_i \exp(i \operatorname{Pd} \operatorname{Fo}) + \bar{N}_{-i} \exp(-i \operatorname{Pd} \operatorname{Fo})] - \\ &- \sum_{n=1}^{\infty} \frac{\mu_n^4}{\mu_n^4 + \operatorname{Pd}^2} D_{nr} \exp(-\mu_n^2 \operatorname{Fo}); \end{aligned}$$

where  $A_{nr}$  and  $D_{nr}$  are the previously mentioned coefficients for a plate ( $\Gamma=0$ ), cylinder ( $\Gamma=1$ ), and sphere ( $\Gamma=2$ ); and where  $\Phi_r(\mu_n X)$  is  $(\cos \mu_n X)$  for a plate,  $[J_0(\mu_n X)]$  for a cylinder, and  $\left(\frac{\sin \mu_n X}{\mu_n X}\right)$  for a sphere. The expressions defining  $N_i$ ,  $N_{-i}$ ,  $\bar{N}_i$ , and  $\bar{N}_{-i}$  for a plate have been given above, and those for a cylinder and sphere are:

TABLE 7-6  
Values of coefficients  $A_c, B_c, A_s,$  and  $B_s$

$z$	$A_c$	$B_c$	$A_s$	$B_s$
0.01	1.000	0.0001	0.0071	0.0071
0.025	1.000	0.0003	0.0177	0.0177
0.05	1.000	0.001	0.035	0.035
0.075	1.000	0.003	0.050	0.050
0.10	1.000	0.005	0.070	0.071
0.15	1.000	0.011	0.106	0.106
0.20	1.000	0.020	0.140	0.142
0.25	1.000	0.031	0.175	0.179
0.30	1.000	0.045	0.209	0.215
0.35	0.999	0.062	0.242	0.253
0.40	0.999	0.080	0.275	0.290
0.45	0.998	0.095	0.307	0.329
0.50	0.997	0.125	0.339	0.369
0.55	0.996	0.151	0.369	0.408
0.60	0.994	0.180	0.398	0.449
0.65	0.993	0.212	0.427	0.492
0.70	0.990	0.245	0.454	0.534
0.75	0.987	0.281	0.479	0.578
0.80	0.983	0.320	0.505	0.625
0.85	0.978	0.362	0.526	0.670
0.90	0.973	0.404	0.547	0.718
0.95	0.966	0.451	0.566	0.771
1.00	0.958	0.499	0.583	0.819
1.05	0.950	0.549	0.599	0.871
1.10	0.938	0.601	0.612	0.924
1.15	0.927	0.658	0.622	0.980
1.20	0.916	0.705	0.629	1.026
1.25	0.898	0.776	0.636	1.096
1.30	0.881	0.839	0.639	1.155
1.35	0.862	0.906	0.639	1.216
1.40	0.834	0.970	0.636	1.271
1.50	0.790	1.108	0.621	1.410
1.60	0.727	1.258	0.590	1.550
1.70	0.654	1.411	0.546	1.692
1.80	0.566	1.571	0.484	1.839
1.90	0.460	1.741	0.402	1.995
2.0	0.340	1.911	0.302	2.151
2.1	0.199	2.087	0.180	2.312
2.2	0.038	2.261	0.036	2.477
2.3	-0.146	2.440	-0.135	2.636
2.4	-0.355	2.616	-0.291	2.798
2.5	-0.587	2.787	-0.554	2.954
2.6	-0.851	2.953	-0.808	3.107
2.7	-1.105	3.093	-1.057	3.234
2.8	-1.468	3.259	-1.413	3.386
2.9	-1.821	3.389	-1.761	3.504
3.0	-2.206	3.502	-2.143	3.604
3.2	-3.086	3.656	-3.021	3.639
3.4	-4.129	3.695	-4.057	3.752
3.6	-5.304	3.553	-5.239	3.602
3.8	-6.629	3.256	-6.567	3.239
4.0	-8.093	2.589	-8.038	2.607
4.2	-9.628	1.660	-9.578	1.669
4.4	-11.227	0.353	-11.183	0.355
4.6	-12.839	-1.393	-12.800	-1.393
4.8	-14.437	-3.715	-14.404	-3.724
5.0	-15.851	-6.568	-15.824	-6.759
5.5	-17.939	-16.982	-17.984	-16.996
6.0	-15.794	-30.897	-15.787	-30.911
6.5	-5.796	-49.149	-5.795	-49.159
7.0	16.616	-68.601	16.614	-68.601
7.5	55.532	-83.365	55.529	-83.369
8.0	115.788	-83.919	115.786	-83.921
8.5	196.188	-54.969	196.186	-54.970
9.0	289.033	23.791	289.019	23.791
9.5	375.199	175.924	375.197	175.025
10.0	414.391	417.288	414.525	417.289

for an infinite cylinder,

$$N_i = \frac{I_0(\sqrt{i \text{Pd}} X)}{I_0(\sqrt{i \text{Pd}}) + \frac{1}{\text{Bi}_q} \sqrt{i \text{Pd}} I_1(\sqrt{i \text{Pd}})};$$

$$N_{-i} = \frac{I_0(\sqrt{-i \text{Pd}} X)}{I_0(\sqrt{-i \text{Pd}}) + \frac{1}{\text{Bi}_q} \sqrt{-i \text{Pd}} I_1(\sqrt{-i \text{Pd}})};$$

$$\bar{N}_i = \frac{2I_1(\sqrt{i \text{Pd}})}{[I_0(\sqrt{i \text{Pd}}) + \frac{1}{\text{Bi}_q} \sqrt{i \text{Pd}} I_1(\sqrt{i \text{Pd}})] \sqrt{i \text{Pd}}};$$

$$\bar{N}_{-i} = \frac{2I_1(\sqrt{-i \text{Pd}})}{[I_0(\sqrt{-i \text{Pd}}) + \frac{1}{\text{Bi}_q} \sqrt{-i \text{Pd}} I_1(\sqrt{-i \text{Pd}})] \sqrt{-i \text{Pd}}};$$

and for  $\text{Bi}_q = \infty$

$$N_i = \frac{I_0(\sqrt{i \text{Pd}} X)}{I_0(\sqrt{i \text{Pd}})}; \quad N_{-i} = \frac{I_0(\sqrt{-i \text{Pd}} X)}{I_0(\sqrt{-i \text{Pd}})};$$

$$\bar{N}_i = \frac{2I_1(\sqrt{i \text{Pd}})}{\sqrt{i \text{Pd}} I_0(\sqrt{i \text{Pd}})}, \quad \bar{N}_{-i} = \frac{2I_1(\sqrt{-i \text{Pd}})}{\sqrt{-i \text{Pd}} I_0(\sqrt{-i \text{Pd}})};$$

for a sphere,

$$N_i = \frac{\text{Bi}_q \text{sh} \sqrt{i \text{Pd}} X / X}{(\text{Bi}_q - 1) \text{sh} \sqrt{i \text{Pd}} + \sqrt{i \text{Pd}} \text{ch} \sqrt{i \text{Pd}}};$$

$$N_{-i} = \frac{\text{Bi}_q \text{sh} \sqrt{-i \text{Pd}} X / X}{(\text{Bi}_q - 1) \text{sh} \sqrt{-i \text{Pd}} + \sqrt{-i \text{Pd}} \text{ch} \sqrt{-i \text{Pd}}};$$

$$\bar{N}_i = \frac{3\text{Bi}_q (\sqrt{i \text{Pd}} - \tanh \sqrt{i \text{Pd}})}{i \text{Pd} [(\text{Bi}_q - 1) \tanh \sqrt{i \text{Pd}} + \sqrt{i \text{Pd}}]};$$

$$\bar{N}_{-i} = \frac{3\text{Bi}_q (\sqrt{-i \text{Pd}} - \tanh \sqrt{-i \text{Pd}})}{-i \text{Pd} [(\text{Bi}_q - 1) \tanh \sqrt{-i \text{Pd}} + \sqrt{-i \text{Pd}}]};$$

and for  $\text{Bi}_q = \infty$

$$N_i = \frac{\text{sh} \sqrt{i \text{Pd}} X / X}{\text{sh} \sqrt{i \text{Pd}}}; \quad N_{-i} = \frac{\text{sh} \sqrt{-i \text{Pd}} X / X}{\text{sh} \sqrt{-i \text{Pd}}};$$

$$\bar{N}_i = \frac{3(\sqrt{i \text{Pd}} - \tanh \sqrt{i \text{Pd}})}{i \text{Pd} \tanh \sqrt{i \text{Pd}}}; \quad \bar{N}_{-i} = \frac{3(\sqrt{-i \text{Pd}} - \tanh \sqrt{-i \text{Pd}})}{-i \text{Pd} \tanh \sqrt{-i \text{Pd}}}.$$

For a quasisteady state the average temperature  $\bar{T}$  will be a periodic function of time:

$$\bar{T} = (\bar{N}_i \bar{N}_{-i}) \cos(\text{Pd Fo} - \bar{M}),$$

where

$$\bar{M} = \arctan i \frac{\bar{N}_i - \bar{N}_{-i}}{\bar{N}_i + \bar{N}_{-i}}.$$

The specific heat consumption during any time interval  $\Delta \tau$  is

$$\Delta Q = c_q \gamma [\bar{T}(\text{Fo}_2) - \bar{T}(\text{Fo}_1)] t_m.$$

Thus, for quasisteady conditions

$$\begin{aligned}\Delta Q &= c_q \gamma t_m (\bar{N}_i \bar{N}_{-i})^{1/2} [\cos(\text{Pd } F_{0_2} - \bar{M}) - \cos(\text{Pd } F_{0_1} - \bar{M})] = \\ &= 2c_q \gamma t_m (\bar{N}_i \bar{N}_{-i})^{1/2} \sin \frac{\text{Pd}}{2} (F_{0_2} - F_{0_1}) \sin \left[ \bar{M} - \frac{1}{2} \text{Pd} (F_{0_2} + F_{0_1}) \right].\end{aligned}\quad (7-3-22)$$

It follows from (7-3-22) that zero heat is consumed during an oscillation period  $\Delta\tau = \frac{1}{\nu}$ , since

$$\frac{1}{2} \text{Pd} (F_{0_2} - F_{0_1}) = \pi \nu (\tau_2 - \tau_1) = \pi \text{ and } \sin \pi = 0.$$

The heat consumed during a time interval  $\Delta\tau$  equal to a half period  $\left(\Delta\tau = \frac{1}{2\nu}\right)$  is

$$\Delta Q = 2c_q \gamma t_m (\bar{N}_i \bar{N}_{-i})^{1/2} \cos(\text{Pd } F_{0_1} + \bar{M}).$$

Thus, the specific heat consumption is determined by the quantity  $(\bar{N}_i \bar{N}_{-i})^{1/2}$  and is a function of the initial time, since the quasisteady state is not established all at once but only after some definite time interval.

In conclusion, let us note that the solutions obtained for the various body shapes can be derived directly from a single general solution, in a way similar to that demonstrated in Chapter VI. A detailed account of the method for generalizing the solution is given in [7], page 398. It will not be given here, since more general problems will be considered in § 7-5.

#### 7-4. Some Generalized Problems of Heat and Mass Transfer in a Medium with Variable Temperature

##### a) Heat and mass transfer when the temperature of the surroundings is an arbitrary function of time

Let us now generalize the previously considered problems to include all conditions in which a body is situated in a medium whose temperature (or some other potential of the medium) is a function of time. In this generalization Duhamel's theorem may be used, a theorem which, on the basis of solutions for a constant potential of the medium (in particular, a constant temperature), enables us to obtain solutions for conditions when the medium potential is a given function of time. For this to be possible, the theorem requires that the function fulfil certain conditions: the function and its derivative must be piecewise continuous for  $F_0 > 0$ . We should also mention here the well-known difference existing between the generalizations of solutions of differential equations for "coupled" and uncoupled transfer. Whereas for uncoupled transfer it is not necessary to treat further the solution which is obtained initially, the solutions of systems of coupled equations demand such an additional processing. To illustrate this method, let us first consider uncoupled transfer. The solution for an infinite plate will be considered in more detail.

1. Let us solve the one-dimensional differential transfer equation

$$\frac{\partial Z}{\partial F_0} = \frac{\partial^2 Z}{\partial X^2} + \frac{F}{X} \cdot \frac{\partial Z}{\partial X} \quad (7-4-1)$$

with the following limiting conditions:

$$Z(X, 0) = 0; \quad (7-4-2)$$

$$\frac{\partial Z(0, F_0)}{\partial X} = 0; \quad (7-4-3)$$

$$-\frac{\partial Z(1, F_0)}{\partial X} + \text{Bi}[Z_e - Z(1, F_0)] = 0, \quad (7-4-4)$$

where

$$Z(X, F_0) = \frac{z - z_0}{z_e - z_0}; \quad Z_e = F(F_0).$$

In § 6-5 a solution was obtained for an infinite plate ( $\Gamma = 0$ ), with the condition that the potential of the surroundings  $Z_e = 1$ . The transform solution is in this case

$$\{Z_1(X, s)\}_L = \frac{1}{s} \left[ \frac{\text{ch} \sqrt{s} X}{\text{ch} \sqrt{s} + \frac{1}{\text{Bi}} \sqrt{s} \text{sh} \sqrt{s}} \right] = \frac{1}{s} G_L(s),$$

where  $\frac{1}{s}$  is the transform of the constant  $Z_e = 1$ .

If  $Z_e = F(F_0)$  and its transform is some function  $F_L(s)$ , then the solution for the plate becomes

$$Z_L = F_L G_L. \quad (7-4-5')$$

Now, if we multiply and divide (7-4-5') by  $s$ , then solution (7-4-5) can be represented as a product of the two transforms  $sF_L$  and  $\{Z_1\}_L$ :

$$Z_L(X, s) = sF_L(s) \{Z_1(X, s)\}_L. \quad (7-4-5)$$

The inverse of the second transform  $\{Z_1(X, s)\}_L$  is known to be

$$Z_1(X, F_0) = L^{-1}[\{Z_1(X, s)\}_L] = 1 - \sum_{n=1}^{\infty} A_n \cos \mu_n X \exp(-\mu_n^2 F_0). \quad (7-4-6)$$

The inverse of the first transform, as is easily shown by differentiating the function  $Z_e = F(F_0)$ , is

$$sF_L(s) = L[F'(F_0)] + F(0).$$

If  $F(0) = 0$ , then the inverse of  $sF_L(s)$  is  $F'(F_0)$ , so that from the theorem of transform multiplication it follows that

$$Z(X, F_0) = \int_0^{F_0} F'(F_0 - F_0^*) Z_1(X, F_0^*) dF_0^*.$$

If  $F(0)$  is constant, then  $Z_L(X, s) = L[F'(F_0)] \{Z_1(X, s)\}_L + F(0) \{Z_1(X, s)\}_L$ . In this case we obtain

$$Z(X, F_0) = \int_0^{F_0} F'(F_0 - F_0^*) Z_1(X, F_0^*) dF_0^* + F(0) Z_1(X, F_0). \quad (7-4-7)$$

Consequently, after substituting (7-4-6) into equation (7-4-7) we have

$$\begin{aligned} Z(X, F_0) = & F(0) \left[ 1 - \sum_{n=1}^{\infty} A_n \cos \mu_n X \exp(-\mu_n^2 F_0) \right] + \int_0^{F_0} F'(F_0 - F_0^*) dF_0^* - \\ & - \sum_{n=1}^{\infty} A_n \cos \mu_n X \int_0^{F_0} \exp(-\mu_n^2 F_0^*) F'(F_0 - F_0^*) dF_0^*. \end{aligned} \quad (7-4-8)$$

If solution (7-4-8) is generalized to include other body shapes (see Chapter VI, equation (6-10)), then we obtain

$$Z(X, Fo) = F(0) \left[ 1 - \sum_{n=1}^{\infty} A_n \Phi_n(\mu_n X) \exp(-\mu_n^2 Fo) \right] + \\ + \int_0^{Fo} F'(Fo - Fo^*) dFo^* - \sum_{n=1}^{\infty} A_n \Phi_n(\mu_n X) \int_0^{Fo} \exp(-\mu_n^2 Fo^*) F'(Fo - Fo^*) dFo^*. \quad (7-4-9)$$

By expressing  $Z_c = F(Fo)$  as different laws of variation with time, it is possible to derive a series of particular solutions from solution (7-4-9). If we set  $Z_c = 1$ , we obtain solution (6-5-10); for  $Z_c = 1 - Pd Fo$ , we obtain solution (7-1-9); etc. If we set  $Bi = \infty$  in (7-4-9), then we obtain the corresponding solutions for boundary conditions of the first kind.

Because of its practical importance, let us consider separately the generalized solution for a hollow cylinder. We assume that exchange with the surroundings, whose potentials vary according to the arbitrary given laws  $Z_{c\alpha} = F_2(Fo)$  and  $Z_{c1} = F_1(Fo)$ , takes place at the internal and external surfaces of the cylinder. Boundary conditions (7-4-4) are in this case

$$\left. \begin{aligned} \frac{\partial Z(1, Fo)}{\partial X} - Bi_1 [F_1(Fo) - Z(1, Fo)] &= 0; \\ \frac{\partial Z(x, Fo)}{\partial X} + Bi_2 [F_2(Fo) - Z(x, Fo)] &= 0. \end{aligned} \right\} \quad (7-4-10)$$

Taking into account (7-4-2), we obtain the transform solution of equation (7-4-1) as

$$Z_L = B_1 I_0(\sqrt{s}X) + B_2 K_0(\sqrt{s}X). \quad (7-4-11)$$

Now let us introduce the following notation:

$$\left. \begin{aligned} J^*(\sqrt{s}; \pm Bi; X) &= \sqrt{s} J_1(\sqrt{s}X) \pm Bi J_0(\sqrt{s}X); \\ Y^*(\sqrt{s}; \pm Bi; X) &= \sqrt{s} Y_1(\sqrt{s}X) \pm Bi Y_0(\sqrt{s}X); \\ I^*(\sqrt{s}; \pm Bi; X) &= \sqrt{s} I_1(\sqrt{s}X) \pm Bi I_0(\sqrt{s}X); \\ K^*(\sqrt{s}; \pm Bi; X) &= \sqrt{s} K_1(\sqrt{s}X) \pm Bi K_0(\sqrt{s}X). \end{aligned} \right\} \quad (7-4-12)$$

When the arbitrary constants  $B_1$  and  $B_2$  are determined from (7-4-10) and then substituted into (7-4-11), we obtain, taking into account (7-4-12),

$$Z_L(X, s) = Bi_2 \frac{P(\sqrt{s})}{N(\sqrt{s})} \{F_2(s)\}_L + Bi_1 \frac{Q(\sqrt{s})}{N(\sqrt{s})} \{F_1(s)\}_L, \quad (7-4-13)$$

where

$$\begin{aligned} P(\sqrt{s}) &= I_0(\sqrt{s}X) K^*(\sqrt{s}; -Bi_1) + K_0(\sqrt{s}X) I^*(\sqrt{s}; Bi_1); \\ Q(\sqrt{s}) &= I_0(\sqrt{s}X) K^*(\sqrt{s}; Bi_2) + K_0(\sqrt{s}X) I^*(\sqrt{s}; -Bi_2); \\ N(\sqrt{s}) &= I^*(\sqrt{s}; Bi_1) K^*(\sqrt{s}; Bi_2) - I^*(\sqrt{s}; -Bi_2) K^*(\sqrt{s}; -Bi_1). \end{aligned}$$

Solution (7-4-13) is the sum of two transform products. Therefore, for each term we must obtain the convolution of the functions. The inverse transform of the first factors can be obtained in either of two ways, by means of the expansion theorem or by using Cauchy's residue theorem (by solving the Riemann-Mellin integrals). After the necessary calculations, the final solution of the problem is found to be /8/:

$$Z(X, Fo) = \pi \sum_{n=1}^{\infty} \frac{\mu_n^2 J_n \exp(-\mu_n^2 Fo)}{c_n^2 (\mu_n^2 + Bi_1^2 + Bi_1) - (\mu_n^2 + Bi_2^2 - \frac{Bi_2}{x})} \times \\ \times \left[ Bi_2 \int_0^{Fo} \exp(\mu_n^2 Fo^*) F_1(Fo^*) dFo^* + Bi_1 c_n \int_0^{Fo} \exp(\mu_n^2 Fo^*) F_2(Fo^*) dFo^* \right]. \quad (7-4-14)$$

where

$$J_n = J^*(\mu_n \text{Bi}_2) Y_0(\mu_n X) - J_0(\mu_n X) Y^*(\mu_n \text{Bi}_2);$$

$$\sigma_n = \frac{J^*(\mu_n; \text{Bi}_2 x)}{J^*(\mu_n; -\text{Bi}_1)} = \frac{Y^*(\mu_n; \text{Bi}_2 x)}{Y^*(\mu_n; -\text{Bi}_1)};$$

and  $\mu_n$  are the roots of the equation

$$J^*(\mu; -\text{Bi}_1) Y^*(\mu; \text{Bi}_2 x) - J^*(\mu; \text{Bi}_2 x) Y^*(\mu; -\text{Bi}_1) = 0.$$

The solutions considered by Danilova /9, 10/, Carslaw and Jaeger /6/, Crank /11/, and others represent particular cases of solution (7-4-14). A similar general solution, but with a source term (depending on the spatial coordinate and on time) present in equation (7-4-1), was given by Moskvitin /12/.

Let us now go on to solve system of heat-transfer and mass-transfer equations (4-1-2) and (4-1-3) with constant initial conditions  $T(X, 0) = \theta(X, 0) = 0$  and with an arbitrary temperature of the surroundings. The boundary conditions are, accordingly,

$$\frac{\partial T(1, \text{Fo})}{\partial X} - \text{Bi}_q [T_c - T_1(1, \text{Fo})] + (1 - \epsilon) \text{Ko Lu Bi}_m [1 - \theta(1, \text{Fo})] = 0; \quad (7-4-15)$$

$$-\frac{\partial \theta(1, \text{Fo})}{\partial X} + \text{Pn} \frac{\partial T(1, \text{Fo})}{\partial X} + \text{Bi}_m [1 - \theta(1, \text{Fo})] = 0, \quad (7-4-16)$$

where  $T_c = F(\text{Fo})$  and  $T = \frac{t - t_c}{t_{c0} - t_c}$ .

It will be convenient to analyze the difference between the new solution and the corresponding solution in Chapter VI. To be specific, let us consider an infinite plate. The transform solutions for this problem are easily obtained, using the corresponding solutions of Chapter VI. These are

$$T_L = \sum_{i=1}^2 B_i \text{ch} \sqrt{s v_i} X; \quad (7-4-17)$$

$$\theta_L = -\frac{1}{\epsilon \text{Ko}} \sum_{i=1}^2 B_i (1 - v_i^2) \text{ch} \sqrt{s v_i} X; \quad (7-4-18)$$

Here  $B_i = B_i^I + B_i^{II}$  ( $i=1,2$ ), where  $B_i^I$  corresponds to the solutions for  $T_c = 1$ :

$$B_1^I = \frac{(\epsilon \text{Ko} K_1 - 1) P_2 - \epsilon \text{Ko} Q_2}{s(P_1 Q_2 - P_2 Q_1)}; \quad B_2^I = -\frac{(\epsilon \text{Ko} K_1 - 1) P_1 - \epsilon \text{Ko} Q_1}{s(P_1 Q_2 - P_2 Q_1)}, \quad (7-4-19)$$

and  $B_i^{II}$  represents the contribution to the coefficients due to the variable temperature:

$$\left. \begin{aligned} B_1^{II} &= B_1^a + B_1^b = \frac{P_2}{s(P_1 Q_2 - P_2 Q_1)} - s F_L(s) \frac{P_2}{s(P_1 Q_2 - P_2 Q_1)}; \\ B_2^{II} &= B_2^a + B_2^b = -\left[ \frac{P_1}{s(P_1 Q_2 - P_2 Q_1)} - s F_L(s) \frac{P_1}{s(P_1 Q_2 - P_2 Q_1)} \right]. \end{aligned} \right\} \quad (7-4-20)$$

The inverse transforms of solutions (7-4-17) and (7-4-18) for the first pair of coefficients (defined by (7-4-19)) are given by equations (6-3-1) and (6-3-2):

$$T^I(X, \text{Fo}) = 1 - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni}^I \cos v_i \mu_n X \exp(-\mu_n^2 \text{Fo}); \quad (7-4-21)$$

$$\theta^I(X, \text{Fo}) = 1 + \frac{1}{\epsilon \text{Ko}} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni}^I (1 - v_i^2) \cos v_i \mu_n X \exp(-\mu_n^2 \text{Fo}), \quad (7-4-22)$$



where  $C_{ni}^I$  are the coefficients defined by equations (6-3-3) and (6-3-4). Let us now find the inverse transforms for the first terms  $B_i^a$  of the second pair of coefficients. These are

$$T^a(X, Fo) = -1 + \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni}^a \cos v_i \mu_n X \exp(-\mu_n^2 Fo); \quad (7-4-23)$$

$$\theta^a(X, Fo) = -\frac{1}{\varepsilon K_0} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni}^a (1 - v_i^2) \cos v_i \mu_n X \exp(-\mu_n^2 Fo). \quad (7-4-24)$$

The inverse transforms for the second terms in equations (7-4-20) will be convolutions. Since

$$\{T^b(X, s)\}_L = -s F_L(s) \{T^a(X, Fo)\}_L$$

and

$$\{\theta^b(X, s)\}_L = -s F_L(s) \{\theta^a(X, Fo)\}_L$$

are analogous to (7-4-5), their inverse transforms may be found using formula (7-4-7). Consequently,

$$T^b(X, Fo) = -\left[ \int_0^{Fo} F'(Fo - Fo^*) T^a(X, Fo^*) dFo^* + F(0) T^a(X, Fo) \right]; \quad (7-4-25)$$

$$\theta^b(X, Fo) = -\left[ \int_0^{Fo} F'(Fo - Fo^*) \theta^a(X, Fo^*) dFo^* + F(0) \theta^a(X, Fo) \right]. \quad (7-4-26)$$

If at the initial moment  $T_0$  is zero, then the last terms in (7-4-25) and (7-4-26) drop out.

The general solution of the problem is

$$T(X, Fo) = T^I(X, Fo) + T^a(X, Fo) + T^b(X, Fo); \quad (7-4-27)$$

$$\theta(X, Fo) = \theta^I(X, Fo) + \theta^a(X, Fo) + \theta^b(X, Fo). \quad (7-4-28)$$

Now, if we substitute solutions (7-4-21), (7-4-23), and (7-4-25) into (7-4-27), and substitute solutions (7-4-22), (7-4-24), and (7-4-26) into (7-4-28), then after certain simple transformations we finally obtain.

$$\begin{aligned} T(X, Fo) = & F(0) + \int_0^{Fo} F'(Fo - Fo^*) dFo^* - \\ & - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \cos v_i \mu_n X \exp(-\mu_n^2 Fo) - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni}^a \cos v_i \mu_n X \times \\ & \times \int_0^{Fo} F'(Fo - Fo^*) \exp(-\mu_n^2 Fo^*) dFo^*; \end{aligned} \quad (7-4-29)$$

$$\begin{aligned} \theta(X, Fo) = & 1 + \frac{1}{\varepsilon K_0} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2) \cos v_i \mu_n X \exp(-\mu_n^2 Fo) + \\ & + \frac{1}{\varepsilon K_0} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni}^a (1 - v_i^2) \cos v_i \mu_n X \int_0^{Fo} F'(Fo - Fo^*) \exp(-\mu_n^2 Fo^*) dFo^*, \end{aligned} \quad (7-4-30)$$

where

$$C_{ni} = C_{ni}^I - C_{ni}^a + F(0) C_{ni}^a = C_{ni}^I + [F(0) - 1] C_{ni}^a;$$

$$C_{n1} = 2 \frac{(1 - \epsilon K_0 K_1) P_{n2} + \epsilon K_0 Q_{n2}}{\mu_n \psi_n};$$

$$C_{n2} = -2 \frac{(1 - \epsilon K_0 K_1) P_{n1} + \epsilon K_0 Q_{n1}}{\mu_n \psi_n};$$

$$C_{n1}^a = 2 \frac{P_{n2}}{\mu_n \psi_n}; \quad C_{n2}^a = -2 \frac{P_{n1}}{\mu_n \psi_n}.$$

The generalization of solutions (7-4-29) and (7-4-30) to include other body shapes gives

$$T(X, Fo) = F(0) + \int_0^{Fo} F'(Fo - Fo^*) dFo^* -$$

$$- \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \Phi_r(v_i \mu_n X) \exp(-\mu_n^2 Fo) - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni}^a \Phi_r(v_i \mu_n X) \times$$

$$\times \int_0^{Fo} F'(Fo - Fo^*) \exp(-\mu_n^2 Fo^*) dFo^*; \quad (7-4-31)$$

$$\theta(X, Fo) = 1 + \frac{1}{\epsilon K_0} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (1 - v_i^2) \Phi_r(v_i \mu_n X) \exp(-\mu_n^2 Fo) +$$

$$+ \frac{1}{\epsilon K_0} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni}^a (1 - v_i^2) \Phi_r(v_i \mu_n X) \int_0^{Fo} F'(Fo - Fo^*) \exp(-\mu_n^2 Fo^*) dFo^*. \quad (7-4-32)$$

In solutions (7-4-31) and (7-4-32), for an infinite plate ( $\Gamma=0$ )  $\Phi_0 = \cos v_i \mu_n X$ , for a cylinder ( $\Gamma=1$ )  $\Phi_1 = J_0(v_i \mu_n X)$ , and for a sphere ( $\Gamma=2$ )  $\Phi_2 = -\frac{\sin v_i \mu_n X}{X}$ . The quantities ( $\psi_{ni}$ ,  $A_{ni}$ ,  $B_{ni}$ ,  $P_{ni}$ ,  $Q_{ni}$ ) entering into coefficients  $C_{ni}$  and  $C_{ni}^a$  for the various body shapes, as well as the equations obeyed by the characteristic-equation roots  $\mu_n$ , are given in § 7-1. Numerical values of  $\mu_n$  and  $C_{ni}$  for various combinations of dimensionless numbers are given in the corresponding sections of Chapter VI. Values of  $C_{ni}^a$  for an infinite plate, for various values of  $Lu$  and  $Bi_q$ , are listed in Table 7-7. By assuming that the temperature of the surroundings obeys various laws of variation with the Fourier number, it is possible to obtain from solutions (7-4-31) and (7-4-32) various particular solutions, for example, the previously derived solutions for linear and exponential laws of the surrounding temperature as a function of  $Fo$ .

In conclusion let us note that here only problems with zero initial distribution of the dimensionless transfer potentials have been considered. However, it can be shown that problems with complex initial conditions may be solved in a similar manner. For such cases it is convenient, by means of an appropriate substitution, to reduce the solution to two problems: one problem with zero initial conditions but with complete boundary conditions (variable potentials in the surroundings), and one problem with arbitrary initial conditions but with zero boundary conditions.

It was in this way that Gamayunov solved the system of heat-transfer and mass-transfer equations, subject to the limiting conditions:

$$-\frac{\partial T(1, Fo)}{\partial X} + Bi_q [T_c(Fo) - T(1, Fo)] -$$

$$-(1 - \epsilon) K_0 Lu Bi_m [\theta_c(Fo) - \theta(1, Fo)] = 0; \quad (7-4-33)$$

$$-\frac{\partial \theta(1, Fo)}{\partial X} + Pn \frac{\partial T(1, Fo)}{\partial X} + Bi_m [\theta_c(Fo) - \theta(1, Fo)] = 0; \quad (7-4-34)$$

$$\frac{\partial T(0, Fo)}{\partial X} = \frac{\partial \theta(0, Fo)}{\partial X} = 0; \quad (7-4-35)$$

$$T(X, 0) = F_1(X); \quad \theta(X, 0) = F_2(X). \quad (7-4-36)$$

The dimensionless transfer potentials in this case are

$$T = \frac{t - t_*}{t_*} \quad \text{and} \quad \theta = \frac{\theta_* - \theta}{\theta_*}.$$

TABLE 7-7

Values of coefficients  $C_{ni}^a$  for an infinite plate, for various  $Lu$  and  $Bi_q$

Parameter	$C_{n1}$		$C_{n2}$	
	$n=1$	$n=2$	$n=1$	$n=2$
Lu	0.15	0.00113	0.135	0.0208
	0.30	0.00635	0.390	0.0224
	0.50	0.01993	0.935	0.0392
	0.70	0.03599	2.227	0.0490
	1.00	0.05238	2.132	0.0768
	2.00	0.05444	-3.944	0.1200
Bi <sub>q</sub>	1	0.2808	-0.606	-0.0191
	5	0.0195	0.424	0.0117
	7	0.0113	0.442	0.0144
	10	0.0058	0.390	0.0184
	15	0.0018	0.412	0.0200
	20	-0.0003	0.420	0.0207

A solution of system (4-1-2) and (4-1-3) with limiting conditions (7-4-33) through (7-4-36) was obtained in the following form:

$$\left. \begin{aligned} T(X, Fo) &= T_1(X, Fo) + T_2(X, Fo); \\ \theta(X, Fo) &= \theta_1(X, Fo) + \theta_2(X, Fo), \end{aligned} \right\} \quad (7-4-37)$$

where  $T_1$  and  $\theta_1$  are the solutions of the system of equations with zero boundary conditions and arbitrary initial conditions (7-4-36), and  $T_2$  and  $\theta_2$  are the solutions with boundary conditions (7-4-33) and (7-4-34) and zero initial conditions.

In solution (7-4-37)

$$\begin{aligned} T_1(X, Fo) &= \frac{1}{v_2^2 - v_1^2} \left\{ \sum_{m=1}^{\infty} E_{m2} \exp(-\mu_m^2 v_2^2 Lu Fo) - \sum_{m=1}^{\infty} E_{m1} \exp(-\mu_m^2 v_1^2 Lu Fo) \right\}; \\ \theta_1(X, Fo) &= \frac{1}{v_2^2 - v_1^2} \left\{ \sum_{m=1}^{\infty} F_{m2} \exp(-\mu_m^2 v_2^2 Lu Fo) - \sum_{m=1}^{\infty} F_{m1} \exp(-\mu_m^2 v_1^2 Lu Fo) \right\}; \\ T_2(X, Fo) &= 2 \sum_{n=1}^{\infty} \left\{ (P_{n2} L_{n1} - P_{n1} L_{n2}) \frac{\mu_n}{\psi_n} \int_0^{Fo} T_c(Fo^*) \exp[-\mu_n^2 (Fo - Fo^*)] dFo^* + \right. \\ &\quad \left. + \epsilon K_0 (S_{n2} L_{n1} - S_{n1} L_{n2}) \frac{\mu_n}{\psi_n} \int_0^{Fo} \theta_c(Fo^*) \exp[-\mu_n^2 (Fo - Fo^*)] dFo^* \right\}; \\ \theta_2(X, Fo) &= 2 \sum_{n=1}^{\infty} \left\{ (P_{n2}^* L_{n1} - P_{n1}^* L_{n2}) \frac{\mu_n}{\psi_n} \frac{1}{\epsilon K_0} \int_0^{Fo} T_c(Fo^*) \exp[-\mu_n^2 (Fo - Fo^*)] \times \right. \\ &\quad \left. \times dFo^* + (S_{n2}^* L_{n1} - S_{n1}^* L_{n2}) \frac{\mu_n}{\psi_n} \int_0^{Fo} \theta_c(Fo^*) \exp[-\mu_n^2 (Fo - Fo^*)] dFo^* \right\}, \end{aligned}$$

where  $\mu_n$  are the roots of the characteristic equation

$$Q_{n2}P_{n1} - Q_{n1}P_{n2} = 0;$$

and where

$$\begin{aligned}\psi_n &= v_1 A_{n1} P_{n2} + v_2 B_{n2} Q_{n1} - v_2 A_{n2} P_{n1} - v_1 B_{n1} Q_{n2}; \\ S_{ni} &= Q_{ni} - K_i P_{ni} \quad (i = 1, 2); \end{aligned}$$

$P_{ni}$ ,  $Q_{ni}$ ,  $A_{ni}$ ,  $B_{ni}$  and  $v_i$  being defined by the corresponding expressions in § 7-1.

In addition, for an infinite plate

$$\begin{aligned}E_{mi} &= 2 \cos \mu_m X \int_0^1 [(v_i^2 - 1) F_1(X) + \varepsilon K_o F_2(X)] \cos \mu_m X dX; \\ F_{mi} &= 2 \cos \mu_m X \int_0^1 \left[ P_n F_1(X) + \left( v_i^2 - \frac{1}{Lu} - \varepsilon K_o P_n \right) F_2(X) \right] \cos \mu_m X dX; \\ L_{ni} &= \operatorname{ch} v_i \mu_n X; \\ P_{ni}^* &= -\varepsilon K_o P_n \cos v_i \mu_n + \frac{\varepsilon K_o P_n}{v_i^2 Lu Bl_m} v_i \mu_n \sin v_i \mu_n; \\ S_{ni}^* &= \left( 1 - \frac{1}{v_i^2 Lu} \right) \cos v_i \mu_n - \left[ \frac{1}{Bl_q} - \frac{1}{v_i^2 Lu} \left( \frac{1}{Bl_q} - \frac{\varepsilon K_o P_n K_1}{Bl_m} \right) \right] v_i \mu_n \sin v_i \mu_n; \end{aligned}$$

where  $\mu_m$  are the roots of equation  $\cos \mu_m = 0$ .

For an infinite cylinder

$$\begin{aligned}E_{mi} &= \frac{2J_0(\mu_m X)}{J_1^2(\mu_m)} \int_0^1 [(v_i^2 - 1) F_1(X) + \varepsilon K_o F_2(X)] X J_0(\mu_m X) dX; \\ F_{mi} &= \frac{2J_0(\mu_m X)}{J_1^2(\mu_m)} \int_0^1 \left[ P_n F_1(X) + \left( v_i^2 - \frac{1}{Lu} - \varepsilon K_o P_n \right) F_2(X) \right] X J_0(\mu_m X) dX; \\ L_{ni} &= I_0(v_i \mu_n X); \\ P_{ni}^* &= -\varepsilon K_o P_n J_0(v_i \mu_n) + \frac{\varepsilon K_o P_n}{v_i^2 Lu Bl_m} v_i \mu_n J_1(v_i \mu_n); \\ S_{ni}^* &= \left( 1 - \frac{1}{v_i^2 Lu} \right) J_0(v_i \mu_n) - \left[ \frac{1}{Bl_q} - \frac{1}{v_i^2 Lu} \left( \frac{1}{Bl_q} - \frac{\varepsilon K_o P_n K_1}{Bl_m} \right) \right] v_i \mu_n J_1(v_i \mu_n); \end{aligned}$$

where  $\mu_m$  are the roots of the equation  $J_0(\mu_m) = 0$ .

For a sphere

$$\begin{aligned}E_{mi} &= \frac{2 \sin \mu_m X}{X} \int_0^1 [(v_i^2 - 1) F_1(X) + \varepsilon K_o F_2(X)] \sin \mu_m X dX; \\ F_{mi} &= \frac{2 \sin \mu_m X}{X} \int_0^1 \left[ P_n F_1(X) + \left( v_i^2 - \frac{1}{Lu} - \varepsilon K_o P_n \right) F_2(X) \right] \sin \mu_m X dX; \\ L_{ni} &= \frac{\operatorname{sh} v_i \mu_n X}{X}; \\ P_{ni}^* &= -\frac{\varepsilon K_o P_n}{v_i^2 Lu Bl_m} - \varepsilon K_o P_n \left( 1 - \frac{1}{v_i^2 Lu Bl_m} \right) v_i \mu_n \cos v_i \mu_n; \end{aligned}$$

$$S_{ni}^* = \left[ \frac{1}{Bi_q} - \frac{1}{v_i^2 Lu} \left( \frac{1}{Bi_q} - \frac{\epsilon Ko Pn K_i}{Bi_m} \right) \right] \sin v_i \mu_n + \\ + \left[ 1 - \frac{1}{Bi_q} - \frac{1}{v_i^2 Lu} \left( 1 - \frac{1}{Bi_q} + \frac{\epsilon Ko Pn K_i}{Bi_m} \right) \right] v_i \mu_n \cos v_i \mu_n;$$

where  $\mu_m$  are the roots of the equation  $\sin \mu_m = 0$ .

**b) Unsteady fields of transfer potential for a variable exchange coefficient and variable potentials in the surroundings**

In a real process a variation of the potentials in the surroundings leads to a variation in the heat-exchange coefficient ( $\alpha_q$ ) and the mass-exchange coefficient ( $\alpha_m$ ). It is therefore interesting to consider how unsteady-state transport processes may be calculated when both of these factors vary simultaneously. As an example let us determine an unsteady potential field for an infinite plate, assuming that the basic transfer is not coupled with the transfer of other potentials. A method for solving this problem has been given by Kiselev and Lazarev /13/.

We will solve the equation

$$\frac{\partial Z(X, Fo)}{\partial Fo} = \frac{\partial^2 Z(X, Fo)}{\partial X^2} \quad (7-4-38)$$

with the following symmetrical limiting conditions:

$$Z(X, 0) = F(X); \quad (7-4-39)$$

$$-\frac{\partial Z(1, Fo)}{\partial X} + Bi(Fo)[Z_o(Fo) - Z(1, Fo)] = 0; \quad (7-4-40)$$

$$\frac{\partial Z(-1, Fo)}{\partial X} + Bi(Fo)[Z_c(Fo) - Z(-1, Fo)] = 0, \quad (7-4-41)$$

where  $Bi(Fo) = \frac{\alpha(Fo)R}{\lambda}$  is the Biot number,  $F(X)$  is a continuous even function of  $X$ , and  $Bi(Fo)$  and  $Z_o(Fo)$  are arbitrary differentiable functions.

The solution may be expressed as the sum of a Poisson integral plus the integrals of two thermal potentials /14/:

$$Z(X, Fo) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{Fo}} \exp \left[ -\frac{(X-X^*)^2}{4Fo} \right] g(X^*) dX^* + \\ + \frac{1}{\sqrt{\pi}} \int_0^{Fo} \frac{\varphi_1(Fo^*)}{\sqrt{Fo-Fo^*}} \exp \left[ -\frac{(X-1)^2}{4(Fo-Fo^*)} \right] dFo^* + \\ + \frac{1}{\sqrt{\pi}} \int_0^{Fo} \frac{\varphi_2(Fo^*)}{\sqrt{Fo-Fo^*}} \exp \left[ -\frac{(X+1)^2}{4(Fo-Fo^*)} \right] dFo^*. \quad (7-4-42)$$

Here the function  $g(X)$  is equal to  $F(X)$  in the interval  $-1 \leq X \leq 1$  and equal to  $F(1) = F(-1)$  outside this interval. The densities  $\varphi_1(Fo)$  and  $\varphi_2(Fo)$  of the thermal potentials are differentiable functions which are still unknown.

Because of the properties of the Poisson integral and the integrals of the thermal potentials, equation (7-4-42) for  $Z(X, Fo)$  satisfies equation (7-4-38) and initial condition (7-4-39). In addition,

$$\lim_{X \rightarrow X_0 \pm 0} \frac{\partial}{\partial X} \frac{1}{\sqrt{\pi}} \int_0^{Fo} \frac{\varphi(Fo^*)}{\sqrt{Fo-Fo^*}} \exp \left[ -\frac{(X-X_0)^2}{4(Fo-Fo^*)} \right] dFo^* = \mp \varphi(Fo). \quad (7-4-43)$$

The substitution of (7-4-42) into conditions (7-4-40) and (7-4-41), at the same time taking into account (7-4-43), shows that the boundary conditions will be satisfied if the thermal-potential densities  $\varphi_1$  and  $\varphi_2$  satisfy the following system of two Volterra-type integral equations:

$$\begin{aligned} & -\frac{\partial}{\partial X} I(1, F_0) - \varphi_1(F_0) + \frac{1}{\sqrt{\pi}} \int_0^{F_0} \frac{\varphi_1(F_0^*)}{(F_0 - F_0^*)^{3/2}} \exp\left[-\frac{1}{F_0 - F_0^*}\right] dF_0^* + \\ & + \text{Bi}(F_0) \left[ Z_0(F_0) - I(1, F_0) - \frac{1}{\sqrt{\pi}} \int_0^{F_0} \frac{\varphi_1(F_0^*)}{\sqrt{F_0 - F_0^*}} dF_0^* - \right. \\ & \left. - \frac{1}{\sqrt{\pi}} \int_0^{F_0} \frac{\varphi_2(F_0^*)}{\sqrt{F_0 - F_0^*}} \exp\left(-\frac{1}{F_0 - F_0^*}\right) dF_0^* \right] = 0; \end{aligned} \quad (7-4-44)$$

$$\begin{aligned} & -\frac{\partial}{\partial X} I(-1, F_0) - \varphi_2(F_0) + \frac{1}{\sqrt{\pi}} \int_0^{F_0} \frac{\varphi_1(F_0^*)}{(F_0 - F_0^*)^{3/2}} \exp\left(-\frac{1}{F_0 - F_0^*}\right) dF_0^* + \\ & + \text{Bi}(F_0) \left[ Z_0(F_0) - I(-1, F_0) - \frac{1}{\sqrt{\pi}} \int_0^{F_0} \frac{\varphi_1(F_0^*)}{\sqrt{F_0 - F_0^*}} dF_0^* - \right. \\ & \left. - \frac{1}{\sqrt{\pi}} \int_0^{F_0} \frac{\varphi_2(F_0^*)}{\sqrt{F_0 - F_0^*}} \exp\left(-\frac{1}{F_0 - F_0^*}\right) dF_0^* \right] = 0, \end{aligned} \quad (7-4-45)$$

where  $I(X, F_0)$  is the Poisson integral

$$I(X, F_0) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{F_0}} \exp\left[-\frac{(X - X^*)}{4F_0}\right] g(X^*) dX^*.$$

It should be noted that because of the evenness of  $g(X)$  the function  $I(X, F_0)$  is also an even function of  $X$  while, consequently,  $\frac{\partial}{\partial X} I(X, F_0)$  is an odd function of  $X$ . If we subtract (7-4-45) from (7-4-44), then we find that the difference  $\varphi_1 - \varphi_2$  satisfies a homogeneous Volterra-type integral equation, so that because of certain well-known properties of this equation, this difference is identically zero. Thus, denoting  $\varphi_1 = \varphi_2 = \varphi$ , we find that  $\varphi$  satisfies the integral equation

$$\varphi(F_0) = \frac{1}{\sqrt{\pi}} \int_0^{F_0} K(F_0, F_0^*) \varphi(F_0^*) dF_0^* + \Psi(F_0), \quad (7-4-46)$$

where

$$\begin{aligned} \Psi(F_0) &= -\frac{\partial}{\partial X} I(1, F_0) + \text{Bi}(F_0) [Z_0(F_0) - I(1, F_0)]; \\ K(F_0 - F_0^*) &= -\text{Bi}(F_0) / \sqrt{F_0 - F_0^*} - \\ & - \frac{\text{Bi}(F_0) - 1/(F_0 - F_0^*)}{\sqrt{F_0 - F_0^*}} \exp\left(-\frac{1}{F_0 - F_0^*}\right). \end{aligned}$$

It should be noted that the kernel  $K(F_0, F_0^*)$  is singular.

The function  $\varphi(F_0)$  can be determined from equation (7-4-46) in different ways, for example by reducing the integral in (7-4-46) to a system of

algebraic equations and solving the latter by approximate integration /13/. Once  $\varphi(F_0)$  is determined, we can calculate the potentials  $Z(X, F_0)$  from equation (7-4-42). After introducing the new variable  $\eta = F_0 - F_0^*$  and taking into account that  $\varphi_1 = \varphi_2 = \varphi$ , we obtain equation (7-4-42) in the form

$$Z(X, F_0) = I(X, F_0) + \frac{1}{\sqrt{\pi}} \int_0^{F_0} \frac{\varphi(F_0 - \eta)}{\eta^{1/2}} \exp\left[-\frac{(1-X)^2}{4\eta}\right] d\eta + \\ + \frac{1}{\sqrt{\pi}} \int_0^{F_0} \frac{\varphi(F_0 - \eta)}{\eta^{1/2}} \exp\left[-\frac{(1+X)^2}{4\eta}\right] d\eta.$$

As an analysis of the solution shows, it is possible to calculate the potentials at the surface of the system by means of the following expressions: for low  $F_0$ ,

$$Z(1, F_0) = Z_0(F_0) - \frac{1}{Bi(F_0)} \left[ \varphi(F_0) + \frac{\partial}{\partial X} I(1, F_0) \right];$$

and for high  $F_0$ ,

$$Z(1, F_0) = Z_0(F_0) - \frac{1}{Bi(F_0)} \left[ \varphi(F_0) + \frac{\partial}{\partial X} I(1, F_0) - \right. \\ \left. - \frac{1}{\sqrt{\pi}} \int_0^{F_0} \frac{\varphi(F_0 - \eta)}{\eta^{3/2}} \exp(-1/\eta) d\eta \right].$$

The integral in the latter expression can be calculated using the familiar methods of approximate integration.

### c) Some problems in mass-transfer theory

In the previously considered problems the variation in the potentials of the external surroundings was essentially independent of the unsteady distribution of potentials in the body. However, for many mass-exchange processes such a representation is not suitable. For instance, during dissolution or during the extraction of matter from a porous body using the diffusion method, the body is in contact with a liquid (or gas) possessing a lower concentration of the substance being extracted from it. Due to the difference in mass-transfer potentials (to a first approximation, a concentration difference), matter will continually move to the surface of the body and even further, into the surroundings. The rate of mass exchange in a limited volume of liquid decreases as time goes by, due to a decrease in the potential gradient in the body and particularly to the saturability of the surroundings. However, from the technological point of view it is very desirable to raise this concentration to a maximum without an appreciable reduction in the rate of the process. A rise in concentration in the surroundings is associated with a decrease in the average concentration of the substance which is extracted from the body. Thus, the internal potentials are coupled with the external potentials, not only with respect to the surface but also with respect to the surroundings.

The boundary conditions for uncoupled transfer in this type of problem should be written in terms of dimensional quantities as

$$-\lambda \frac{\partial z}{\partial n} \Big|_s = a(z_s - z_0), \quad (7-4-47')$$

where

$$z_c = z_m + \beta(z_0 - \bar{z}).$$

As previously, "s" refers to the surface, "m" refers to the equalized value of the potential, and "c" refers to the surroundings. The quantity  $\beta$  is a dimensionless coefficient, which for an extraction process characterizes the ratio between the liquid volume in the pores (flowing into the equipment per unit time) and the volume of extracting liquid.

If we set  $Z = \frac{z_0 - z}{z_0 - z_m}$ , then condition (7-4-47') for a one-dimensional body will be, in dimensionless form,

$$\frac{\partial Z(1, Fo)}{\partial X} = Bi [1 - Z(1, Fo) + \beta \bar{Z}]. \quad (7-4-47)$$

This condition may serve to characterize the various possible directions of the process. Aksel'rud has shown that for a periodic extraction process  $\beta > 0$ , while  $z_m$  corresponds to the average initial concentration of matter in the pores. For a uniflow process  $\beta > 0$  and  $z_m = \{z_0\}_c$ ; for a counterflow process  $\beta < 0$  and  $z_m = \{z_k\}_c$ , where the subscript "k" refers to the final average concentration in the surroundings.

The solution of equation (7-4-38), subject to boundary conditions (7-4-47), symmetry conditions, and a zero initial distribution of potentials, does not involve special complications. Therefore, we will give here the final results only, referring those who are interested in a more detailed analysis to reference /15/:

$$\frac{z_0 - z}{z_0 - z_m} = \frac{1}{1 + \beta} + \sum_{n=1}^{\infty} A_n \left[ (\Gamma + 1) \beta - \frac{\mu_n^2}{Bi} \right] \frac{\Phi_{\Gamma}(\mu_n X)}{\varphi_{\Gamma}(\mu_n)} \exp(-\mu_n^2 Fo), \quad (7-4-48)$$

$$\frac{z_0 - z_c}{z_0 - z_m} = \frac{1}{1 + \beta} + \sum_{n=1}^{\infty} A_n (\Gamma + 1) \beta \exp(-\mu_n^2 Fo), \quad (7-4-49)$$

$$\frac{z_0 - \bar{z}}{z_0 - z_m} = \frac{1}{1 + \beta} - \sum_{n=1}^{\infty} A_n (\Gamma + 1) \exp(-\mu_n^2 Fo), \quad (7-4-50)$$

where for a plate

$$\Gamma = 0; \Phi_0 = \cos \mu_n X; \varphi_0(\mu_n) = \cos \mu_n; A_n = \frac{2}{\left(\frac{\mu_n^2}{Bi} - \beta\right)^2 + \mu_n^2 \left(\frac{1}{Bi} + 1\right) + \beta};$$

and  $\mu_n$  are the roots of the characteristic equation

$$\cot \mu = \frac{\mu}{Bi} - \frac{\beta}{\mu};$$

for a cylinder

$$\Gamma = 1; \Phi_1 = J_0(\mu_n X); \varphi_1(\mu_n) = J_0(\mu_n); A_n = \frac{2}{\mu_n^2 + 4\beta + \left(2\beta - \frac{\mu_n^2}{Bi}\right)^2};$$

and  $\mu_n$  are the roots of the characteristic equation

$$\frac{J_1(\mu)}{J_0(\mu)} = \frac{1}{\mu/Bi - 2\beta/\mu};$$

and for a sphere

$$\Gamma = 2; \Phi_2 = \frac{\sin \mu_n X}{X}; \varphi_2 = \sin \mu_n; A_n = \frac{2}{\left(3\beta - \frac{\mu_n^2}{Bi}\right)^2 + \mu_n^2 \left(1 - \frac{1}{Bi}\right) + 9\beta};$$



and  $\mu_n$  are the roots of the characteristic equation

$$\cot \mu = \frac{1}{\frac{3\beta}{\mu} - \frac{\mu}{Bi}} + \frac{1}{\mu}.$$

If we exclude from consideration the range in which  $Fo$  is close to zero, then it is possible to neglect all but the first term in the infinite sum. In this case the first characteristic-equation root is determined by the following expressions:

for a plate

$$\mu_1 = \sqrt{\frac{1 + \beta}{1/3 + 1/Bi}};$$

for a cylinder

$$\mu_1 = \sqrt{\frac{2(1 + \beta)}{1/4 + 1/Bi}};$$

and for a sphere

$$\mu_1 = \sqrt{3 Bi (1 + \beta)}.$$

If we set  $Bi = \infty$  in (7-4-48) through (7-4-50), then we obtain solutions for boundary conditions of the first kind.

## 7-5. Heat and Mass Exchange in a Layer

System of differential equations (4-1-2) and (4-1-3), together with the corresponding boundary conditions, provide a description of the transfer of heat and mass in an isolated body. It will be useful to generalize this system of equations to include the exchange of heat and mass in a layer. The resulting system of equations makes it possible, first of all, to give up several restrictions which have been imposed on an analytical description of transfer phenomena in a layer. In most works on heat exchange and drying the particles of the layer have been assumed to be so small, or the heat conductivity and potential conductivity in them have been assumed to be so high, that the gradients of the heat-transfer and mass-transfer potentials in the particles were neglected. As a result, the layer of material was represented as a continuous medium with continually and unidirectionally varying temperature and mass-transfer potential. Real processes often take place in layers whose elements are not small. If, in addition, the temperature of the heat carrier is high, then the transfer-potential gradients in the material are considerable. Thus, a model of the process which does not take into account coupling between heat exchange and mass exchange as well as the resistance to transfer phenomena is an incorrect model. A generalization of system (4-1-2) and (4-1-3) and the boundary conditions must lead to relations between the properties of the body and those of the surroundings, primarily relations between the temperature at the body surface and the temperature of the heat carrier [16]. Such an approach will make it possible at the same time to describe more fully and correctly the variation in the surrounding temperature. Thus, this section may be regarded as a closer examination of the material which was presented earlier.

a) The problem posed. The system of differential equations

Let us consider a layer of dispersed (moist) material, consisting of spherical particles of radius  $R$ . A heat-carrying fluid is blown through the layer in the direction  $\eta$  with a constant velocity  $v$ . The temperature of the medium ahead of the layer is  $t_{e0}$ . The initial mass-transfer potentials in the layer particles (that is, before the particles contact the heat carrier) are  $t_0$  and  $\theta_0$ . The porosity of the layer is  $m$ .

If we neglect the effects which radiation inside the layer and contact thermal conduction between the particles have on the potential field, then the system of transfer equations can be written as

$$\frac{\partial [t(r, \tau)]}{\partial \tau} = a_q \frac{\partial^2 [t(r, \tau)]}{\partial r^2} + \frac{\epsilon \rho c_m}{c_q} \cdot \frac{\partial [r \theta(r, \tau)]}{\partial \tau}; \quad (7-5-1)$$

$$\frac{\partial [r \theta(r, \tau)]}{\partial \tau} = a_m \frac{\partial^2 [r \theta(r, \tau)]}{\partial r^2} + a_m \delta \frac{\partial^2 [t(r, \tau)]}{\partial r^2}; \quad (7-5-2)$$

$$-\lambda_q \frac{\partial t(R, \tau)}{\partial r} + a_q [t_0(\eta, \tau) - t(R, \tau)] - (1 - \epsilon) \rho j_m = 0; \quad (7-5-3)$$

$$\lambda_m \left[ \frac{\partial \theta(R, \tau)}{\partial r} + \delta \frac{\partial t(R, \tau)}{\partial r} \right] + j_m = 0; \quad (7-5-4)$$

$$\frac{\partial t_0(\eta, \tau)}{\partial \tau} + v \frac{\partial t_0(\eta, \tau)}{\partial \eta} + \frac{a_q F}{c_0 \gamma_0 m} [t_0(\eta, \tau) - t(R, \tau)] = 0; \quad (7-5-5)$$

$$\frac{\partial \theta(0, \tau)}{\partial r} = \frac{\partial \theta(0, \tau)}{\partial r} = 0; \quad (7-5-6)$$

with

$$\eta = 0 \quad t_c = t_{c0} = \text{const};$$

$$\eta = v\tau \quad t = t_0 = \text{const}; \quad \theta = \theta_0 = \text{const}. \quad (7-5-7)$$

b) Dimensionless parameters for heat and mass exchange in a layer

From system of equations (7-5-1) through (7-5-6) some new similarity criteria may be obtained:

the homochronicity number, characterizing the time-scale transformation,

$$Ho = \frac{v\tau}{R};$$

the Margulis number (or Stanton number  $St$ ), which is the ratio of the specific heat transferred to the material to the heat transferred by the heat carrier through the layer,

$$Mr = \frac{a_q}{v c_0 \gamma_0};$$

and the Miniovich number, which is made up of quantities characterizing the geometry of the layer,

$$Mn = \frac{FR}{m}.$$

Since  $Ho = FoPe$ , where  $Pe = \frac{vR}{a_q}$  is the Péclet number, the potentials of the layer of material, taking into account which dimensionless quantities pertain to the system, can be written as the following dimensionless functions:

$$T = T(Fo, Lu, Bi_q, Bi_m, Ko, Pn, \epsilon, Pe, Mr, Mn, W, V, x/R, \eta/R); \quad (7-5-8)$$

$$\theta = \theta(\text{Fo}, \text{Lu}, \text{Bi}_q, \text{Bi}_m, \text{Ko}, \text{Pn}, \epsilon, \text{Pe}, \text{Mr}, \text{Mn}, \mathbb{W}, V, x/R, \eta/R). \quad (7-5-9)$$

The number of independent dimensionless quantities in equations (7-5-8) and (7-5-9) is considerably smaller, since relations exist between the individual similarity criteria.

c) Solutions for the temperature field in the material  
and in the heat carrier for  $\rho = 0$

$$\begin{aligned} T = \frac{t - t_o}{t_{co} - t_o} &= 1 - \sum_{n=1}^{\infty} A_n \left( 1 - \text{Mr Mn} \frac{\eta}{R} \right) \frac{\sin \mu_n X}{\mu_n X} \exp[-\mu_n^2 K (\text{Pe Fo} - \eta/R)] - \\ &- \sum_{n=1}^{\infty} A_n^2 \frac{\sin \mu_n}{\mu_n} \left\{ \left[ \frac{\mu_n^2}{\mu_n^2 + \text{Bi}_q (\text{Bi}_q - 1)} + \mu_n^2 K (\text{Pe Fo} - \eta/R) + \frac{\text{Bi}_q}{2} \right] \times \right. \\ &\times \left. \frac{\sin \mu_n X}{\mu_n X} - \cos \mu_n X \right\} \text{Mr Mn} \frac{\eta}{R} \exp \left[ -\mu_n^2 K \left( \text{Pe Fo} - \frac{\eta}{R} \right) \right] + \dots; \\ T_o = \frac{t_o(\eta/R, \text{Fo}) - t_o}{t_{co} - t_o} &= 1 - \sum_{n=1}^{\infty} A_n \text{Mr Mn} \frac{\eta}{R} \exp \left[ -\mu_n^2 K \left( \text{Pe Fo} - \frac{\eta}{R} \right) \right] + \dots \end{aligned}$$

In these equations

$$\begin{aligned} A_n &= \frac{2(\sin \mu_n - \mu_n \cos \mu_n)}{\mu_n - \sin \mu_n \cos \mu_n} = \frac{2\text{Bi}_q \sqrt{\mu_n^2 + (\text{Bi}_q - 1)^2}}{\mu_n^2 + \text{Bi}_q (\text{Bi}_q - 1)}; \\ K &= \frac{\text{Mr } c_o \gamma_o}{\text{Bi}_q c_q \gamma}; \end{aligned}$$

and  $\mu_n$  are the roots of the characteristic equation

$$\tan \mu = -\frac{\mu}{\text{Bi}_q - 1}.$$

By means of the previously described methods of solution it is possible to find more general solutions of the system of heat-exchange and mass-exchange equations for a layer and to analyze the process on this basis.

If the heat exchange is not complicated by mass exchange then the solution is simplified considerably, since instead of the system of differential equations we obtain two independent transfer equations. Solutions of this type of problem have been given by Ivantsev and Lyubov /17/, Gol'dfarb /18, 19/, Aksel'rud /15/, and others. Here, only the method for solving one-dimensional problems, proposed by Gol'dfarb /19/, will be given, and for solutions of other problems the reader is referred to the references, especially /15/.

We begin with the familiar heat-conduction equation

$$\frac{\partial t}{\partial \tau} = a_q \frac{\partial^2 t}{\partial x^2} + \frac{\Gamma}{x} \cdot \frac{\partial t}{\partial x}, \quad (7-5-10)$$

where  $\Gamma = 2\nu + 1$  (for a plate  $\nu = -\frac{1}{2}$ , for a cylinder  $\nu = 0$ , and for a sphere  $\nu = 1/2$ ).

The limiting conditions are

$$t(x, 0) = t_{ceq} = \text{const}; \quad t_{co}(0) = t_{co} = \text{const}; \quad (7-5-11)$$

$$\frac{\partial t(0, \tau)}{\partial x} = 0; \quad (7-5-12)$$

$$-\lambda_q \frac{\partial t(R, \tau)}{\partial x} + \alpha_q [t_c(\tau) - t(R, \tau)] = 0; \quad (7-5-13)$$

$$\pm w \frac{\partial t_c(\tau)}{\partial \tau} + \alpha_q [t_c(\tau) - t(R, \tau)] = 0, \quad (7-5-14)$$

where  $w$  is the water equivalent of the gases (taken over a unit heating surface), and where a plus sign in front of the water equivalent corresponds to uniflow while a minus sign indicates counterflow.

We now apply a Laplace transformation to equation (7-5-10), taking into account limiting conditions (7-5-11) through (7-5-14), to obtain the following solution in terms of transforms:

$$t_L(x, s) - \frac{t_{ceq}}{s} = \frac{(t_{c0} - t_{ceq}) \left(\frac{R}{x}\right)' I_1\left(\sqrt{\frac{s}{\alpha_q}} x\right)}{s \left[ I_1\left(\sqrt{\frac{s}{\alpha_q}} R\right) + \frac{1}{Bi_q} \sqrt{\frac{s}{\alpha_q}} RI_{1/2+1}\left(\sqrt{\frac{s}{\alpha_q}} R\right) \right] \pm \frac{\alpha_q}{w} \frac{1}{Bi_q} \sqrt{\frac{s}{\alpha_q}} RI_{1/2+1}\left(\sqrt{\frac{s}{\alpha_q}} R\right)}; \quad (7-5-15)$$

$$\{t_c(s)\}_L = \frac{\alpha_q t_L(R, s) \pm w t_{c0}}{\alpha_q \pm ws}. \quad (7-5-16)$$

The corresponding inverse transforms are

$$T(X, Fo) = \frac{t(x, \tau) - t_{ceq}}{t_{c0} - t_{ceq}} = \frac{1}{1 \pm \frac{x}{2v+2}} - \sum_{n=1}^{\infty} P_v(\mu_n) \frac{1}{X^v} \frac{J_v(\mu_n X)}{J_v(\mu_n)} \exp(-\mu_n^2 Fo); \quad (7-5-17)$$

$$T_c(Fo) = \frac{t_c(\tau) - t_{ceq}}{t_{c0} - t_{ceq}} = \frac{1}{1 \pm \frac{x}{2v+2}} + \sum_{n=1}^{\infty} \frac{P_v(\mu_n)}{\frac{\mu_n^2}{\pm x Bi_q} - 1} \exp(-\mu_n^2 Fo). \quad (7-5-18)$$

In these equations

$$P_v(\mu_n) = \frac{2Bi_q(\mu_n^2 \pm x Bi_q)}{Bi_q(Bi_q - 2v)\mu_n^2 \pm x(Bi_q)^2(2v+2) + (\mu_n^2 \pm x Bi_q)^2};$$

$$x = \frac{R\gamma_{ce}}{w};$$

and  $\mu_n$  are the roots of the characteristic equation

$$\frac{J_v(\mu_n)}{J_{v+1}(\mu_n)} = \frac{\mu_n}{Bi_q} - \frac{(\pm x)}{\mu_n}.$$

For  $x=0$  solution (7-5-17) becomes solution (6-5-45'), while for both  $x=0$  and  $Bi_q=\infty$  it becomes solution (6-5-43') in § 6-5. Particular solutions for the individual body shapes /19/ may be obtained from (7-5-17) and (7-5-18) by substituting in them  $v=-1/2, 0, 1/2$  and by taking into account the relation between Bessel functions of fractional order and the trigonometric functions, just as was indicated previously in § 6-5. For example, for spherical particles solutions (7-5-17) and (7-5-18) are

$$T(X, Fo) = \frac{1}{1 \pm \frac{x}{3}} - \sum_{n=1}^{\infty} A_n \frac{\sin \mu_n X}{X} \exp(-\mu_n^2 Fo);$$

$$T_c(Fo) = \frac{1}{1 \pm \frac{x}{3}} + \sum_{n=1}^{\infty} \frac{A_n}{\pm \frac{\mu_n^2}{x Bi_q} - 1} \sin \mu_n \exp(-\mu_n^2 Fo),$$

where

$$A_n = \frac{2\mu_n^2 (\sin \mu_n - \mu_n \cos \mu_n)}{\mu_n^3 (\mu_n - \sin \mu_n \cos \mu_n) \pm 2x \sin \mu_n (\sin \mu_n - \mu_n \cos \mu_n)};$$

and  $\mu_n$  are the roots of the equation

$$\frac{\mu_n}{1 - \mu_n \cot \mu_n} = \frac{\mu_n}{Bi_q} - \frac{\pm x}{\mu_n}.$$

If we neglect the resistance to the transfer of heat and mass in individual particles of the layer, then system of equations (7-5-1) and (7-5-2) together with limiting conditions (7-5-3) through (7-5-7) is transformed into a system of equations for a continuous medium. A detailed analysis of various analytical solutions of problems of this kind has been made by Chudnovskii /20/ and Kantorovich /21/. We will consider here only the most typical methods for describing unsteady uncoupled transfer in a layer.

As a representation of the unsteady heating of a layer of material (consisting of lumps) made up of particles of small dimensions or high thermal conductivity, no heat sources being present, Anzelius /22/ and Schumann /23/ proposed the following system of equations:

$$\frac{\partial t_0}{\partial \tau} + v \frac{\partial t_0}{\partial x} = -\frac{\alpha_v}{c_e m} (t_0 - t); \quad (7-5-19)$$

$$\frac{\partial t}{\partial \tau} = \frac{\alpha_v}{c_q (1 - m)} (t_0 - t), \quad (7-5-20)$$

where  $\alpha_v$  is the heat-exchange coefficient, referred to unit volume of the layer, and  $x$  is the spatial coordinate in the gas in the flow direction along the layer. If we introduce new variables

$$Y = \frac{\alpha_v x}{vc_e m} \quad \text{and} \quad Z = \frac{\alpha_v}{c_q (1 - m)} \left( \tau - \frac{x}{v} \right), \quad (7-5-21)$$

then (7-5-19) and (7-5-20) can be reduced to the system of equations

$$\frac{\partial t}{\partial Z} = t_0 - t; \quad (7-5-22)$$

$$\frac{\partial t_0}{\partial Y} = t - t_0. \quad (7-5-23)$$

A whole series of solutions of equations (7-5-22) and (7-5-23) have been derived, both for heat exchange and for diffusion (see /22/ through /27/). These solutions give the temperatures  $t$  and  $t_0$  as functions of the dimensionless quantities  $Y$  and  $Z$ .

The above method has also been used by many other authors to investigate heat exchange and mass exchange in a layer, especially to determine the exchange coefficients (Saunders and Ford, Chukhanov and Shapatina, Kichkina and Zhuchkov, Vetrov and Todes, among others).

Consequently, it will be interesting to demonstrate a means of solving system of equations (7-5-22) and (7-5-23). Here we will show what is, in our opinion, a quite simple method for integrating the system. In this method, which was proposed by Smirnov /27/, we first consider the system of equations

$$\frac{\partial T}{\partial Z} = b(T_0 - T) \quad \text{and} \quad \frac{\partial T_0}{\partial Y} = -a(T_0 - T) \quad (7-5-24)$$

subject to the conditions

$$T_0|_{Y=0}=1 \quad \text{and} \quad T|_{Z=0}=0, \quad (7-5-25)$$

where  $T_c(Z, Y)$  and  $T(Z, Y)$  are the dimensionless temperatures of the surroundings and the body (or of the two fluid flows), which are moving respectively in the  $y$  and  $x$  directions;  $a$  and  $b$  are constants depending on the area of the heating surface over which flow takes place and on the thermal conductivity and specific heat; and  $Z$  and  $Y$  are the dimensionless variables defined by (7-5-21).

If  $T$  is eliminated from system (7-5-24), then we obtain the following equation in  $T_c$ :

$$\frac{\partial^2 T_c}{\partial Z \partial Y} + a \frac{\partial T_c}{\partial Z} + b \frac{\partial T_c}{\partial Y} = 0$$

subject to the conditions

$$T_c|_{Z=0} = \exp(-aY) \quad \text{and} \quad T_c|_{Y=0} = 1.$$

If we define

$$T_c = \exp[-aY - bZ] u; \quad \xi = aZ; \quad \eta = bY,$$

then we obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = u; \quad (7-5-26)$$

$$u|_{\xi=0} = 1; \quad u|_{\eta=0} = \exp \frac{b\xi}{a}. \quad (7-5-27)$$

The following general solution of equation (7-5-26) derived by Eugin /28/ may be used to solve (7-5-26) and (7-5-27):

$$u(\xi, \eta) = \int_0^\xi \phi_1(\xi) J_0(2i\sqrt{\eta(\xi - \xi^*)}) d\xi^* + \int_0^\eta \phi_2(\eta^*) J_0(2i\sqrt{\xi(\eta - \eta^*)}) d\eta^* + u(0, 0) J_0(2i\sqrt{\xi\eta}), \quad (7-5-28)$$

where  $\phi_1(\xi^*)$  and  $\phi_2(\eta^*)$  are arbitrary functions. By means of conditions (7-5-27) it is easy to show that

$$\phi_1(\xi^*) = \frac{b}{a} \exp \frac{b\xi^*}{a} \quad \text{and} \quad \phi_2(\eta^*) = 0.$$

Consequently,

$$u(\xi, \eta) = \int_0^\xi \frac{b}{a} J_0(2i\sqrt{\eta(\xi - \xi^*)}) \exp \frac{b\xi^*}{a} d\xi^* + J_0(2i\sqrt{\xi\eta})$$

or, reversing the substitutions made earlier, we have

$$T_c(Z, Y) = \exp[-aY - bZ] \left[ b \int_0^Z \exp(b\xi^*) J_0(2i\sqrt{abY(Z - \xi^*)}) d\xi^* + J_0(2i\sqrt{abZY}) \right]. \quad (7-5-29)$$

Then, from the second equation of system (7-5-24) it follows that

$$T(Z, Y) = -i \exp[-aY - bZ] \left[ b^2 \int_0^Z \exp(b\xi^*) J_1(2i\sqrt{abY(Z - \xi^*)}) \times \right. \\ \left. \times \frac{(Z - \xi^*) d\xi^*}{\sqrt{abY(Z - \xi^*)}} + bZ \frac{J_1(2i\sqrt{abZY})}{\sqrt{abZY}} \right]. \quad (7-5-30)$$

Equations (7-5-29) and (7-5-30) represent solutions of system (7-5-24).

In the study of heat-exchange processes involving phase or chemical transformations, a solution of system of equations (7-5-19) and (7-5-20) when the latter system includes a source term is much more interesting. Brinkley /29/, Rabinovich /30, 31/, and others have derived solutions of this problem. The solution methods for these problems do not differ from those of the problems considered above.

## 7-6. Regular Thermal Conditions

An analysis of the solutions derived above shows that an unsteady process of heat and mass transfer develops in three stages. The first stage is a disordered period during which the initial state of the system has a strong effect on the potential fields. To a certain extent this effect is a random one, since any nonuniformities in the initial distributions of the potentials are reflected in the subsequent distributions. When the heat and mass transfer are coupled, the disordered nature and the instability of the process are intensified, due to superposition of the force fields of the various potentials. As an example, we may mention the first stage of heat and mass transfer when the rate of mass exchange at the surface of a body is constant (§§ 6-2 and 6-6). A correct analytical description of this stage of a process is achieved only by taking into account a considerably large number of terms in the infinite sum.

As time goes by, the effect of the initial potential distributions on their subsequent variations levels off. During the development of coupled processes greater directivity and uniformity are established. From the first, disordered stage the process passes to the second, ordered stage. For an analytical description of this stage, it is sufficient to retain from one to three terms of the infinite sum. It should be noted that there is some difference between the second stages of "coupled" and "uncoupled" transfer phenomena. Processes of uncoupled transfer are particularly interesting, since during these the so-called regular conditions are established. For regular conditions the relation between the dimensionless potential and the Fourier number is a simple exponential law.

For the coupled processes determined by system of equations (4-1-2) and (4-1-3) actual regular conditions are not established. Due to the coupling between heat transfer and mass transfer, the relation between the potentials and the Fourier number is not exponential. The apparent simplicity of the process is a result of the establishment of a relatively stable coupling between the potentials. Mathematically, the difference between the two situations consists only in that a single term of the sum is sufficient to describe regular conditions, whereas to describe the stable stage of a coupled transfer up until the steady state two or three terms must be retained.

The third stage corresponds to the steady state ( $Fo = \infty$ ), during which the potentials at all points in the body are equal to the potentials of the surroundings.

Let us examine in more detail the laws describing the regular regime, since at present these laws are widely used in various fields of engineering and physics to calculate not only thermal processes /32, 33/ but also diffusion processes /34, 35/.

If the heat exchange between a body and surroundings at constant temperature ( $t_c$  constant) obeys the convection law (boundary conditions of the third kind), then the solution for the simplest body shapes may be written as (see equation (6-5-10))

$$1 - T(X, Fo) = \frac{t_c - t}{t_c - t_0} = \sum_{n=1}^{\infty} A_{nr} \Phi_r(\mu_n X) \exp(-\mu_n^2 Fo), \quad (7-6-1)$$

where  $A_{nr}$  are the initial thermal amplitudes and  $\mu_n$  are the roots of the corresponding characteristic equations. In deriving formula (7-6-1) a very

simple initial condition has been specified (uniform initial temperature distribution in the body). However, this simplification does not impose any restrictions on the basic conclusions to follow, and they will be valid even for complex initial conditions.

As we have noted over and over, with increasing  $Fo$  the terms of series (7-6-1) decrease in absolute value but not at the same rate, so that the terms from the second term on soon become negligibly small in comparison with the first term. Therefore, the temperature  $T$  at any point of the body may, long before it becomes essentially equal to the temperature of the surroundings ( $T_c=1$ ), be represented by a monomial exponential relation:

$$\frac{t_0 - t}{t_0 - t_0} = A_{1r} \Phi_r(\mu_1 X) \exp(-\mu_1^2 Fo). \quad (7-6-2)$$

From this moment on, the body is subjected to special heating conditions, characterized by a constant rate of variation of the logarithm of the excess temperature:

$$-\frac{\partial [\ln(t_0 - t)]}{\partial \tau} = m. \quad (7-6-3)$$

The quantity  $m$ , known as the heating rate, is of central importance in the theory describing the regular regime, since it is a measure of the rapidity of the regular heating (or cooling). The rate of heating for the average temperature of the body may be defined similarly as

$$\frac{1}{t_0 - \bar{t}} \cdot \frac{d\bar{t}}{d\tau} = m. \quad (7-6-4)$$

Let us note that the heating and cooling of multidimensional bodies (prisms of rectangular cross section, rectangular parallelepipeds, short cylinders, etc.) also obey these laws. It will be shown in the next chapter that for such multidimensional bodies a solution can be obtained by multiplying together the solutions corresponding to the three unidimensional thermal fluxes (for the three-dimensional case) in the directions of each coordinate axis:

$$1 - T = \frac{t_0 - t}{t_0 - t_0} = \sum_{n_1} \sum_{n_2} \sum_{n_3} A_{n_1, n_2, n_3} \Phi\left(\mu_{n_1, i} \frac{x_i}{R_i}\right) \exp\left[-\left(\mu_{n_1, i} \frac{R_V}{R_i}\right)^2 Fo_V\right],$$

where  $A_{n_1} = (A_{n_1}, A_{n_2}, A_{n_3})$  are the initial thermal amplitudes,  $\Phi\left(\mu_{n_1, i} \frac{x_i}{R_i}\right)$  are functions which take into account the temperature variations along the coordinates ( $x=x_1, y=x_2, z=x_3$ ),  $R_i$  are the dimensions of the body, and  $R_V$  is the generalized dimension of the body, equal to the body volume ( $V$ ) divided by the body surface area ( $F$ ).

In this case equation (7-6-2) becomes

$$\frac{t_0 - t}{t_0 - t_0} = \prod_{l=1}^3 A_{l, i} \Phi\left(\mu_{l, i} \frac{x_i}{R_i}\right) \exp\left[-\left(\mu_{l, i} \frac{R_V}{R_i}\right)^2 Fo_V\right]. \quad (7-6-2')$$

Temkin and others have shown that the problem of the heating of a body of complex configuration can be reduced to a problem of the heating of a body of basic shape (plate, cylinder, or sphere) by introducing a criterion for approximate similarity. This makes it possible to extend the laws describing regular conditions to bodies of any shape.



From (7-6-4) it is possible to determine the value of  $m$  for bodies of any shape, using the balance equation

$$c_q \gamma V \frac{dt}{d\tau} = a_q F (t_0 - t_s),$$

from which we obtain

$$m = \frac{a_q F}{c_q \gamma V} \cdot \frac{t_0 - t_s}{t_0 - t_s} = \frac{a_q}{R_V^2} \cdot \frac{a_q R_V}{\lambda_q} \psi = \frac{a_q}{R_V^2} \text{Bi}_V \psi. \quad (7-6-5)$$

Here  $\psi = \frac{t_0 - t_s}{t_0 - t_s}$  is the dimensionless parameter characterizing the nonuniformity of the temperature field. The quantity  $\text{Bi}_V = \frac{a_q R_V}{\lambda_q}$  is the Biot number, the characteristic body dimension  $R_V$  being the ratio of the body volume to the body surface area. Consequently, the numerical value of  $m$  is determined by the thermal coefficients, the dimensions, the body shape, and the heat-exchange conditions.

From equation (7-6-5) we obtain a relation which is very important in the theory describing the regular regime:

$$\text{Kn} = \psi \text{Bi}_V, \quad (7-6-6)$$

where  $\text{Kn} = \frac{m R_V}{a_q} = \sum_{i=1}^3 \left( \mu_{1,i} \frac{R_V}{R_i} \right)^2$  is the Kondrat'ev number, a dimensionless

quantity determined by the body shape and by the characteristic numbers:  $\mu_{1,1}$ ,  $\mu_{1,2}$ , and  $\mu_{1,3}$  (and thus by the Biot number as well, since the characteristic numbers are functions of the Biot number).

It turns out that the curves of  $\text{Kn} = f(\text{Bi}_V)$  for bodies of completely different geometrical shapes (spheres, parallelepipeds, cylinders, etc.) lie so close to each other that their whole family can practically be replaced by a single average curve (Figure 7-18). An analytical expression for the curve has been given by Yaryshev /33/:

$$\psi = \frac{\text{Kn}}{\text{Bi}_V} = \frac{1}{\sqrt{\text{Bi}_V^2 + 1.437 \text{Bi}_V + 1}}.$$

If  $\text{Bi}_V \rightarrow 0$  (in practice, if  $\text{Bi}_V < 0.1$ ), then  $t_s \rightarrow t$  ( $\psi = 1$ ). Consequently, the Kondrat'ev number will equal the Biot number ( $\text{Kn}_0 = \text{Bi}_V$ ) and the heating rate will be  $m = a_q / c_q \gamma R_V$ , so that we obtain the Newtonian law for the heating of a body.

If  $\text{Bi}_V \rightarrow \infty$  (in practice, if  $\text{Bi}_V > 100$ ), then the Kondrat'ev number  $(\text{Kn})_\infty = \sum_{i=1}^3 (\mu_{1,i}^2)_\infty \frac{R_V^2}{R_i^2}$  will be constant. In this case the heating rate will be directly proportional to the thermal diffusivity (Kondrat'ev's first theorem), namely  $(m)_\infty = \frac{a_q}{R_V^2} (\text{Kn})_\infty$ . Thus, the value of  $\text{Kn}$  lies between zero and some constant value  $(\text{Kn})_\infty$ , the latter being determined by the shape of the body.

Following the suggestion of Kondrat'ev, we call the given type of heating kinetics regular conditions of the first kind. For the center of a plate or cylinder the beginning of this regime is a function of  $\text{Fo}$  and  $\text{Bi}$ , of the temperature simplex [dimensionless parametric temperature]  $T(0, \text{Fo}) = \frac{t_c - t_{ce}}{t_c - t_0}$ , and of the allowable calculation error  $\varepsilon$  (see Figures 7-19 and 7-20 and also /36/). These figures show that, for low values of the Biot

number, at the central points of the given bodies regular conditions are established quite rapidly. As the Biot number increases, the relative time required to establish regular conditions increases, then attains a maximum at  $Bi \approx 2$ , and subsequently decreases again somewhat. For given  $\epsilon$  and  $Bi$ ,

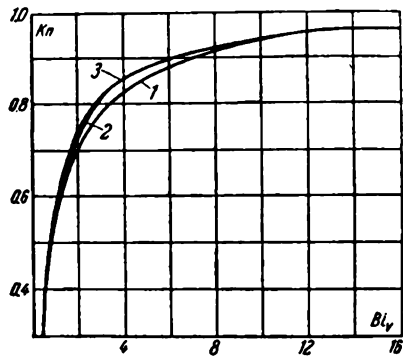


FIGURE 7-18. Universal approximate curve for  $Kn=f(Bi_0)$ , showing the rate of regular heating or cooling of a body as a function of the generalized Biot number  
1) plate; 2) sphere; 3) cylinder

the regular-regime stage begins earlier on a cylinder axis than at the center of a plate.

If the temperature  $T_0(\tau)$  of the external surroundings varies at a constant rate, then after a certain time the temperatures at all points in the system begin to vary at this same rate. For example, if the temperature of the surroundings varies according to the linear law  $t_0(\tau)=t_0+b\tau$ , then the coefficient  $b$  represents the constant rate of variation in the environment.

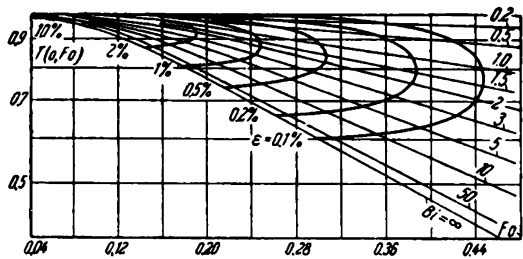


FIGURE 7-19. The beginning of regular thermal conditions in the midplane of a plate, expressed as a functional relation between  $F_0$ ,  $Bi$ ,  $T(0, F_0)$ , and the allowable calculation error  $\epsilon$

In dimensionless form, this rate is expressed by the Predvoditelev number  $Pd=\frac{bR^2}{a_0\Delta t}$ . The general solution of problems of uncoupled transfer is

in this case

$$T(X, Fo) = \frac{t - t_0}{t_0} = \text{Pd} \left[ Fo - \frac{1}{2(\Gamma + 1)} \left( 1 - X^2 + \frac{2}{\text{Bi}} \right) \right] + \sum_{n=1}^{\infty} \frac{A_{n\Gamma}}{\mu_n^2} \Phi_{\Gamma}(\mu_n X) \exp(-\mu_n^2 Fo). \quad (7-6-7)$$

An analysis of (7-6-7) shows that, for all values  $Fo > Fo_0$ , the series sum is negligibly small in comparison with the expression in square brackets. From this moment the temperature at any point of the body is a linear function of time, and the temperature distribution may be represented by a simple parabola (the temperature gradient at any point of the body does not vary with time). Analytically, this result indicates that so-called "quasi-steady" conditions have been established in the body. This stage of the heating development is called regular conditions of the second kind.

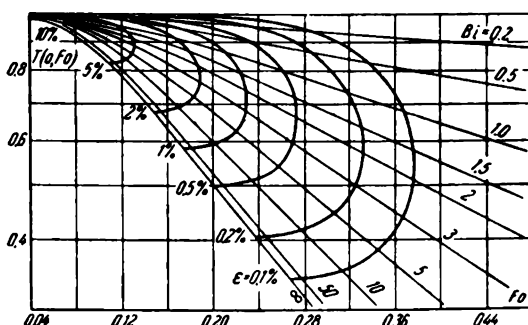


FIGURE 7-20. The beginning of regular thermal conditions on the axis of an infinite cylinder, expressed as a functional relation between  $Fo$ ,  $Bi$ ,  $T(t, Fo)$ , and the allowable calculation error  $\epsilon$

It is interesting to note that a similar result is obtained if the body is heated by a constant or varying thermal flow at the surface of the body (boundary conditions of the second kind). For example, for a constant thermal flux, the solution for the plate can be written as

$$T(X, Fo) = \frac{t - t_0}{t_0 - t_s} = \text{Ki}_q \left[ Fo - \frac{1}{6} (1 - 3X^2) \right] + \text{Ki}_q \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\mu_n^2} \cos \mu_n X \exp(-\mu_n^2 Fo). \quad (7-6-8)$$

During the stage of regular conditions of the second kind, as indicated by (7-6-8), the temperature gradient at any point of the body does not vary and the temperature distribution is parabolic. Consequently, regular conditions of the heating development are established not only with respect to the temperature fields but also with respect to the heat fluxes.

Regular conditions of the first and second kinds have a property in common, namely that the ratio between the specific heat flux  $j_q(x, \tau)$  at any point in the body and the specific heat flux at the body surface is time-independent (is a function of the spatial coordinates of the body only) [37]. For example,

with boundary conditions of the third kind we have:

a) for a constant surrounding temperature ( $t_c$  constant),  
for a plate,

$$\frac{j_q(x, \tau)}{(j_q)_s} = \frac{\sin \mu_1 x / R}{\sin \mu_1};$$

for a cylinder,

$$\frac{j_q(r, \tau)}{(j_q)_s} = \frac{J_1(\mu_1 r / R)}{J_1(\mu_1)};$$

for a sphere,

$$\frac{j_q(r, \tau)}{(j_q)_s} = \frac{(r \cos \mu_1 r / R - \sin \mu_1 r / R)}{r^2 / R^2 (R \cos \mu_1 - \sin \mu_1)};$$

b) for a variable surrounding temperature ( $t_c = t_c(\tau)$ ); if  $t_c(\tau) = t_m - (t_m - t_0) \exp(-\kappa \tau)$ , then the characteristic numbers [eigenvalues]  $\mu_1$

for a plate, cylinder, and sphere will be the same, namely  $\mu_1 = \sqrt{\frac{\kappa}{a_q}} R$ ;

if the surrounding temperature varies linearly with time, then the heat-flux ratio will be

$$\frac{j_q(x, \tau)}{(j_q)_s} = \frac{x}{R} \quad \text{or} \quad \frac{j_q(r, \tau)}{(j_q)_s} = \frac{r}{R}. \quad (7-6-9)$$

Thus, in one-dimensional heat-conduction problems the ratio between the heat flux at any point of a body and the surface heat flux is, during the regular-regime stage, equal to the dimensionless coordinate of the body. For boundary conditions of the second kind we obtain exactly the same relation, which is valid for both constant and variable heat fluxes at the body surface. Relation (7-6-9) for one-dimensional heat-conduction problems, as well as the general characteristic equation for regular conditions of the first and second kinds, remain valid even when a continually acting heat source (constant source strength  $\omega$ ) is present. The above general property of regularization of the kinetics when a solid body is heated can be given a firm physical basis by means of the thermodynamics of irreversible processes /6/.

## Chapter VIII

### TWO-DIMENSIONAL AND THREE-DIMENSIONAL FIELDS OF HEAT-TRANSFER AND MASS-TRANSFER POTENTIAL

#### 8-1. Heat and Mass Transfer in a Semi-Infinite Medium. Boundary Conditions of the First and Second Kinds

In many practical problems two-dimensional and three-dimensional bodies or media are dealt with. Under such conditions a one-dimensional mathematical model of heat and mass transfer is unsuitable. Thus, in the present chapter methods for solving the systems of heat-transfer and mass-transfer equations

$$\frac{\partial \theta_1}{\partial \tau} = a_v \nabla^2 \theta_1 + sp \frac{c_m}{c_v} \cdot \frac{\partial \theta_2}{\partial \tau}; \quad (8-1-1)$$

$$\frac{\partial \theta_2}{\partial \tau} = a_m \nabla^2 \theta_2 + a_m \delta \nabla^2 \theta_1 \quad (8-1-2)$$

will be considered, where  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  or  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$  is the Laplace operator. In addition, methods for solving generalized systems of differential transfer equations (see also § 9-6) such as

$$\frac{\partial \theta_1}{\partial \tau} = K_{11} \nabla^2 \theta_1 + K_{12} \nabla^2 \theta_2 + w_1(x, y, z, \tau); \quad (8-1-1')$$

$$\frac{\partial \theta_2}{\partial \tau} = K_{21} \nabla^2 \theta_1 + K_{22} \nabla^2 \theta_2 + w_2(x, y, z, \tau) \quad (8-1-2')$$

will be studied. System (8-1-1) and (8-1-2) is a particular case of system (8-1-1') and (8-1-2'). Here  $w_l$  ( $l=1, 2$ ) are certain sources corresponding to the transfer potentials  $\theta_l$ . To simplify the calculations, we have denoted the temperature as  $\theta_1$  and the mass-transfer potential as  $\theta_2$ .

Once again integral-transformation methods are the most effective means of obtaining rapidly the desired solutions. Along with the previously considered methods of integral transformation, we will often use complex Fourier transformations in their various forms to solve multidimensional problems. In contrast to the previous chapters, the problems will be solved mainly in terms of dimensional quantities.

The systems of heat-transfer and mass-transfer equations have been solved by Tsoi /1, 2, 3/ for a semi-infinite medium, by Kim and Ivanova /4/ for an infinite two-dimensional plate, and by Prudnikov /5, 6/ for a finite plate. Solutions of the differential equations for uncoupled transfer, with various boundary conditions and for various body shapes, have been derived by many Soviet and non-Soviet authors. Summaries of some of

these solutions are given in references /8/ through /10/. Several interesting works which have been carried out in recent years will be discussed in §§ 8-4 and 8-5.

Let us find the potentials  $\theta_l(x, y, z, \tau)$  for a semi-infinite three-dimensional medium in the domain  $\Omega$ :

$$\begin{aligned} 0 &\leq x < \infty; \\ -\infty &< y < \infty; \\ -\infty &< z < \infty; \\ \tau &> 0, \end{aligned}$$

where these potentials satisfy system of equations (8-1-1) and (8-1-2), initial conditions

$$\theta_l(x, y, z, 0) = f_l(x, y, z) \quad (l=1, 2), \quad (8-1-3)$$

and boundary conditions of the first or second kind.

### 1. Unsteady three-dimensional fields of transfer potential with boundary conditions of the second kind

Boundary conditions of the second kind state that

$$\frac{\partial \theta_l(0, y, z, \tau)}{\partial x} = \varphi_l(y, z, \tau) \quad (l=1, 2). \quad (8-1-4)$$

Let us seek potential functions  $\theta_l$  and  $\bar{\theta}_l$  for which a Fourier transformation is applicable with respect to the spatial coordinates and for which a Laplace transformation is applicable with respect to the time  $\tau$  in a domain  $\Omega$ . It is assumed that the functions being defined are regular at infinity:

$$\theta_l(x, y, z, \tau) \rightarrow 0 \text{ for } R^2 = x^2 + y^2 + z^2 \rightarrow \infty.$$

If we define

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_l(x, y, z, \tau) \exp[i(\eta y + \zeta z)] dy dz = u_l(x, \eta, \zeta, \tau) \quad (l=1, 2), \quad (8-1-5)$$

then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\partial^2 \theta_l}{\partial y^2} + \frac{\partial^2 \theta_l}{\partial z^2} \right) \exp[i(\eta y + \zeta z)] dy dz = -(\eta^2 + \zeta^2) u_l(x, \eta, \zeta, \tau).$$

Now, with respect to the  $x$  coordinate the Fourier cosine transformation

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} u_l(x, \eta, \zeta, \tau) \cos x\xi dx = u_l(\xi, \eta, \zeta, \tau) = \{u_l\}_\xi \quad (8-1-6)$$

is applied, so that for the second derivative

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u_l}{\partial x^2} \cos x\xi dx = -\xi^2 \{u_l\}_\xi - \sqrt{\frac{2}{\pi}} \Phi_l,$$

where

$$\Phi_l = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_l(y, z, \tau) \exp[i(\eta y + \zeta z)] dy dz.$$

When transformations (8-1-5) and (8-1-6) are applied to system of equations (8-1-1) and (8-1-2), then taking into account the expressions for the second derivatives we obtain

$$\left. \begin{aligned} \frac{\partial}{\partial \tau} \{u_1\}_e &= -a_q r^2 \{u_1\}_e + \sigma \frac{c_m}{c_q} \cdot \frac{\partial}{\partial \tau} \{u_1\}_e - a_q \sqrt{\frac{2}{\pi}} \Phi_1; \\ \frac{\partial}{\partial \tau} \{u_2\}_e &= -a_m r^2 \{u_2\}_e - a_m \delta r^2 \{u_1\}_e - \sqrt{\frac{2}{\pi}} (a_m \delta \Phi_1 + a_m \Phi_2), \end{aligned} \right\} \quad (8-1-7)$$

where  $r^2 = \xi^2 + \eta^2 + \zeta^2$ .

The initial conditions for system (8-1-7) are

$$\{F_l\}_e = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_l(x, \eta, \zeta) \cos x\xi dx, \quad (8-1-8)$$

where

$$F_l = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_l(x, y, z) \exp[i(\eta y + \zeta z)] dy dz.$$

Next, we apply to system (8-1-7) the Laplace transformation

$$\{u_l\}_{eL} = \{u_l(\xi, \eta, \zeta, \tau)\}_L = \int_0^{\infty} u_l(\xi, \eta, \zeta, \tau) \exp(-s\tau) d\tau, \quad (8-1-9)$$

in order to obtain, taking into account initial conditions (8-1-8),

$$\left. \begin{aligned} (a_m r^2 + s) \{u_2\}_{eL} + a_m \delta r^2 \{u_1\}_{eL} &= \\ &= -\sqrt{\frac{2}{\pi}} [a_m \{\Phi_2\}_L + a_m \delta \{\Phi_1\}_L] + \{F_2\}_e; \\ -\sigma \frac{c_m}{c_q} \{u_2\}_{eL} + (a_q r^2 + s) \{u_1\}_{eL} &= \\ &= -\sqrt{\frac{2}{\pi}} a_q \{\Phi_1\}_L - \sigma \frac{c_m}{c_q} \{F_2\}_e + \{F_1\}_e. \end{aligned} \right\} \quad (8-1-10)$$

Consequently, the transforms  $\{u_l\}_{eL}$  ( $l=1, 2$ ) can be written as

$$\{u_l\}_{eL} = \frac{D_l(\xi, \eta, \zeta, s)}{D(\xi, \eta, \zeta, s)} \quad (l=1, 2), \quad (8-1-11)$$

where

$$D(\xi, \eta, \zeta, s) = \begin{vmatrix} (s + a_m r^2) & a_m \delta r^2 \\ -\sigma \frac{c_m}{c_q} s & (s + a_q r^2) \end{vmatrix};$$

$$D_1(\xi, \eta, \zeta, s) = \begin{vmatrix} (s + a_m r^2) & -\sqrt{\frac{2}{\pi}} [a_m \{\Phi_2\}_L + a_m \delta \{\Phi_1\}_L] + \{F_2\}_e \\ -\sigma \frac{c_m}{c_q} s & -\sqrt{\frac{2}{\pi}} a_q \{\Phi_1\}_L + \{F_1\}_e - \sigma \frac{c_m}{c_q} \{F_2\}_e \end{vmatrix};$$

$$D_2(\xi, \eta, \zeta, s) = \begin{vmatrix} \left(-\sqrt{\frac{2}{\pi}}[a_m\{\Phi_2\}_L + a_m\delta\{\Phi_1\}_L] + \{F_2\}_c\right) & a_m\delta r^2 \\ \left(-\sqrt{\frac{2}{\pi}}[a_q\{\Phi_1\}_L + \{F_1\}_c - \varepsilon\rho\frac{c_m}{c_q}\{F_2\}_c\right) & (s+a_qr^2) \end{vmatrix}.$$

Let us now expand the principal determinant of system of equations (8-1-10):

$$D(\xi, \eta, \zeta, s) = s^2 + s \left[ a_m + a_q + a_m\delta\varepsilon\rho\frac{c_m}{c_q} \right] r^2 + a_m a_q r^4.$$

The roots of this determinant may be denoted as  $s_1 = -ar^2$  and  $s_2 = -br^2$ , where  $a = a_m v_1^2$ ,  $b = a_m v_2^2$  and

$$v_i^2 = \frac{1}{2} \left[ (1 + \varepsilon \text{Ko Pn} + 1/\text{Lu}) + (-1)^i \sqrt{(1 + \varepsilon \text{Ko Pn} + 1/\text{Lu})^2 - 4/\text{Lu}} \right].$$

The analysis of  $v_i^2$  and  $a_m$  carried out in the previous chapters indicates that  $a$  and  $b$  are both positive.

Let us now find the inverse transforms of the transfer potentials. As an example of the methods used, we will find the inverse transform of the mass-transfer potential. For  $\{u_2\}_{cL}$  equation (8-1-11) is

$$\begin{aligned} \{u_2\}_{cL} = & \frac{-(s+a_qr^2)\sqrt{\frac{2}{\pi}}\{\Phi\}_L}{D(\xi, \eta, \zeta, s)} + \frac{a_q a_m \delta \sqrt{\frac{2}{\pi}}\{\Phi_1\}_L r^2}{D(\xi, \eta, \zeta, s)} + \\ & + \frac{(s+a_qr^2)\{F_2\}_c + r^2\{F\}_c}{D(\xi, \eta, \zeta, s)}, \end{aligned}$$

where

$$\{\Phi\}_L = a_m \{\Phi_2\}_L + a_m \delta \{\Phi_1\}_L \quad \text{and} \quad \{F\}_c = a_m \delta \left[ \varepsilon\rho\frac{c_m}{c_q}\{F_2\}_c - \{F_1\}_c \right].$$

In order to invert the transform with respect to the parameter  $\tau$ , let us use the formula for the determination of the inverse transform when the transform is a regular fractional-rational function with simple roots:

$$L^{-1} \left[ \frac{Q_m(s)}{R_n(s)} \right] = \sum_{i=1}^n \frac{Q_m(s_i)}{R'_n(s_i)} \exp(s_i \tau),$$

where  $s_i$  are the simple roots of the polynomial  $R_n(s)$ , and where  $n > m$ .

Using this formula and the corresponding convolution theorem, we obtain in this case

$$\begin{aligned} \{u_2\}_c = & \sum_{i=1}^2 A_i^{(2)} \left\{ \sqrt{\frac{2}{\pi}} \int_0^\tau \left[ \frac{a_q a_m \delta}{a_q - a_m v_i^2} \Phi_1(\eta, \zeta, \tau^*) - \right. \right. \\ & \left. \left. - \Phi(\eta, \zeta, \tau^*) \right] \exp[-r^2 a_m v_i^2 (\tau - \tau^*)] d\tau^* + \right. \\ & \left. + \left[ \{F_2\}_c + \frac{\{F\}_c}{a_q - a_m v_i^2} \right] \exp(-r^2 a_m v_i^2 \tau) \right\}, \end{aligned}$$

where

$$A_i^{(2)} = \frac{a_q - a_m v_i^2}{a_m + a_q + a_m \delta \varepsilon \rho \frac{c_m}{c_q} - 2a_m v_i^2} = \frac{1/\text{Lu} - v_i^2}{1 + 1/\text{Lu} + \varepsilon \text{Ko Pn} - 2v_i^2}.$$



Then, we may apply to the function  $\{u_2\}_\epsilon$  an inverse Fourier cosine transformation with respect to the variable  $\xi$  to obtain

$$u_2(x, \eta, \zeta, \tau) = \sum_{i=1}^2 \frac{2}{\pi} A_i^{(2)} \left\{ \int_0^\tau \left[ \frac{a_q a_m \delta}{a_q - a_m v_i^2} \Phi_1 - \Phi \right] \exp [-(\eta^2 + \zeta^2)(\tau - \tau^*) a_m v_i^2] d\tau^* \int_0^\infty \exp [-\xi^2 a_m v_i^2 (\tau - \tau^*)] \cos x\xi d\xi + \right. \\ \left. + \int_0^\infty \left[ F_2(a, \eta, \zeta) + \frac{F(a, \eta, \zeta)}{a_q - a_m v_i^2} \right] \exp [-a_m v_i^2 (\eta^2 + \zeta^2) \tau] d\alpha \times \right. \\ \left. \times \int_0^\infty \exp (-\xi^2 a_m v_i^2 \tau) \cos \alpha \xi \cos x\xi d\xi \right\}. \quad (8-1-12)$$

The evaluation of the following improper integral:

$$\int_0^\infty \exp(-a^2 x^2) \cos(2bx) dx = \frac{\sqrt{\pi}}{2a} \exp\left(-\frac{b^2}{a^2}\right)$$

and the trigonometric relation  $2 \cos a\xi \cos x\xi = \cos \xi(x+a) + \cos \xi(x-a)$  makes it possible to rewrite equation (8-1-12) as

$$u_2(x, \eta, \zeta, \tau) = \sum_{i=1}^2 \frac{A_i^{(2)}}{a_m v_i^2 \sqrt{\pi}} \left\{ \int_0^\tau \left[ \frac{a_q a_m \delta}{a_q - a_m v_i^2} \Phi_1 - \Phi \right] \frac{\exp [-(\eta^2 + \zeta^2)(\tau - \tau^*) a_m v_i^2]}{\sqrt{\tau - \tau^*}} \exp \left[ -\frac{x^2}{4a_m v_i^2 (\tau - \tau^*)} \right] d\tau^* + \right. \\ \left. + \frac{1}{2\sqrt{\tau}} \int_0^\infty \left[ F_2(a, \eta, \zeta) + \frac{F(a, \eta, \zeta)}{a_q - a_m v_i^2} \right] \exp [-a_m v_i^2 (\eta^2 + \zeta^2) \tau] \right. \\ \left. + \zeta^2 \tau \left[ \exp \left[ -\frac{(x-a)^2}{4a_m v_i^2 \tau} \right] + \exp \left[ -\frac{(x+a)^2}{4a_m v_i^2 \tau} \right] \right] d\alpha \right\}. \quad (8-1-13)$$

To obtain the inverse transform with respect to the other coordinates, it is convenient to use the formula for the two-dimensional Fourier transformation:

$$F^{-1}\{\exp[-a^2(\eta^2 + \zeta^2)\tau]\} = \frac{1}{2a^2\tau} \exp\left(-\frac{y^2 + z^2}{4a^2\tau}\right).$$

Thus, after inverse transformation with respect to the variables  $\eta$  and  $\zeta$  and after application of the convolution theorem, equation (8-1-13) becomes

$$\theta_2(x, y, z, \tau) = \sum_{i=1}^2 A_i^{(2)} \left\{ \frac{1}{8(a_m v_i^2 \pi)^{3/2}} \times \right. \\ \times \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\varphi^*_{2i}(\beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} \exp \left[ -\frac{x^2 + (y - \beta)^2 + (z - \gamma)^2}{4a_m v_i^2 (\tau - \tau^*)} \right] d\tau^* d\beta d\gamma + \\ \left. + \frac{1}{(2V \pi a_m v_i^2)^3} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f^*_{2i}(a, \beta, \gamma) \exp \left[ -\frac{(x-a)^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_m v_i^2 \tau} \right] \times \right. \\ \left. \times \left[ 1 + \exp\left(-\frac{ax}{a_m v_i^2 \tau}\right) \right] da d\beta d\gamma \right\}, \quad (8-1-14)$$

where

$$\varphi^*_{2i}(y, z, \tau) = \frac{a_m \delta a_m v_i^2}{a_q - a_m v_i^2} \Phi_1(y, z, \tau) - a_m \varphi_2(y, z, \tau);$$

and

$$f^*_{\pm}(x, y, z) = \left(1 + \frac{a_m \delta \epsilon \rho \frac{c_m}{c_q}}{a_q - a_m v_i^2}\right) f_{\pm}(x, y, z) - \frac{a_m^2}{a_q - a_m v_i^2} f_1(x, y, z).$$

Equation (8-1-14) represents the solution of the given problem for the mass-transfer potential  $\theta_s$ .

Inversion of the transform of the heat-transfer potential is carried out by means of the same sequence of operations performed during the determination of  $\theta_s$ . Here these transformations will be omitted, and only the final result

$$\begin{aligned} \theta_1(x, y, z, \tau) = & \sum_{i=1}^2 A_i^{(1)} \left\{ \frac{1}{8(\pi a_m v_i^2)^{3/2}} \times \right. \\ & \times \int_0^{\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi^{**}_i(\beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} \exp \left[ -\frac{x^2 + (y - \beta)^2 + (z - \gamma)^2}{4a_m v_i^2 (\tau - \tau^*)} \right] d\tau^* d\beta d\gamma + \\ & + \frac{1}{(2\sqrt{\pi a_m v_i^2})^3} \int_0^{\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^{**}_i(a, \beta, \gamma) \exp \left[ -\frac{(x - a)^2 + (y - \beta)^2 + (z - \gamma)^2}{4a_m v_i^2 \tau} \right] \times \\ & \times \left[ 1 + \exp \left( -\frac{ax}{a_m v_i^2 \tau} \right) \right] da d\beta d\gamma \right\} \end{aligned} \quad (8-1-15)$$

will be given, where

$$\begin{aligned} \varphi^{**}_i(y, z, \tau) = & \left[ \frac{a_m \delta \epsilon \rho \frac{c_m}{c_q} a_m v_i^2}{a_m - a_m v_i^2} - a_q \right] \times \\ & \times \varphi_1(y, z, \tau) + \frac{a_m \epsilon \rho \frac{c_m}{c_q} a_m v_i^2}{a_m - a_m v_i^2} \varphi_2(y, z, \tau); \end{aligned}$$

$$\begin{aligned} f^{**}_i(x, y, z) = & f_1(x, y, z) - \frac{a_m \epsilon \rho \frac{c_m}{c_q}}{a_m - a_m v_i^2} f_2(x, y, z); \\ A_i^{(1)} = & \frac{a_m - a_m v_i^2}{a_m + a_q + a_m \delta \epsilon \rho \frac{c_m}{c_q} - 2a_m v_i^2} = \frac{1 - v_i^2}{1 + 1/Lu + \epsilon Ko Pn - 2v_i^2}. \end{aligned}$$

Equations (8-1-14) and (8-1-15) represent a solution of the problem for the very general initial conditions (8-1-3) and for boundary conditions of the second kind (conditions (8-1-4)). These solutions provide a basis for the derivation of several particular solutions.

As we have emphasized previously, in many processes the coefficient  $\delta$  is small. In such cases solutions (8-1-15) and (8-1-14) may be written as [3]:

$$\begin{aligned} \theta_1(x, y, z, \tau) = & \frac{1}{4\sqrt{a_q \pi^3}} \int_0^{\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi(\beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} \exp \left[ -\frac{x^2 + (y - \beta)^2 + (z - \gamma)^2}{4a_q (\tau - \tau^*)} \right] d\tau^* d\beta d\gamma - \\ & - \frac{a_m \epsilon \rho \frac{c_m}{c_q}}{4(a_m - a_q)\sqrt{\pi^3}} \int_0^{\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi_2(\beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} \exp \left[ -\frac{x^2 + (y - \beta)^2 + (z - \gamma)^2}{4a_m (\tau - \tau^*)} \right] d\tau^* d\beta d\gamma + \\ & + \frac{1}{(2\sqrt{\pi a_q \tau})^3} \int_0^{\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a, \beta, \gamma) \exp \left[ -\frac{(x - a)^2 + (y - \beta)^2 + (z - \gamma)^2}{4a_q \tau} \right] \times \end{aligned}$$

$$\begin{aligned} & \times \left[ 1 + \exp \left( -\frac{ax}{a_m \tau} \right) \right] da d\beta d\gamma + \frac{e^{\frac{c_m}{c_q}}}{\sqrt{a_m(a_m - a_q)(2\sqrt{\pi\tau})}} \times \\ & \times \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f_2(a, \beta, \gamma) \exp \left[ -\frac{(x-a)^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_m \tau} \right] \times \\ & \times \left[ 1 + \exp \left( -\frac{ax}{4a_m \tau} \right) \right] da d\beta d\gamma; \end{aligned} \quad (8-1-15')$$

$$\begin{aligned} \theta_2(x, y, z, \tau) &= \frac{1}{(2\sqrt{a_m \pi \tau})^3} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f_2(a, \beta, \gamma) \times \\ & \times \exp \left[ -\frac{(x-a)^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_m \tau} \right] \left[ 1 + \exp \left( -\frac{ax}{a_m \tau} \right) \right] da d\beta d\gamma - \\ & - \frac{1}{4\sqrt{\pi^3 a_m}} \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\varphi_2(\beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} \exp \left[ -\frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_m(\tau - \tau^*)} \right] d\tau^* d\beta d\gamma, \end{aligned} \quad (8-1-16)$$

where

$$\begin{aligned} \varphi(y, z, \tau) &= \frac{e^{\frac{c_m}{c_q}} a_m}{a_m - a_q} \varphi_2(y, z, \tau) - \varphi_1(y, z, \tau); \\ f(x, y, z) &= f_1(x, y, z) - \frac{e^{\frac{c_m}{c_q}} a_m}{a_m - a_q} f_2(x, y, z). \end{aligned}$$

Relation (8-1-16) is the familiar classical solution of the heat-conduction equation.

## 2. Unsteady three-dimensional fields of transfer potential with boundary conditions of the first kind

Let us now solve system of equations (8-1-1) and (8-1-2) for initial conditions (8-1-3) and for the boundary conditions

$$\theta_l(0, y, z, \tau) = \psi_l(y, z, \tau) \quad (l=1, 2), \quad (8-1-17)$$

where the potentials  $\theta_l$  are defined, as previously, in the domain  $\Omega$ .

As in the previous problem, the transforms of the functions  $\theta_l$  being determined are obtained by means of the following successive integral transformations: a double Fourier integral transformation with respect to  $y$  and  $z$ , a sine transformation with respect to  $x$ , and a Laplace transformation with respect to the time  $\tau$ . When these transformations are applied to system of equations (8-1-1) and (8-1-2) and to boundary conditions (8-1-3) and (8-1-17), a transform solution of the system of differential heat-transfer and mass-transfer equations is obtained. The inverse transforms are then found by means of the corresponding inverse integral transformations (carried out, of course, in reverse order).

Once the above transformations have been made, the final solution is found to be:

$$\begin{aligned} \theta_1(x, y, z, \tau) &= \sum_{i=1}^2 \frac{1}{(2\sqrt{\pi} a_m v_i^2)^3} \left\{ \frac{x}{v_i^2 a_m} \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{a_i \psi_1(\beta, \gamma, \tau^*) - b_i \psi_2(\beta, \gamma, \tau^*)}{\sqrt{(\tau - \tau^*)^3}} \times \right. \\ & \times \exp \left[ -\frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_m v_i^2 (\tau - \tau^*)} \right] d\tau^* d\beta d\gamma + \frac{1}{\sqrt{\tau^3}} \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty [c_i f_1(a, \beta, \gamma) - d_i f_2(a, \beta, \gamma)] \times \\ & \times \exp \left[ -\frac{(x-a)^2 + (y-\beta)^2 + (z-\gamma)^2}{4a_m v_i^2 \tau} \right] \left[ 1 - \exp \left( -\frac{ax}{a_m v_i^2 \tau} \right) \right] da d\beta d\gamma \Big\}; \end{aligned}$$

$$\begin{aligned} \theta_2(x, y, z, \tau) = & \sum_{i=1}^2 \frac{1}{(2\sqrt{\pi} a_m v_i^2)^3} \left\{ \frac{x}{a_m v_i^2} \int_0^\tau \int_0^\infty \frac{m_i \psi_2(\beta, \gamma, \tau^*) - n_i \psi_1(\beta, \gamma, \tau^*)}{V(\tau - \tau^*)^5} \times \right. \\ & \times \exp \left[ -\frac{x^2 + (y - \beta)^2 + (z - \gamma)^2}{4a_m v_i^2 (\tau - \tau^*)} \right] d\tau^* d\beta d\gamma + \\ & + \frac{1}{V\tau^3} \int_0^\infty \int_0^\infty [k_i f_2(\alpha, \beta, \gamma) - p_i f_1(\alpha, \beta, \gamma)] \exp \left[ -\frac{(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2}{4a_m v_i^2 \tau} \right] \times \\ & \times \left[ 1 - \exp \left( -\frac{\alpha x}{a_m v_i^2 \tau} \right) \right] d\alpha d\beta d\gamma \Big\}, \end{aligned}$$

where

$$\begin{aligned} m_i &= \frac{a_m (1/Lu - v_i^2)}{1 + 1/Lu + \epsilon Ko Pn - 2v_i^2}; \quad a_i = \frac{a_m [(1 - v_i^2)/Lu - \epsilon Ko Pn v_i^2]}{1 + 1/Lu + \epsilon Ko Pn - 2v_i^2}; \\ n_i &= \frac{a_m b v_i^2}{1 + 1/Lu + \epsilon Ko Pn - 2v_i^2}; \quad b_i = \frac{a_m \epsilon p c_m / c_q v_i^2}{1 + 1/Lu + \epsilon Ko Pn - 2v_i^2}; \\ k_i &= 1 - \frac{(1 - v_i^2)}{1 + 1/Lu + \epsilon Ko Pn - 2v_i^2}; \quad c_i = \frac{1 - v_i^2}{1 + 1/Lu + \epsilon Ko Pn - 2v_i^2}; \\ p_i &= \frac{1}{1 + 1/Lu + \epsilon Ko Pn - 2v_i^2}; \quad d_i = \frac{\epsilon p c_m / c_q}{1 + 1/Lu + \epsilon Ko Pn - 2v_i^2}. \end{aligned}$$

It should be noted that, during inversion of the transform with respect to the parameter  $\xi$ , it is necessary to use the following familiar relation for the improper integral:

$$\int_0^\infty \xi \exp(-a^2 \xi^2 \tau) \sin x \xi d\xi = \frac{x \sqrt{\pi}}{4a^2 \sqrt{\tau^3}} \exp\left(-\frac{x^2}{4a^2 \tau}\right).$$

### 3. A generalized system of heat-transfer and mass-transfer equations. Boundary conditions of the second kind

Let us now solve generalized system of differential transfer equations (8-1-1') and (8-1-2') for initial conditions (8-1-3) and boundary conditions (8-1-4). It should be noted that the class of problems represented by this system of equations is wider than that represented by system of differential heat-transfer and mass-transfer equations (8-1-1) and (8-1-2). Only for the special conditions  $K_{11} = a_q + a_m \frac{\epsilon p d c_m}{c_q}$ ;  $K_{12} = \epsilon p \frac{c_m}{c_q} a_m$ ;  $K_{21} = a_m \delta$ ;  $K_{22} = a_m$ ; and  $w_1 = w_2 = 0$  are the two systems identical.

Just as in subsection 1, we seek a solution of the problem for a class of functions to which Fourier integral transformations with respect to the spatial coordinates and a Laplace transformation with respect to time are applicable. Thus, it must be true that  $\theta_i \rightarrow 0$  for  $R^2 = x^2 + y^2 + z^2 \rightarrow \infty$ .

Following Tsoi [1], we apply to system (8-1-1') and (8-1-2') and to limiting conditions (8-1-3) and (8-1-4) Fourier transformation (8-1-5) with respect to  $y$  and  $z$ , cosine transformation (8-1-6) with respect to  $x$ , and Laplace transformation (8-1-9) with respect to the time  $\tau$ . After

several simplifications we obtain

$$(s + K_{11}r^2)\{u_1\}_{cL} + K_{12}r^2\{u_2\}_{cL} = -\sqrt{\frac{2}{\pi}} [K_{11}\{\Phi_1\}_L + K_{12}\{\Phi_2\}_L] + \{\omega_1\}_{cL} + \{F_1\}_c; \quad (8-1-18)$$

$$K_{21}r^2\{u_1\}_{cL} + (s + K_{22}r^2)\{u_2\}_{cL} = -\sqrt{\frac{2}{\pi}} [K_{21}\{\Phi_1\}_L + K_{22}\{\Phi_2\}_L] + \{\omega_2\}_{cL} + \{F_2\}_c, \quad (8-1-19)$$

where the notation is that of subsection 1 of this section.

The solution of system (8-1-18) and (8-1-19) for the transforms of the transfer potentials  $\theta_l$  is

$$\{u_l\}_{cL} = \frac{D_l(\xi, \eta, \zeta, s)}{D(\xi, \eta, \zeta, s)} \quad (l=1, 2). \quad (8-1-20)$$

The principal determinant for system (8-1-18) and (8-1-19) may be expanded to

$$D(\xi, \eta, \zeta, s) = s^2 + s(K_{11} + K_{22})r^2 + (K_{11}K_{22} - K_{12}K_{21})r^4.$$

The roots of this determinant are denoted as  $s_1 = -ar^2$  and  $s_2 = -br^2$ , where

$$-a = \frac{-(K_{11} + K_{22}) + \sqrt{(K_{11} - K_{22})^2 + 4K_{12}K_{21}}}{2};$$

$$-b = \frac{-(K_{11} + K_{22}) - \sqrt{(K_{11} - K_{22})^2 + 4K_{12}K_{21}}}{2}.$$

For  $K_{11}K_{22} \geq K_{12}K_{21}$ , the factors  $a$  and  $b$  are real and positive. Therefore, we set  $a = c_1^2$  and  $b = c_2^2$ .

The inverse transform of solution (8-1-20) is found by carrying out the respective inverse transformations in the same sequence as that specified in the first problem, and also by using the convolution theorem for Fourier transformations. The final solution of the given problem, after the indicated operations, is

$$\begin{aligned} \theta_1(x, y, z, \tau) = & \sum_{i=1}^2 \frac{1}{(2c_i \sqrt{\pi})^3} \left\{ - \int_0^\tau \int_{-\infty}^\infty \frac{A_i^1 \varphi_1(\beta, \gamma, \tau^*) - B_i^1 \varphi_2(\beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} \times \right. \\ & \times E_i(x, y, z, \tau; 0, \beta, \gamma, \tau^*) d\tau^* d\beta d\gamma + \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{A_i^1 \omega_1(\alpha, \beta, \gamma, \tau^*) - B_i^1 \omega_2(\alpha, \beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} \times \\ & \times E_i(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) d\tau^* d\alpha d\beta d\gamma + \frac{1}{\sqrt{\tau^3}} \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty [A_i^1 f_1(\alpha, \beta, \gamma) - B_i^1 f_2(\alpha, \beta, \gamma)] \times \\ & \left. \times E_i(x, y, z, \tau; \alpha, \beta, \gamma, 0) d\alpha d\beta d\gamma \right\}; \quad (8-1-21) \end{aligned}$$

$$\begin{aligned} \theta_2(x, y, z, \tau) = & \sum_{i=1}^2 \frac{1}{2c_i(\sqrt{\pi})^3} \left\{ - \int_0^\tau \int_{-\infty}^\infty \frac{A_i^2 \varphi_2(\beta, \gamma, \tau^*) - B_i^2 \varphi_1(\beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} \times \right. \\ & \times E_i(x, y, z, \tau; 0, \beta, \gamma, \tau^*) d\tau^* d\beta d\gamma + \\ & + \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{A_i^2 \omega_2(\alpha, \beta, \gamma, \tau^*) - B_i^2 \omega_1(\alpha, \beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} E_i(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) d\tau^* d\alpha d\beta d\gamma + \\ & \left. + \frac{1}{\sqrt{\tau^3}} \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty [A_i^2 f_2(\alpha, \beta, \gamma) - B_i^2 f_1(\alpha, \beta, \gamma)] E_i(x, y, z, \tau; \alpha, \beta, \gamma, 0) d\alpha d\beta d\gamma \right\}. \quad (8-1-22) \end{aligned}$$

In solutions (8-1-21) and (8-1-22) the following notation is used:

$$E_i(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) = \exp \left[ + \frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4c_i^2(\tau-\tau^*)} \right] \times \\ \times \left[ 1 + \exp \left( - \frac{\alpha x}{c_i^2(\tau-\tau^*)} \right) \right]; \\ A_i^1 = \frac{K_{22} - c_i^2}{K_{11} + K_{22} - 2c_i^2}; \quad B_i^1 = \frac{K_{12}}{K_{11} + K_{22} - 2c_i^2}; \\ A_i^2 = \frac{K_{11} - c_i^2}{K_{11} + K_{22} - 2c_i^2}; \quad B_i^2 = \frac{K_{21}}{K_{11} + K_{22} - 2c_i^2}.$$

The potentials  $\theta_i(x, y, z, \tau)$  determined by (8-1-21) and (8-1-22) represent solutions of system of differential equations (8-1-1') and (8-1-2') with limiting conditions (8-1-3) and (8-1-4). The presence of kernels of the form  $\exp(-a^2 r^2)$  in these solutions guarantees convergence of the integrals for a wide range of integrand functions.

In those cases when functions  $w_i$ ,  $\varphi_i$ , and  $f_i$  are not included in the class of functions to which Fourier integral transformations are applicable, the solutions are called "generalized solutions." For example, for constant limiting conditions of the second kind the solution is a particular case of the "generalized solution."

For constant limiting conditions, that is, for

$$w_i(x, y, z, \tau) = w_i^0 = \text{const};$$

$$\varphi_i(y, z, \tau) = q_i = \text{const};$$

$$f_i(x, y, z) = \theta_i^0 = \text{const},$$

solutions (8-1-21) and (8-1-22) may be written as /1/:

$$\theta_i(x, y, z, \tau) = \sum_{i=1}^2 \left\{ - \frac{q_{ii}^*}{c_i \sqrt{\pi}} \int_0^\tau \frac{\exp \left[ - \frac{x^2}{4c_i^2(\tau-\tau^*)} \right]}{\sqrt{\tau-\tau^*}} d\tau^* + w_{ii}^* \tau + \theta_{ii}^* \right\}, \quad (8-1-23)$$

where

$$q_{ii}^* = n_{ii} q_i + m_{ii} q_{2i}; \quad w_{ii}^* = n_{ii} w_i^0 + m_{ii} w_{2i}^0;$$

$$\theta_{ii}^* = n_{ii} \theta_i^0 + m_{ii} \theta_{2i}^0$$

$$n_{ii} = A_i^1 K_{11} - B_i^1 K_{21}; \quad n_{2i} = A_i^2 K_{21} - B_i^2 K_{11};$$

$$m_{ii} = A_i^1 K_{12} - B_i^1 K_{22}; \quad m_{2i} = A_i^2 K_{22} - B_i^2 K_{12}.$$

Since we know that

$$\frac{c_i}{\sqrt{\pi}} \int_0^\tau \frac{\exp \left[ - \frac{x^2}{4c_i^2(\tau-\tau^*)} \right]}{\sqrt{\tau-\tau^*}} d\tau^* = \frac{x}{\sqrt{\pi}} \left[ \frac{\exp(-a_i^2)}{a_i} - 2 \int_0^\infty \exp(-z^2) dz \right],$$

equation (8-1-23) can also be written as

$$\theta_i(x, y, z, \tau) = \sum_{i=1}^2 \left\{ w_{ii}^* \tau + \theta_{ii}^* + \frac{q_{ii}^* x}{c_i^2} \left[ \text{erfc}(a_i) - \frac{\exp(-a_i^2)}{a_i \sqrt{\pi}} \right] \right\},$$

where

$$a_i = \frac{x}{2c_i \sqrt{\tau}}.$$

If the heat transfer is not coupled to the mass transfer (if there are no effects of mutual superposition of transfer phenomena), then  $K_{12}=K_{21}=0$ . When this is true, system of equations (8-1-1') and (8-1-2') breaks up into two nonhomogeneous equations of the type of the heat-conduction equation, the two equations being independent of one another.

It is evident that  $c_1^2=K_{22}$  and  $c_2^2=K_{11}$ . When these values are substituted into (8-1-21) and (8-1-22), then after certain transformations we obtain the solution of the nonhomogeneous differential heat-conduction (diffusion) equation

$$\begin{aligned} \theta_i(x, y, z, \tau) = & \frac{1}{(2\sqrt{\pi\tau}K_{ii})^3} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f_i(\alpha, \beta, \gamma) E_i(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) d\alpha d\beta d\gamma + \\ & + \frac{1}{(2\sqrt{\pi}K_{ii})^3} \int_0^\tau \int_0^\infty \int_{-\infty}^\infty \frac{w_i(\alpha, \beta, \gamma, \tau^*)}{V(\tau-\tau^*)^3} E_i(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) d\tau^* d\alpha d\beta d\gamma - \\ & - \frac{1}{(2\sqrt{\pi})^3 K_{ii}} \int_0^\tau \int_0^\infty \int_{-\infty}^\infty \frac{\varphi_i(\beta, \gamma, \tau^*)}{V(\tau-\tau^*)^3} E_i(x, y, z, \tau; 0, \beta, \gamma, \tau^*) d\tau^* d\beta d\gamma. \end{aligned} \quad (8-1-24)$$

In formula (8-1-24)

$$\begin{aligned} E_i(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) = \\ = \exp \left[ -\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4K_{ii}(\tau-\tau^*)} \right] \left[ 1 + \exp \left( -\frac{\alpha x}{K_{ii}(\tau-\tau^*)} \right) \right]. \end{aligned}$$

## 8-2. Heat and Mass Transfer in a Two-Dimensional Infinite Plate. Boundary Conditions of the Third Kind

Let us consider the solution of system of equations (8-1-1') and (8-1-2') for a two-dimensional body in the absence of sources:

$$\frac{\partial \theta_l}{\partial \tau} = \sum_{k=1}^2 K_{lk} \nabla^2 \theta_k \quad (k=1, 2), \quad (8-2-1)$$

where  $K_{lk}$  are constant quantities which satisfy the conditions

$$K_{11} > 0; K_{22} > 0; \left| \frac{K_{11} K_{12}}{K_{21} K_{22}} \right| > 0.$$

Following reference /4/, let us solve system of equations (8-2-1) with initial conditions

$$\theta_l(x, y, 0) = f_l(x, y) \quad (0 < x < L; -\infty < y < +\infty) \quad (8-2-2)$$

and boundary conditions

$$\left[ \frac{\partial \theta_l}{\partial x} + \sum_{k=1}^2 (\gamma_{lk})_0 \theta_k \right]_{x=0} = \varphi_l(y, \tau) \quad (l=1, 2); \quad (8-2-3)$$

$$\left[ \frac{\partial \theta_l}{\partial x} + \sum_{i=1}^2 (\gamma_{lh})_L \theta_h \right]_{x=L} = \psi_l(y, \tau), \quad (8-2-4)$$

where  $(\gamma_{lh})_0$ ,  $(\gamma_{lh})_L$ ,  $\Phi_l$ ,  $\psi_l$ , and  $f_l$  are bounded continuous functions. The first two functions are the coefficients of the heat exchange and mass exchange between the body and its surroundings.

Let us first consider the particular problem with the boundary conditions

$$\frac{\partial \theta_l(0, y, \tau)}{\partial x} = 0; \quad \frac{\partial \theta_l(L, y, \tau)}{\partial x} = \psi_l(y, \tau) \quad (l=1, 2). \quad (8-2-5)$$

If it is assumed that the functions  $f$  and  $\Phi$ , as well as the potentials  $\theta_l$  and their derivatives, are bounded and continuous, then a Fourier transformation can be applied to the system of equations and to the limiting conditions. When we apply to equations (8-2-1), (8-2-2), and (8-2-5) the transformation

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta(x, y, \tau) \exp(i\eta y) dy = u(x, \eta, \tau), \quad (8-2-6)$$

we obtain

$$\frac{\partial u_l}{\partial \tau} = \sum_{i=1}^2 K_{lh} \frac{\partial^2 u_l}{\partial x^2} + (i\eta)^2 \sum_{k=1}^2 K_{lk} u_k; \quad (8-2-7)$$

$$u_l(x, \eta, 0) = g_l(x, \eta); \quad (8-2-8)$$

$$\frac{\partial u_l(0, \eta, \tau)}{\partial x} = 0; \quad \frac{\partial u_l(L, \eta, \tau)}{\partial x} = \Phi_l(\eta, \tau). \quad (8-2-9)$$

With respect to the variable  $x$ , the finite Fourier cosine transformation

$$\int_0^L u_l(x, \eta, \tau) \cos \Lambda_n x dx = u_l(p, \eta, \tau) = \{u_l\}_c \quad (8-2-10)$$

may be applied, where

$$\Lambda_n = \left( n + \frac{1}{2} \right) \pi / L.$$

If transformation (8-2-10) is applied to equations and conditions (8-2-7) through (8-2-9), then we obtain

$$\begin{aligned} \frac{d}{d\tau} \{u_l\}_c &= -(\Lambda_n^2 + \eta^2) \sum_{k=1}^2 K_{lk} \{u_k\}_c + \sum_{k=1}^2 (-1)^n \Lambda_n K_{lk} \Phi_k(\eta, \tau); \\ \{u_l(p, \eta, 0)\}_c &= \{g_l(p, \eta)\}_c \quad (k=1, 2). \end{aligned} \quad (8-2-11)$$

Equations (8-2-11) are a system of ordinary linear nonhomogeneous differential equations with constant coefficients. The solution of this system is

$$\begin{aligned} \{u_l(p, \eta, \tau)\}_c &= \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k \exp \{ -\mu_m^2 (\Lambda_n^2 + \eta^2) \tau \} g_k(p, \eta)_c + \\ &+ (-1)^n \Lambda_n \sum_{k=1}^2 \sum_{m=1}^2 \mu_m^2 A_{lm}^k \int_0^\tau \exp [ -\mu_m^2 (\Lambda_n^2 + \eta^2) (\tau - \tau^*) ] \Phi_k(\eta, \tau^*) d\tau^*, \end{aligned} \quad (8-2-12)$$

where  $\mu_m^2$  are the roots of the equation

$$\begin{vmatrix} K_{11} - \mu^2 & K_{12} \\ K_{21} & K_{22} - \mu^2 \end{vmatrix} = 0,$$



and coefficients  $A_{lm}^k$  are defined as

$$A_{1m}^1 = \frac{\mu_m^2 - K_{22}}{\mu_1^2 - \mu_2^2}; \quad A_{1m}^2 = (-1)^{m+1} \frac{K_{11}}{\mu_1^2 - \mu_2^2};$$

$$A_{2m}^1 = (-1)^{m+1} \frac{K_{21}}{\mu_1^2 - \mu_2^2}; \quad A_{2m}^2 = \frac{\mu_m^2 - K_{11}}{\mu_1^2 - \mu_2^2} \quad (m=1, 2).$$

It is easy to show that coefficients  $A_{lm}^k$  satisfy the equations

$$\sum_{m=1}^2 A_{lm}^k = \begin{cases} 0 & (k \neq l); \\ 1 & (k = l); \end{cases} \quad \sum_{l=1}^2 A_{lm}^k K_{li} = \mu_m^2 A_{im}^k, \quad (8-2-13)$$

Since the inverse transformation of (8-2-10) is

$$u_l(x, \eta, \tau) = \frac{2}{L} \sum_{n=0}^{\infty} \{u_l\}_0 \cos \Lambda_n x, \quad (8-2-14)$$

we may substitute (8-2-12) into (8-2-14), and after some transformations we obtain

$$u_l(x, \eta, \tau) = \frac{2}{L} \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k \int_0^L g_k(\alpha, \eta) \sum_{n=0}^{\infty} \exp[-\mu_m^2(\Lambda_n^2 + \eta^2)\tau] \cos \Lambda_n \alpha \times$$

$$\times \cos \Lambda_n x \, d\alpha + \frac{2}{L} \sum_{k=1}^2 \sum_{m=1}^2 \mu_m^2 A_{lm}^k \int_0^1 \Phi_k(\eta, \tau^*) \sum_{n=0}^{\infty} (-1)^n \Lambda_n \times$$

$$\times \exp[-\mu_m^2(\Lambda_n^2 + \eta^2)(\tau - \tau^*)] \cos \Lambda_n x \, d\tau^*$$

or

$$u_l(x, \eta, \tau) = \frac{1}{L} \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k \int_0^L g_k(\alpha, \eta) \sum_{n=0}^{\infty} \exp[-\mu_m^2(\Lambda_n^2 + \eta^2)\tau] \times$$

$$\times [\cos \Lambda_n(\alpha + x) + \cos \Lambda_n(\alpha - x)] \, d\alpha + \frac{2}{L} \sum_{k=1}^2 \sum_{m=1}^2 \mu_m^2 A_{lm}^k \frac{\partial}{\partial x} \int_0^1 \Phi_k(\eta, \tau^*) \times$$

$$\times \sum_{n=0}^{\infty} \exp[-\mu_m^2(\Lambda_n^2 + \eta^2)(\tau - \tau^*)] \cos \Lambda_n(L - x) \, d\tau^*. \quad (8-2-15)$$

If the theta function

$$\vartheta_2(y, x) = \frac{1}{\sqrt{\pi x}} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left[-\frac{(n+y)^2}{x}\right] =$$

$$= 2 \sum_{n=0}^{\infty} \exp\left[-\left(n + \frac{1}{2}\right)^2 \pi^2 x\right] \cos(2n+1)\pi y \quad (0 \leq y < 1)$$

is introduced, then (8-2-15) becomes

$$u_l(x, \eta, \tau) = \frac{1}{2L} \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k \int_0^L g_k(\alpha, \eta) \exp(-\mu_m^2 \eta^2 \tau) \left[ \vartheta_2\left(\frac{\alpha+x}{2L}, \frac{\mu_m^2 \tau}{L^2}\right) + \right.$$

$$\left. + \vartheta_2\left(\frac{\alpha-x}{2L}, \frac{\mu_m^2 \tau}{L^2}\right) \right] d\alpha + \frac{1}{L} \sum_{k=1}^2 \sum_{m=1}^2 \mu_m^2 A_{lm}^k \frac{\partial}{\partial x} \int_0^1 \Phi_k(\eta, \tau^*) \exp[-\mu_m^2 \eta^2 (\tau - \tau^*)] \times$$

$$\times \vartheta_2\left[\frac{L-x}{2L}, \frac{\mu_m^2 (\tau - \tau^*)}{L^2}\right] d\tau^*. \quad (8-2-16)$$

The inverse transformation for the function  $u(x, \eta, \tau)$ , which is defined by (8-2-6), is

$$\theta_i(x, y, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-i\eta y) u_i(x, \eta, \tau) d\eta. \quad (8-2-17)$$

When inverse transformation (8-2-17) is applied to (8-2-16), we obtain the final solution of the problem with boundary conditions (8-2-5):

$$\begin{aligned} \theta_i(x, y, \tau) = & \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^L d\alpha \int_{-\infty}^{\infty} \frac{f_k(\alpha, \beta)}{4\sqrt{\pi\mu_m L V \tau}} \exp \left[ -\frac{(y-\beta)^2}{4\mu_m^2 \tau} \right] \times \\ & \times \left[ \theta_2 \left( \frac{\alpha+x}{2L}, \frac{\mu_m^2 \tau}{L^2} \right) + \theta_2 \left( \frac{\alpha-x}{2L}, \frac{\mu_m^2 \tau}{L^2} \right) \right] d\beta + \\ & + \sum_{k=1}^2 \sum_{m=1}^2 \mu_m A_{im}^k \frac{\partial}{\partial x} \int_0^{\tau} \frac{\psi_k(\beta, \tau^*)}{2\sqrt{\pi L V \tau - \tau^*}} \exp \left[ -\frac{(y-\beta)^2}{4\mu_m^2 (\tau - \tau^*)} \right] \times \\ & \times \theta_2 \left( \frac{L-x}{2L}, \frac{\mu_m^2 (\tau - \tau^*)}{L^2} \right) d\beta. \end{aligned} \quad (8-2-18)$$

A check of solution (8-2-18) shows that it actually does satisfy system of equations (8-2-1), initial conditions (8-2-2), and boundary conditions (8-2-5), for any continuous bounded functions  $f_k$  and  $\psi_k$ .

Let us now solve system of equations (8-2-1) with boundary conditions (8-2-3) and (8-2-4). To do this, we must determine which terms in equation (8-2-18) govern the satisfaction of solution (8-2-18) subject to boundary conditions (8-2-5).

It may be demonstrated that the satisfaction of (8-2-18) with boundary conditions (8-2-5) is governed by the integrals in (8-2-18) whose principal terms are theta functions, namely by the integrals

$$v_i^{(0)} = \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^L d\alpha \int_{-\infty}^{\infty} \frac{f_k(\alpha, \beta)}{4\pi\mu_m^2 \tau} \exp \left[ -\frac{(x-\alpha)^2 + (y-\beta)^2}{4\mu_m^2 \tau} \right] d\beta; \quad (8-2-19)$$

$$v_i = \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^{\tau} d\tau^* \int_{-\infty}^{\infty} \frac{\psi_k(\beta, \tau^*)}{2\pi(\tau - \tau^*)} \exp \left[ -\frac{(L-x)^2 + (y-\beta)^2}{4\mu_m^2 (\tau - \tau^*)} \right] d\beta. \quad (8-2-20)$$

First of all, quantities (8-2-19) and (8-2-20) satisfy system of equations (8-2-1). Actually,

$$G_m(x-\alpha, y-\beta, \tau) = \frac{1}{4\pi\mu_m^2 \tau} \exp \left[ -\frac{(x-\alpha)^2 + (y-\beta)^2}{4\mu_m^2 \tau} \right]$$

satisfies the equation

$$\frac{\partial G_m}{\partial \tau} = \mu_m^2 \left( \frac{\partial^2 G_m}{\partial x^2} + \frac{\partial^2 G_m}{\partial y^2} \right) = \mu_m^2 \nabla^2 G_m,$$

so that as a result, taking into account (8-2-13), we have

$$\sum_{i=1}^2 K_{vi} \nabla^2 v_i^{(0)} = \sum_{k=1}^2 \sum_{m=1}^2 \left( \sum_{i=1}^2 K_{vi} A_{im}^k \right) \int_0^L d\alpha \int_{-\infty}^{\infty} f_k(\alpha, \beta) \nabla^2 [G_m(x-\alpha, y-\beta, \tau)] d\beta =$$

$$\begin{aligned}
&= \sum_{k=1}^2 \sum_{m=1}^2 \left( \sum_{l=1}^2 A_{lm}^k K_{vl} \right) \int_0^L d\alpha \int_{-\infty}^{\infty} f_k(\alpha, \beta) \frac{1}{\mu_m^2} \frac{\partial}{\partial \tau} [G_m(x - \alpha, y - \beta, \tau)] d\beta = \\
&= \sum_{k=1}^2 \sum_{m=1}^2 A_{vm}^k \int_0^L d\alpha \int_{-\infty}^{\infty} f_k(\alpha, \beta) \frac{\partial}{\partial \tau} [G_m(x - \alpha, y - \beta, \tau)] d\beta = \frac{\partial \theta_v}{\partial \tau}.
\end{aligned}$$

The same is true for  $v_l$ . In addition,

$$\begin{aligned}
\lim_{\tau \rightarrow 0} v_l^{(0)} &= \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k \lim_{\tau \rightarrow 0} \int_0^L d\alpha \int_{-\infty}^{\infty} f_k(\alpha, \beta) G_m(x - \alpha, y - \beta, \tau) d\beta = \\
&= \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k f_k(x, y) = f_1(x, y) \sum_{m=1}^2 A_{lm}^1 + f_2(x, y) \sum_{m=1}^2 A_{lm}^2 = f_l(x, y); \\
\lim_{x \rightarrow L} \frac{\partial v_l}{\partial x} &= \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k \lim_{x \rightarrow L} \int_0^L d\tau^* \int_{-\infty}^{\infty} \frac{\psi_k(\beta, \tau^*)(L - x)}{(\tau - \tau^*)^2} G_m(x - L, y - \beta, \tau - \tau^*) d\beta = \\
&= \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k \psi_k(y, \tau) = \psi_l(y, \tau).
\end{aligned}$$

The other terms of equation (8-2-18) also satisfy system of equations (8-2-1), but in contrast to terms (8-2-19) and (8-2-20) they do not possess the latter properties. Thus, the function  $v_l$  acts as the thermal potential of a simple layer with respect to the line  $x=L/11/$ . In exactly the same way it can be shown that the function

$$v_l = 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k \mu_m^2 \int_0^{\tau} d\tau^* \int_{-\infty}^{\infty} \psi_k(\beta, \tau^*) G_m(x, y - \beta, \tau - \tau^*) d\beta$$

satisfies system (8-2-1) and

$$\lim_{x \rightarrow +0} \frac{\partial v_l}{\partial x} = -\psi_l(y, \tau).$$

Let us consider the functions

$$v_l(x, y, \tau) = 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k \mu_m^2 \int_0^{\tau} d\tau^* \int_c \psi_k(\sigma, \tau^*) G_m(p, \tau; p_1, \tau^*) d\sigma; \quad (8-2-21)$$

$$w_l(x, y, \tau) = 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k \mu_m^2 \int_0^{\tau} d\tau^* \int_c \psi_k(\sigma, \tau^*) \frac{\partial}{\partial n_{p_1}} G_m(p, \tau; p_1, \tau^*) d\sigma, \quad (8-2-22)$$

where

$$\begin{aligned}
G_m(p, \tau; p_1, \tau^*) &= \frac{1}{4\pi\mu_m^2(\tau - \tau^*)} \exp\left(-\frac{r^2}{4\mu_m^2(\tau - \tau^*)}\right); \\
r &= \sqrt{(x - \alpha)^2 + (y - \beta)^2};
\end{aligned}$$

$\sigma$  is the coordinate of the point  $p_1(\alpha, \beta)$  in a curvilinear coordinate system; and  $n$  is the internal normal.

On the basis of the properties of the thermal potentials /11/ when the curve is quite smooth in the neighborhood of point  $p_0(x_0, y_0)$ , and taking into account relations (8-2-13), it is possible to derive the following formulas:

$$w_l^{(n)} = \psi_l(s, \tau) + 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{lm}^k \mu_m^2 \int_0^{\tau} d\tau^* \int_c \psi_k(\sigma, \tau^*) \frac{\partial}{\partial n_{p_1}} G(p_0, \tau; p_1, \tau^*) d\sigma;$$

$$\begin{aligned}
w_i^{(L)} &= -\psi_i(s, \tau) + 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \mu_m^2 \int_0^\tau d\tau^* \int_c \psi_k(\sigma, \tau^*) \frac{\partial}{\partial n_{p_1}} G(p_0, \tau; p_1, \tau^*) d\sigma; \\
\left(\frac{\partial v_i}{\partial \theta}\right)_0 &= -\psi_0(s, \tau) + 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \mu_m^2 \int_0^\tau d\tau^* \int_c \psi_k(\sigma, \tau^*) \frac{\partial}{\partial n_{p_0}} G(p_0, \tau; p_1, \tau^*) d\sigma; \\
\left(\frac{\partial v_i}{\partial \theta}\right)_e &= \psi_i(s, \tau) + 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \mu_m^2 \int_0^\tau d\tau^* \int_c \psi_k(\sigma, \tau^*) \frac{\partial}{\partial n_{p_0}} G(p_0, \tau; p_1, \tau^*) d\sigma.
\end{aligned}$$

In addition, it can be shown that the functions  $v_i$  and  $w_i$  defined by formulas (8-2-21) and (8-2-22) satisfy system of differential equations (8-2-1).

The solution of system (8-2-1) with limiting conditions (8-2-2) through (8-2-4) will have the form

$$\begin{aligned}
\theta_i(x, y, \tau) &= \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^L d\alpha \int_{-\infty}^{\infty} f_k(\alpha, \beta) G_m(x - \alpha, y - \beta, \tau) d\beta + \\
&+ 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \mu_m^2 \int_0^\tau d\tau^* \int_{-\infty}^{\infty} \omega_k^{(1)}(\beta, \tau^*) G_m(x, y - \beta; \tau - \tau^*) d\beta + \\
&+ 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \mu_m^2 \int_0^\tau d\tau^* \int_{-\infty}^{\infty} \omega_k^{(2)}(\beta, \tau^*) G_m(L - x, y - \beta; \tau - \tau^*) d\beta. \quad (8-2-23)
\end{aligned}$$

Obviously, functions  $\theta_i(x, y, \tau)$  satisfy system of equations (8-2-1) and initial condition (8-1-3). Let us choose  $\omega_k^{(1)}$  and  $\omega_k^{(2)}$  so that the functions  $\theta_i(x, y, \tau)$  defined by (8-2-23) satisfy boundary conditions (8-2-3) and (8-2-4). To do this, it will be convenient to calculate the boundary values of the functions  $\theta_i$  and of their derivatives:

$$\begin{aligned}
\theta_i(0, y, \tau) &= \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^L d\alpha \int_{-\infty}^{\infty} f_k(\alpha, \beta) G_m(\alpha, y - \beta; \tau) d\beta + \\
&+ 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \mu_m^2 \int_0^\tau d\tau^* \int_{-\infty}^{\infty} \omega_k^{(1)}(\beta, \tau^*) G_m(0, y - \beta; \tau - \tau^*) d\beta + \\
&+ 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \mu_m^2 \int_0^\tau d\tau^* \int_{-\infty}^{\infty} \omega_k^{(2)}(\beta, \tau^*) G_m(L, y - \beta; \tau - \tau^*) d\beta; \quad (8-2-24)
\end{aligned}$$

$$\begin{aligned}
\theta_i(L, y, \tau) &= \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^L d\alpha \int_{-\infty}^{\infty} f_k(\alpha, \beta) G_m(L - \alpha, y - \beta; \tau) d\beta + \\
&+ 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \mu_m^2 \int_0^\tau d\tau^* \int_{-\infty}^{\infty} \omega_k^{(1)}(\beta, \tau^*) G_m(L, y - \beta; \tau - \tau^*) d\beta + \\
&+ 2 \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \mu_m^2 \int_0^\tau d\tau^* \int_{-\infty}^{\infty} \omega_k^{(2)}(\beta, \tau^*) G_m(0, y - \beta; \tau - \tau^*) d\beta; \quad (8-2-25)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial \theta_i}{\partial x}\right)_{x=0} &= \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^L d\alpha \int_{-\infty}^{\infty} \frac{f_k(\alpha, \beta) \alpha}{2\mu_m^2 \tau} G_m(\alpha, y - \beta; \tau) d\beta - \\
&- \omega_i^{(1)}(y, \tau) + \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^\tau d\tau^* \int_{-\infty}^{\infty} \frac{\omega_k^{(2)}(\beta, \tau^*) L}{(\tau - \tau^*)} G_m(L, y - \beta; \tau - \tau^*) d\beta; \quad (8-2-26)
\end{aligned}$$

$$\begin{aligned} \left(\frac{\partial \theta_i}{\partial x}\right)_{x=L-0} = & \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^L d\alpha \int_{-\infty}^{\infty} \frac{f(\alpha, \beta)(L-\alpha)}{2\mu_m^2 \tau} G_m(L-\alpha, y-\beta, \tau-\tau^*) d\beta - \\ & - \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^{\tau} d\tau^* \int_{-\infty}^{\infty} \frac{\omega_k^{(1)}(\beta, \tau^*) L}{(\tau-\tau^*)} G_m(L, y-\beta, \tau-\tau^*) d\beta + \omega_i^{(2)}(y, \tau). \end{aligned} \quad (8-2-27)$$

By substituting boundary values (8-2-24) through (8-2-27) into (8-2-3) and (8-2-4), we obtain the system of integral equations

$$\begin{aligned} \omega_i^{(1)}(y, \tau) = & \sum_{k=1}^2 \sum_{m=1}^2 \int_0^{\tau} d\tau^* \int_{-\infty}^{\infty} R_{im}^k(y, \tau; \beta, \tau^*) \omega_m^{(k)}(\beta, \tau^*) d\beta + F_i^{(1)}(y, \tau); \\ \omega_i^{(2)}(y, \tau) = & \sum_{k=1}^2 \sum_{m=1}^2 \int_0^{\tau} d\tau^* \int_{-\infty}^{\infty} H_{im}^k(y, \tau; \beta, \tau^*) \omega_m^{(k)}(\beta, \tau^*) d\beta + F_i^{(2)}(y, \tau), \end{aligned}$$

where

$$\begin{aligned} R_{im}^1 = & \sum_{k=1}^2 \sum_{l=1}^2 (\gamma_{lk})_0 A_{li}^m G_m(0, y-\beta, \tau-\tau^*); \\ R_{im}^2 = & \sum_{k=1}^2 \sum_{l=1}^2 (\gamma_{lk})_0 A_{li}^m G_m(L, y-\beta, \tau-\tau^*) + \\ & + \sum_{l=1}^2 A_{li}^m \frac{L}{(\tau-\tau^*)} G_m(L, y-\beta, \tau-\tau^*); \\ H_{im}^1 = & - \sum_{k=1}^2 \sum_{l=1}^2 A_{li}^m (\gamma_{lk})_L G_m(L, y-\beta, \tau-\tau^*) + \\ & + \sum_{l=1}^2 A_{li}^m \frac{L}{\tau-\tau^*} G_m(L, y-\beta, \tau-\tau^*); \\ H_{im}^2 = & - \sum_{k=1}^2 \sum_{l=1}^2 A_{li}^m (\gamma_{lk})_L G_m(L, y-\beta, \tau-\tau^*); \\ F_i^{(1)}(y, \tau) = & \varphi_i(y, \tau) + \sum_{k=1}^2 \sum_{l=1}^2 \sum_{m=1}^2 (\gamma_{li})_0 A_{im}^k \int_0^L d\alpha \int_{-\infty}^{\infty} f_k(\alpha, \beta) G_m(\alpha, y-\beta, \tau) d\beta + \\ & + \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^L d\alpha \int_{-\infty}^{\infty} \frac{f_k(\alpha, \beta) \alpha}{2\mu_m^2 \tau} G_m(\alpha, y-\beta, \tau) d\beta; \\ F_i^{(2)}(y, \tau) = & \psi_i(y, \tau) - \sum_{k=1}^2 \sum_{l=1}^2 \sum_{m=1}^2 (\gamma_{li})_L A_{im}^k \int_0^L d\alpha \int_{-\infty}^{\infty} f_k(\alpha, \beta) G_m(\alpha, y-\beta, \tau) d\alpha + \\ & + \sum_{k=1}^2 \sum_{m=1}^2 A_{im}^k \int_0^L d\alpha \int_{-\infty}^{\infty} \frac{f_k(\alpha, \beta)(L-\alpha)}{2\mu_m^2 \tau} G_m(\alpha, y-\beta, \tau) d\beta. \end{aligned}$$

The final solution of system of differential equations (8-2-1) with limiting conditions (8-2-2) through (8-2-4) has the form of a series:

$$\theta_i^{(j)} = \omega_{i0}^{(j)} + \omega_{i1}^{(j)} + \dots + \omega_{in}^{(j)} + \dots, \quad (8-2-28)$$

where

$$\begin{aligned}\omega_{i_0}^{(l)} &= F_i^{(l)}; \\ \omega_{i,n}^{(1)} &= \sum_{m=1}^2 \sum_{k=1}^2 \int_0^{\tau} d\tau^* \int_{-\infty}^{\infty} R_{im}^k(y, \tau; \beta, \tau^*) \omega_{m,n-1}^{(k)}(\beta, \tau^*) d\beta; \\ \omega_{i,n}^{(2)} &= \sum_{m=1}^2 \sum_{k=1}^2 \int_0^{\tau} d\tau^* \int_{-\infty}^{\infty} H_{im}^k(y, \tau; \beta, \tau^*) \omega_{m,n-1}^{(k)}(\beta, \tau^*) d\beta.\end{aligned}$$

The series (8-2-38) is absolutely and uniformly convergent /4/.

### 8-3. Heat and Mass Transfer in a Two-Dimensional Plate. Boundary Conditions of the First and Second Kinds

Let us now consider solutions of the two-dimensional system of differential heat-transfer and mass-transfer equations (8-1-1) and (8-1-2) which satisfy initial conditions (8-1-3), boundary conditions of the first and second kinds, and also the symmetry conditions

$$\frac{\partial \theta_l(0, y, \tau)}{\partial x} = \frac{\partial \theta_l(x, 0, \tau)}{\partial y} = 0 \quad (l=1, 2). \quad (8-3-1)$$

#### 1. Boundary conditions of the second kind

Let us solve system of equations (8-1-1) and (8-1-2) subject to the following boundary conditions:

$$\frac{\partial \theta_l(h, y, \tau)}{\partial x} = \varphi_l(y, \tau); \quad (8-3-2)$$

$$\frac{\partial \theta_l(x, d, \tau)}{\partial y} = \psi_l(x, \tau) \quad (l=1, 2). \quad (8-3-3)$$

First, it will be necessary to apply a double finite complex Fourier transformation, for a domain  $D$  in which

$$(-h < x < h) \text{ and } (-d < y < d):$$

$$\{\theta(x, y, \tau)\}_F = \iint_{(D)} \exp\left(-im \frac{\pi}{h} \xi - in \frac{\pi}{d} \eta\right) \theta(\xi, \eta, \tau) d\xi d\eta = u(m, n, \tau). \quad (8-3-4)$$

The application of this transformation to a real function  $\theta(x, y, \tau)$  which is integrable in the domain  $D$  gives a transform  $u(m, n, \tau)$  which is a function of two integer arguments  $m$  and  $n$ .

The inverse transform is found by means of the formula

$$\begin{aligned}\theta(x, y, \tau) &= \frac{1}{4hd} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp\left(im \frac{\pi}{h} x + in \frac{\pi}{d} y\right) u(m, n, \tau) = \\ &= F^{-1}\{u(m, n, \tau)\}.\end{aligned} \quad (8-3-5)$$

If the function  $\varphi_1(x, y)$  is integrated over a rectangle in which  $-2h < x < 2h$  and  $-2d < y < 2d$ , and the function  $\varphi_2(x, y)$  is integrated in the [narrower] domain  $D$ , then the convolution of these functions is

$$\{\varphi_1(x, y)^* \varphi_2(x, y)\}_F = \iint_{(D)} \varphi_1(x - \xi, y - \eta) \varphi_2(\xi, \eta) d\xi d\eta. \quad (8-3-6)$$

If function  $\varphi_i(x, y)$  is periodic, having a period  $2h$  with respect to its first argument  $[x]$  and having a period  $2d$  with respect to its second argument  $[y]$ , then the basic convolution law

$$\{\varphi_i^*, \varphi_j\}_F = \{\varphi_i\}_F \{\varphi_j\}_F \quad (8-3-7)$$

is valid. It will be convenient to continue functions  $\theta_i$  ( $i=1, 2$ ) into the intervals  $(-h \leq x \leq h)$  and  $(-d \leq y \leq d)$  as even functions, and to continue their partial derivatives  $(\theta_i)_x$  and  $(\theta_i)_y$  as odd functions, which is possible in view of conditions (8-3-1). Then, after the application of Fourier transformation (8-3-4), a system of two ordinary differential equations is obtained:

$$\frac{du_1}{d\tau} = -a_q u_1 + 2a_q P_1(\tau) + \frac{\epsilon \rho c_m}{c_q} \frac{du_2}{d\tau}, \quad (8-3-8)$$

$$\frac{du_2}{d\tau} = -a_m u_2 - a_m \delta u_1 + 2a_m P_2(\tau) + 2 \frac{\epsilon \rho c_m}{c_q} P_1(\tau) \quad (8-3-9)$$

with the initial conditions

$$u_i(m, n, 0) = F_i(m, n) = \{f_i(x, y)\}_F, \quad (8-3-10)$$

where

$$\sigma = \pi^2 \left( \frac{m^2}{h^2} + \frac{n^2}{d^2} \right); \quad (8-3-11)$$

$$P_i(\tau) = (-1)^m \int_{-d}^d \exp\left(-in \frac{\pi}{d} \eta\right) \varphi_i(\tau \eta) d\eta + (-1)^n \int_{-h}^h \exp\left(-im \frac{\pi}{h} \xi\right) \times \\ \times \psi_i(\xi, \tau) d\xi. \quad (8-3-12)$$

In the derivation of system (8-3-8) and (8-3-9) it was assumed that the operations  $\{ \}_F$  and  $\frac{\partial}{\partial \tau}$  are commutative.

System (8-3-8) and (8-3-9) may be solved using a Laplace integral transformation. As a result, the following solutions are obtained for  $u_i(m, n, \tau)$ :

$$u_i(m, n, \tau) = \sum_{l=1}^2 (-1)^{l+1} \{ A_l^i F_i(m, n) \exp[-a_q \text{Lu } v_l^2 \sigma \tau] + \\ + B_l^i F_i(m, n) \exp[-a_q \text{Lu } v_l^2 \sigma \tau] + C_l^i \exp[-a_q \text{Lu } v_l^2 \sigma \tau] P_1(\tau) + \\ + D_l^i \exp[-a_q \text{Lu } v_l^2 \sigma \tau] P_2(\tau) \} \quad (l=1, 2), \quad (8-3-13)$$

where the symbol  $\tau$  denotes a convolution operation with respect to  $\tau$ , from 0 to  $\tau$ ;

$$v_l^2 = \frac{1}{2} [(1 + \epsilon \text{Ko Pn} + 1/\text{Lu}) + (-1)^l \sqrt{(1 + \epsilon \text{Ko Pn} + 1/\text{Lu})^2 - 4/\text{Lu}}];$$

and the constant coefficients are

$$A_l^i = \frac{\epsilon \rho c_m}{c_q (v_l^2 - v_2^2)}; \quad B_l^i = \frac{v_l^2 - 1}{v_l^2 - v_2^2}; \\ C_l^i = \frac{2a_q [v_l^2 (1 + \epsilon \text{Ko Pn Lu}) - 1]}{v_l^2 - v_2^2}; \\ D_l^i = \frac{2a_q \epsilon \text{Ko Pn } v_l^2}{\delta (v_l^2 - v_2^2)};$$

$$A_i^2 = \frac{v_i^2 - \frac{1}{Lu} - \epsilon \text{ Ko Pn}}{v_1^2 - v_2^2}; \quad B_i^2 = \frac{\delta}{v_1^2 - v_2^2};$$

$$C_i^2 = \frac{2a_q \delta Lu v_i^2}{v_1^2 - v_2^2}; \quad D_i^2 = \frac{2a_q (Lu v_i^2 - 1)}{v_1^2 - v_2^2}.$$

Before finding the inverse transform, we should mention certain familiar properties of a theta function of two variables such as

$$\vartheta_s(v, \tau) = \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n} \quad (z = e^{i\pi v}, \quad q = e^{-i\pi \tau}).$$

The function  $\{K(x, y, z)\}_F$ , defined by the relation

$$\{K(x, y, z)\}_F = \exp(-\sigma z), \quad (8-3-14)$$

corresponds to the inverse transform

$$K(x, y, z) = \frac{1}{4hd} \vartheta_s \left[ \frac{x}{2h}, \frac{i\pi z}{h^2} \right] \vartheta_s \left[ \frac{y}{2d}, \frac{i\pi z}{d^2} \right]. \quad (8-3-15)$$

From the properties of these theta functions it follows that the function  $K(x, y, z)$  is periodic, having a period  $2h$  with respect to  $x$  and a period  $2d$  with respect to  $y$ . Once the latter fact and equations (8-3-7), (8-3-14), and (8-3-15) are taken into account, it is easy to obtain the inverse transforms of (8-3-13) with respect to  $m$  and  $n$ .

After the final transformations we obtain

$$\vartheta_l(x, y, \tau) = \sum_{i=1}^2 (-1)^{i+1} \{A_i^l I_{i1} + B_i^l I_{i1} + C_i^l I_{i1}^* + D_i^l I_{i1}^*\} \quad (l=1, 2), \quad (8-3-16)$$

where

$$I_{i1} = \frac{1}{4hd} \iint_{(D)} \vartheta_s \left( \frac{x-\xi}{2h}, i\pi v_i^2 Lu Fo_1 \right) \vartheta_s \left( \frac{y-\eta}{2d}, i\pi v_i^2 Lu Fo_1 \right) f_i(\xi, \eta) d\xi d\eta; \quad (8-3-17)$$

$$I_{i1}^* = \frac{1}{4hd} \int_0^{\tau} d\tau^* \left\{ \vartheta_s \left[ \frac{h-x}{2h}, i\pi v_i^2 Lu Fo_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times \right.$$

$$\times \int_{-d}^{+d} \vartheta_s \left[ \frac{y-\eta}{2d}, i\pi v_i^2 Lu Fo_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \varphi_h(\eta, \tau^*) d\eta +$$

$$+ \vartheta_s \left[ \frac{d-y}{2d}, i\pi v_i^2 Lu Fo_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \int_{-h}^{+h} \vartheta_s \left[ \frac{x-\xi}{2h}, i\pi v_i^2 Lu Fo_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times$$

$$\times \psi_h(\xi, \tau^*) d\xi \Big\}. \quad (8-3-18)$$

In (8-3-17) and (8-3-18), as well as in the equations later in this section,

$$Fo_1 = \frac{a_q \tau}{h^2} \quad \text{and} \quad Fo_2 = \frac{a_q \tau}{d^2}.$$

Equation (8-3-16) represents the solution of the problem at hand.

## 2. Generalized boundary conditions of the second kind

Let us now solve the problem of subsection 1, but instead of boundary conditions (8-3-2) and (8-3-3)



we assume generalized boundary conditions of the second kind:

$$\begin{aligned} -\lambda_q \frac{\partial \theta_1(h, y, \tau)}{dx} + \varphi_1(y, \tau) &= 0; \\ a_m \frac{\partial \theta_2(h, y, \tau)}{\partial x} + a_m \delta \frac{\partial \theta_1(h, y, \tau)}{\partial x} + \varphi_2(y, \tau) &= 0; \\ -\lambda_q \frac{\partial \theta_1(x, d, \tau)}{\partial y} + \psi_1(x, \tau) &= 0; \\ a_m \frac{\partial \theta_2(x, d, \tau)}{\partial y} + a_m \delta \frac{\partial \theta_1(x, d, \tau)}{\partial y} + \psi_2(x, \tau) &= 0. \end{aligned}$$

Once again we will use the finite two-dimensional complex Fourier transformation in the domain  $D$ , as defined in (8-3-4). If we substitute partial derivatives into (8-3-4) instead of  $\theta$  and perform an integration by parts, then we obtain expressions for the transforms of the partial derivatives:

$$\left\{ \frac{\partial \theta}{\partial x} \right\}_F = im \frac{\pi}{h} u(m, n, \tau) + (-1)^m \int_{-d}^{+d} \exp \left( -in \frac{\pi}{d} \eta \right) [\theta(h, \eta, \tau) + \theta(-h, \eta, \tau)] d\eta; \quad (8-3-19)$$

$$\left\{ \frac{\partial \theta}{\partial y} \right\}_F = in \frac{\pi}{d} u(m, n, \tau) + (-1)^n \int_{-h}^{+h} \exp \left( -im \frac{\pi}{h} \xi \right) [\theta(\xi, d, \tau) + \theta(\xi, -d, \tau)] d\xi; \quad (8-3-20)$$

$$\begin{aligned} \left\{ \frac{\partial^2 \theta}{\partial x^2} \right\}_F &= -m^2 \frac{\pi^2}{h^2} u(m, n, \tau) + (-1)^m \int_{-d}^{+d} \exp \left( -in \frac{\pi}{d} \eta \right) \left\{ \frac{\partial \theta(h, \eta, \tau)}{\partial \xi} - \frac{\partial \theta(-h, \eta, \tau)}{\partial \xi} + \right. \\ &\quad \left. + im \frac{\pi}{h} [\theta(h, \eta, \tau) - \theta(-h, \eta, \tau)] \right\} d\eta; \end{aligned} \quad (8-3-21)$$

$$\begin{aligned} \left\{ \frac{\partial^2 \theta}{\partial y^2} \right\}_F &= -n^2 \frac{\pi^2}{d^2} u(m, n, \tau) + (-1)^n \int_{-h}^{+h} \exp \left( -im \frac{\pi}{h} \xi \right) \left\{ \frac{\partial \theta(\xi, d, \tau)}{\partial \eta} - \frac{\partial \theta(\xi, -d, \tau)}{\partial \eta} + \right. \\ &\quad \left. + in \frac{\pi}{d} [\theta(\xi, d, \tau) - \theta(\xi, -d, \tau)] \right\} d\xi. \end{aligned} \quad (8-3-22)$$

It is next convenient to continue the functions  $\theta_l$  ( $l = 1, 2$ ) into the intervals  $(-h \leq x < 0)$  and  $(-d \leq y < 0)$  as even functions, and to continue their partial derivatives  $\frac{\partial \theta_l}{\partial x}$  and  $\frac{\partial \theta_l}{\partial y}$  as odd functions; this can be done on the basis of (8-3-1). Then, after transforming the system of equations and the boundary conditions with respect to  $x$  and  $y$ , according to formulas (8-3-19) through (8-3-22), we obtain, in terms of the transforms, a system of two ordinary differential equations for the functions  $u_l(m, n, \tau)$ :

$$\left. \begin{aligned} \frac{du_1}{d\tau} &= -a_q \sigma u_1 + \frac{2a_q}{\lambda_q} P_1(\tau) + \frac{\epsilon \rho c_m}{c_q} \frac{du_2}{d\tau}; \\ \frac{du_2}{d\tau} &= -a_m \sigma u_2 - a_m \delta \sigma u_1 - 2P_2(\tau), \end{aligned} \right\} \quad (8-3-23)$$

where  $\sigma$  and  $P_l(\tau)$  ( $l = 1, 2$ ) are defined by expressions (8-3-11) and (8-3-12). After transformation (8-3-4) is applied to them, initial conditions (8-1-3) have the form of conditions (8-3-10).

If we now apply to system (8-3-23) a Laplace transformation with respect to  $\tau$ , then we obtain a system of two first-degree algebraic equations in the two unknown functions  $\{u_l(m, n, \tau)\}_L = u_l(m, n, s)$  ( $l = 1, 2$ ). This system having been solved, we find the inverse transform with respect to  $s$ :

$$\begin{aligned} u_l(m, n, \tau) &= \sum_{i=1}^2 \{ \exp(-a_q Lu v_i^2 \sigma \tau) [A_i^l F_1(m, n) + B_i^l F_2(m, n)] + \\ &\quad + \exp(-a_q Lu v_i^2 \sigma \tau) [D_i^l P_2(\tau) + C_i^l P_1(\tau)] \} \quad (l = 1, 2). \end{aligned} \quad (8-3-24)$$

The coefficients  $A_i^l$ ,  $B_i^l$ ,  $C_i^l$ , and  $D_i^l$  are defined as

$$A_i^l = (-1)^{i+1} \frac{v_1^2 - 1}{v_1^2 - v_2^2}; \quad A_i^2 = (-1)^{i+1} \frac{\delta}{v_1^2 - v_2^2}; \quad (8-3-25)$$

$$\left. \begin{aligned} B_i^1 &= (-1)^{i+1} \frac{\varepsilon \text{Ko Pn}}{v_1^2 - v_2^2}; \quad B_i^2 = (-1)^{i+1} \frac{v_i^2 - \varepsilon \text{Ko Pn} - \frac{1}{\text{Lu}}}{v_1^2 - v_2^2}; \\ C_i^1 &= \frac{2}{c_q \gamma_0} A_i; \quad C_i^2 = (-1)^{i+1} \frac{2\delta}{c_q \gamma_0 (v_1^2 - v_2^2)}; \\ D_i^1 &= 2v_i^2 B_i; \quad D_i^2 = (-1)^{i+1} \frac{\left(v_i^2 - \frac{1}{\text{Lu}}\right)}{v_1^2 - v_2^2}. \end{aligned} \right\} \quad (8-3-25)$$

In order to obtain the final solution of the problem, we must still find the inverse transforms with respect to  $m$  and  $n$  of equations (8-3-24). These can be found by means of the convolution theorem, using theta functions. Because of the properties of the theta function  $\theta_8$ , mentioned in the previous problem, the first two terms on the right side of (8-3-24) may be inverted in accordance with convolution theorem (8-3-6), while the remaining terms may be inverted directly according to formula (8-3-5). Thus, we obtain

$$\theta_i(x, y, \tau) = \sum_{i=1}^2 [A_i^l I_{i1} + B_i^l I_{i2} + C_i^l I^{*}_{i1} + D_i^l I^{*}_{i2}], \quad (8-3-26)$$

where  $I_{i1}$  and  $I^{*}_{i1}$  are defined by equations (8-3-17) and (8-3-18).

If we introduce the notation

$$\begin{aligned} A_i^l &= A_{i1}^l; \quad A_2^l = A_{21}^l; \quad B_i^l = A_{i2}^l; \quad B_2^l = A_{22}^l; \\ C_i^l &= B_{i1}^l; \quad C_2^l = B_{21}^l; \quad D_i^l = B_{i2}^l; \quad D_2^l = B_{22}^l, \end{aligned}$$

then equation (8-3-26) can be rewritten as

$$\theta_i(x, y, \tau) = \sum_{l=1}^2 \sum_{k=1}^2 (A_{ik}^l I_{i1k} + B_{ik}^l I^{*}_{i1k}) \quad (l=1, 2).$$

### 3. Boundary conditions of the first and second kinds

Let us solve the same problem when different boundary conditions (those of the first and second kinds) are satisfied at symmetrically situated surfaces. The boundary conditions are

$$\begin{aligned} \theta_l(h, y, \tau) &= \varphi_l(y, \tau) \quad (l=1, 2); \\ -\lambda_q \frac{\partial \theta_1(x, d, \tau)}{\partial y} + \psi_1(x, \tau) &= 0; \\ a_m \left[ \frac{\partial \theta_2(x, d, \tau)}{\partial y} + \delta \frac{\partial \theta_1(x, d, \tau)}{\partial y} \right] + \psi_2(x, \tau) &= 0. \end{aligned}$$

We will apply a finite two-dimensional complex Fourier transformation in the same domain  $D$  as previously:

$$\{\theta(x, y, \tau)\}_F = u(m, n, \tau) = \iint_{(D)} \exp \left[ -i \left( m - \frac{1}{2} \right) \frac{\pi}{h} \xi - i n \frac{\pi}{d} \eta \right] \theta(\xi, \eta, \tau) d\xi d\eta. \quad (8-3-27)$$

The inversion formula for this transform is

$$\theta(x, y, \tau) = \frac{1}{4hd} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp \left[ i \left( m - \frac{1}{2} \right) \frac{\pi}{h} x + i n \frac{\pi}{d} y \right] u(m, n, \tau). \quad (8-3-28)$$

By applying transformation (8-3-27) to system of heat-transfer and mass-transfer equations (8-1-1) and (8-1-2), subject to the boundary conditions of the present problem, we obtain the following system of equations in terms of the transforms:

$$\left. \begin{aligned} \frac{du_1}{d\tau} &= -a_q a u_1 + 2a_q P_1(\tau) + \frac{2a_q}{\lambda_q} Q_1(\tau) + \frac{\varepsilon \rho c_m}{c_q} \frac{du_2}{d\tau}; \\ \frac{du_2}{d\tau} &= -a_m a u_2 - a_m \delta a u_1 + 2a_m P_2(\tau) + 2a_m \delta P_1(\tau) - 2Q_2(\tau), \end{aligned} \right\} \quad (8-3-29)$$

where

$$\sigma = \pi^2 \left[ \frac{\left(m - \frac{1}{2}\right)^2}{h^2} + \frac{n^2}{d^2} \right]; \quad (8-3-30)$$

$$P_l(\tau) = (-1)^m i \int_{-d}^{+d} \exp\left(-in \frac{\pi}{d} \eta\right) \left[ i \left(m - \frac{1}{2}\right) \frac{\pi}{h} \right] \varphi_l(\eta, \tau) d\eta;$$

$$Q_l(\tau) = (-1)^n \int_{-h}^{+h} \exp\left[-i \left(m - \frac{1}{2}\right) \frac{\pi}{h} \xi\right] \psi_l(\xi, \tau) d\xi. \quad (8-3-31)$$

The solution of this system is

$$u_l(m, n, \tau) = \sum_{i=1}^k \left\{ \exp(-a_q \text{Lu } v_i^2 \sigma \tau) [A_i^l F_1(m, n) + B_i^l F_2(m, n) + \right. \\ \left. + C_i^l Q_1(\tau) + D_i^l Q_2(\tau) + a_m D_i^l P_2(\tau) + E_i^{ls} P_1(\tau)] \right\} \quad (l = 1, 2). \quad (8-3-32)$$

Coefficients  $A_i^l$ ,  $B_i^l$ ,  $C_i^l$ , and  $D_i^l$  are defined by formulas (8-3-25), while coefficients  $E_i^l$  are

$$E_i^1 = (-1)^{i+1} \frac{2a_q}{v_1^2 - v_2^2} [v_i^2 (1 + \varepsilon \text{Ko Pn Lu}) - 1];$$

$$E_i^2 = 2a_m \sigma (-1)^{i+1} \frac{v_i^2}{v_1^2 - v_2^2}.$$

The convolution theorem and inversion formula (8-3-28) may now be applied to (8-3-32) to obtain

$$\theta_l(x, y, \tau) = \sum_{i=1}^2 [A_i^l I_{i1} + B_i^l I_{i2} + E_i^l I^{*i1} + a_m D_i^l I^{*i2} + C_i^l M_{i1} + D_i^l M_{i2}] \quad (l = 1, 2),$$

where

$$I_{i1} = \frac{1}{4hd} \iint_{(D)} \vartheta_2 \left( \frac{x-\xi}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \right) \vartheta_2 \left( \frac{y-\eta}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \right) f_1(\xi, \eta) d\xi d\eta; \quad (8-3-33)$$

$$I^{*i1} = \frac{1}{4hd} \int_0^{\tau^*} d\tau^* \left\{ \frac{\partial}{\partial x} \vartheta_2 \left[ \frac{h-x}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times \right. \\ \left. \times \int_{-d}^{+d} \vartheta_2 \left[ \frac{y-\eta}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \varphi_l(\eta, \tau^*) d\eta \right\}; \quad (8-3-34)$$

$$M_{i1} = \frac{1}{4hd} \int_0^{\tau^*} d\tau^* \left\{ \vartheta_2 \left[ \frac{d-y}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times \right. \\ \left. \times \int_{-h}^{+h} \vartheta_2 \left[ \frac{x-\xi}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \psi_l(\xi, \tau^*) d\xi \right\}. \quad (8-3-35)$$

#### 4. Boundary conditions of the first kind

Let us now solve the system of heat-transfer and mass-transfer equations with the boundary conditions

$$\theta_l(h, y, \tau) = \varphi_l(y, \tau);$$

$$\theta_l(x, d, \tau) = \psi_l(x, \tau) \quad (l = 1, 2).$$

The rest of the conditions remain as before.

Here we will apply the following finite two-dimensional complex Fourier transformation in the domain  $D$ :

$$\{\theta(x, y, \tau)\}_F = u(m, n, \tau) = \iint_{(D)} \exp \left[ -i \left( m - \frac{1}{2} \right) \frac{\pi}{h} \xi - i \left( n - \frac{1}{2} \right) \frac{\pi}{d} \eta \right] \times \\ \times \theta(\xi, \eta, \tau) d\xi d\eta, \quad (8-3-36)$$

which has the inversion formula

$$\theta(x, y, \tau) = \frac{1}{4hd} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp \left[ i \left( m - \frac{1}{2} \right) \frac{\pi}{h} x + i \left( n - \frac{1}{2} \right) \frac{\pi}{d} y \right] u(m, n, \tau). \quad (8-3-37)$$

If we write

$$\{\theta_1(x, y, \tau)\}_F = u_1(m, n, \tau), \quad \{\theta_2(x, y, \tau)\}_F = u_2(m, n, \tau),$$

then on the condition that

$$\theta_1(x + 2h, y, \tau) = -\theta_1(x, y, \tau) \quad \text{and} \quad \theta_1(x, y + 2d, \tau) = -\theta_1(x, y, \tau)$$

the convolution theorem

$$F^{-1}\{u \cdot u_2\} = \iint_{(D)} \theta_1(x - \xi, y - \eta, \tau) \theta_2(\xi, \eta) d\xi d\eta$$

is valid.

Next, if we substitute the corresponding partial derivatives into (8-3-36) instead of  $\theta_i$  and perform an integration by parts, we obtain the following system of ordinary differential equations in  $u_l(m, n, \tau)$  ( $l = 1, 2$ ):

$$\left. \begin{aligned} \frac{du_1}{d\tau} &= -a_q u_1 + a_q P_1(\tau) + \frac{e p c_m}{c_q} \frac{du_2}{d\tau}; \\ \frac{du_2}{d\tau} &= -a_m u_2 - a_m \partial u_1 + a_m [P_2(\tau) + P_1(\tau)], \end{aligned} \right\} \quad (8-3-38)$$

where

$$a = \frac{\left(m - \frac{1}{2}\right)^2}{h^2} + \frac{\left(n - \frac{1}{2}\right)^2}{d^2}; \quad (8-3-39)$$

$$P_l(\tau) = 2(-1)^m i \int_{-d}^{+d} \exp \left[ -i \left( n - \frac{1}{2} \right) \frac{\pi}{d} \eta \right] \left[ i \left( m - \frac{1}{2} \right) \frac{\pi}{h} \right] \varphi_l(\eta, \tau) d\eta + \\ + 2(-1)^n i \int_{-h}^h \exp \left[ -i \left( m - \frac{1}{2} \right) \frac{\pi}{h} \xi \right] \left[ i \left( n - \frac{1}{2} \right) \frac{\pi}{d} \right] \phi_l(\xi, \tau) d\xi.$$

The final solution of system (8-3-38) is

$$\theta_l(x, y, \tau) = \sum_{i=1}^2 [A_i^l I_{i1} + B_i^l I_{i2} + 2E_i^l I_{i1}^* + a_m D_i^l I_{i2}^*] \quad (l = 1, 2),$$

where

$$I_{ii} = \frac{1}{4hd} \iint_{(D)} \theta_2 \left( \frac{x - \xi}{2h}, i\pi v_i^2 Lu Fo_i \right) \theta_1 \left( \frac{y - \eta}{2d}, i\pi v_i^2 Lu Fo_i \right) f_i(\xi, \eta) d\xi d\eta; \quad (8-3-40)$$

$$I_{i1}^* = \frac{1}{2hd} \int_0^{\tau} d\tau^* \left\{ \frac{\partial}{\partial x} \theta_2 \left[ \frac{h - x}{2h}, i\pi v_i^2 Lu Fo_i \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times \right.$$

$$\begin{aligned}
& \times \int_{-d}^{+d} \vartheta_2 \left[ \frac{y-\eta}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \varphi_1(\eta, \tau^*) d\eta \Big\} + \\
& + \frac{1}{2hd} \int_0^{\tau} d\tau^* \left\{ \frac{\partial}{\partial x} \vartheta_2 \left[ \frac{d-y}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times \right. \\
& \times \int_{-h}^{+h} \vartheta_2 \left[ \frac{x-\xi}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \psi_1(\xi, \tau^*) d\xi \Big\}. \quad (8-3-41)
\end{aligned}$$

### 5. Boundary conditions of the first and second kinds

Let us now find the solution of system of equations (8-1-1) and (8-1-2) with initial conditions (8-1-3) and the following boundary conditions:

$$\frac{\partial \theta_l(0, y, \tau)}{\partial x} = \theta_l(x, 0, \tau) = 0 \quad (l = 1, 2); \quad (8-3-42)$$

$$\theta_l(x, d, \tau) = \psi_l(x, \tau); \quad (8-3-43)$$

$$-\lambda_g \frac{\partial \theta_1(h, y, \tau)}{\partial x} + \varphi_1(y, \tau) = 0; \quad (8-3-44)$$

$$a_m \left[ \frac{\partial \theta_2(h, y, \tau)}{\partial x} + \delta \frac{\partial \theta_1(h, y, \tau)}{\partial x} \right] + \varphi_2(y, \tau) = 0. \quad (8-3-45)$$

Here we use transformation (8-3-4) and inversion formula (8-3-5). If functions  $\theta_l$  are continued into the interval  $(-h \leq x \leq 0)$  as even functions and into the interval  $(-d \leq y \leq 0)$  as odd functions, which is permissible in view of (8-3-42), then we obtain

$$\begin{aligned}
\theta_l(x, y, \tau) &= \theta_l(-x, y, \tau); \quad \theta_l(x, y, \tau) = -\theta_l(x, -y, \tau); \\
\frac{\partial \theta_l(x, y, \tau)}{\partial x} &= -\frac{\partial \theta_l(-x, y, \tau)}{\partial x}; \quad \frac{\partial \theta_l(x, y, \tau)}{\partial y} = \frac{\partial \theta_l(x, -y, \tau)}{\partial y}; \\
f_l(x, y) &= f_l(-x, y) = -f_l(x, y); \\
\varphi_l(y, \tau) &= -\varphi_l(-y, \tau); \quad \psi_l(x, \tau) = \psi_l(-x, \tau).
\end{aligned}$$

The application of transformation (8-3-4) to system (8-1-1) and (8-1-2) and to boundary conditions (8-3-42) through (8-3-45) gives a system of ordinary differential equations in  $\theta_l$  in terms of transforms. After we apply inverse transformations to the solutions of this system, we obtain finally

$$\theta_l(x, y, \tau) = \sum_{i=1}^2 [A_i^l I_{li} + B_i^l I_{li}^* + C_i^l I_{li}^* + D_i^l I_{li}^* + E_i^l M_{li} + a_m D_i^l M_{li}],$$

where  $I_{li}$  are defined by equation (3-3-17) and  $I_{li}^*$  and  $M_{li}$  are

$$\begin{aligned}
I_{li}^* &= \frac{1}{4hd} \int_0^{\tau} d\tau^* \left\{ \vartheta_2 \left[ \frac{h-x}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times \right. \\
& \times \int_{-d}^{+d} \vartheta_2 \left[ \frac{y-\eta}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \varphi_1(\eta, \tau^*) d\eta \Big\}; \\
M_{li} &= \frac{1}{4hd} \int_0^{\tau} d\tau^* \left\{ \frac{\partial}{\partial y} \vartheta_2 \left[ \frac{d-y}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times \right. \\
& \times \int_{-h}^{+h} \vartheta_2 \left[ \frac{x-\xi}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \psi_1(\xi, \tau^*) d\xi \Big\}.
\end{aligned}$$

## 6. Boundary conditions of the second kind

Let us solve the problem given in subsection 5, but with the following conditions instead of conditions (8-3-43):

$$\begin{aligned} -\lambda_q \frac{\partial \theta_1(x, d, \tau)}{\partial y} + \psi_1(x, \tau) &= 0; \\ a_m \left[ \frac{\partial \theta_2(x, d, \tau)}{\partial y} + \delta \frac{\partial \theta_1(x, d, \tau)}{\partial y} \right] + \psi_2(x, \tau) &= 0. \end{aligned}$$

Here we employ the finite two-dimensional Fourier integral transformation

$$\{\theta(x, y, \tau)\}_F = \iint_{(D)} \exp \left[ -im \frac{\pi}{h} \xi - i \left( n - \frac{1}{2} \right) \frac{\pi}{d} \eta \right] \theta(\xi, \eta, \tau) d\xi d\eta = u(m, n, \tau),$$

which has the inversion formula

$$\theta(x, y, \tau) = \frac{1}{4hd} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp \left[ im \frac{\pi}{h} x + i \left( n - \frac{1}{2} \right) \frac{\pi}{d} y \right] u(m, n, \tau).$$

If we apply the operations repeatedly employed above to system of equations (8-1-1) and (8-1-2) and to the limiting conditions, then we obtain a system of equations in terms of transforms. When this system is solved and the inversion formula and convolution theorem are applied to it, we obtain finally

$$\theta_l(x, y, \tau) = \sum_{i=1}^2 [A_i^l I_{i1} + B_i^l I_{i2} + C_i^l I_{i1}^* + D_i^l I_{i2}^*] \quad (l = 1, 2),$$

where

$$\begin{aligned} I_{i1} &= \frac{1}{4hd} \iint_{(D)} \theta_2 \left( \frac{x-\xi}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \right) \theta_1 \left( \frac{y-\eta}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \right) f_i(\xi, \eta) d\xi d\eta; \\ I_{i1}^* &= \frac{1}{4hd} \int_0^{\tau^*} d\tau^* \left\{ \theta_2 \left[ \frac{h-x}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \int_{-d}^{+d} \theta_2 \left[ \frac{y-\eta}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \varphi_1(\eta, \tau^*) d\eta + \right. \\ &\quad \left. \theta_2 \left[ \frac{d-y}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \int_{-h}^{+h} \theta_2 \left[ \frac{x-\xi}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \varphi_2(\xi, \tau^*) d\xi \right\}. \end{aligned}$$

Coefficients  $A_i^l$ ,  $B_i^l$ ,  $C_i^l$ , and  $D_i^l$  are defined by equations (8-3-25).

## 7. Mixed boundary conditions

Let us solve system of equations (8-1-1) and (8-1-2) with initial conditions (8-1-3) and boundary conditions

$$\left. \begin{aligned} \frac{\partial \theta_l(0, y, \tau)}{\partial x} &= \theta_l(x, 0, \tau) = 0 \\ \theta_l(h, y, \tau) &= \varphi_l(y, \tau) \end{aligned} \right\} \quad (l = 1, 2); \quad (8-3-46)$$

$$(8-3-47)$$

$$-\lambda_q \frac{\partial \theta_1(x, d, \tau)}{\partial y} + \psi_1(x, \tau) = 0; \quad (8-3-48)$$

$$a_m \left[ \frac{\partial \theta_2(x, d, \tau)}{\partial y} + \delta \frac{\partial \theta_1(x, d, \tau)}{\partial y} \right] + \psi_2(x, \tau) = 0. \quad (8-3-49)$$

In this solution transformation formula (8-3-36) and inversion formula (8-3-37) are used. Let us continue the potentials  $\theta_i$  into the interval  $(-h \leq x \leq 0)$  as even functions and into the interval  $(-d \leq y \leq 0)$  as odd functions. Then, in system (8-1-1) and (8-1-2) and in conditions (8-1-3) and (8-3-46) through (8-3-49), we take the transforms with respect to the spatial coordinates to obtain the system of two ordinary differential equations (8-3-29), where in this case

$$\sigma = \pi^2 \left[ \frac{\left(m - \frac{1}{2}\right)^2}{h^2} + \frac{\left(n - \frac{1}{2}\right)^2}{d^2} \right];$$

$$P_i(\tau) = (-1)^m i \int_{-d}^{+d} \exp \left[ -i \left( n - \frac{1}{2} \right) \frac{\pi}{d} \eta \right] \left[ i \left( m - \frac{1}{2} \right) \frac{\pi}{h} \right] \varphi_i(\eta, \tau) d\eta,$$

and  $Q_i(\tau)$  is defined by (8-3-31).

The final solution is

$$\theta_i(x, y, \tau) = \sum_{i=1}^2 [A_i' I_{i1} + B_i' I_{i2} + E_i' I_{i3} + a_m D_i' I_{i4} + C_i' M_{i1} + D_i' M_{i2}],$$

where  $I_{i1}$ ,  $I_{i3}$ , and  $M_{i1}$  are defined by equations (8-3-33), (8-3-34), and (8-3-35). The constant coefficients have the same form as in subsection 3.

### 8. Boundary conditions of the first kind

Let us solve the previous problem, but with the following conditions instead of (8-3-48) and (8-3-49):

$$\theta_i(x, d, \tau) = \psi_i(x, \tau).$$

In this solution we will continue the functions  $\theta_i$  in the interval  $(-h \leq x \leq 0)$  as even functions and in the interval  $(-d \leq y \leq 0)$  as odd functions. Transformation formula (8-3-27) and inversion formula (8-3-28) will be used.

In the system of equations and in the limiting conditions we take the transforms with respect to the spatial coordinates to obtain the system of two ordinary differential equations (8-3-24), where  $\sigma$  is defined by (8-3-30) and  $P_i(\tau)$  is

$$P_i(\tau) = 2(-1)^m i \int_{-d}^{+d} \exp \left( -in \frac{\pi}{d} \eta \right) \left[ i \left( m - \frac{1}{2} \right) \frac{\pi}{h} \right] \psi_i(\eta, \tau) d\eta +$$

$$+ 2(-1)^n i \int_{-h}^{+h} \exp \left[ -i \left( m - \frac{1}{2} \right) \frac{\pi}{h} \xi \right] \left( in \frac{\pi}{d} \right) \phi_i(\xi, \tau) d\xi.$$

This system is solved in the usual way.

Then, employing the inversion formula and the convolution theorem, we obtain

$$\theta_i(x, y, \tau) = \sum_{i=1}^2 [A_i' I_{i1} + B_i' I_{i2} + 2E_i' I_{i3} + a_m D_i' I_{i4}],$$

where  $I_{i1}$  and  $I_{i3}$  are defined by (8-3-40) and (8-3-41). The constant coefficients are identical to those in subsection 4.

### 9. Mixed boundary conditions of the first and second kinds

Let us solve the system of heat-transfer and mass-transfer equations with the boundary conditions

$$\theta_i(0, y, \tau) = \theta_i(x, 0, \tau) = 0; \quad (8-3-50)$$

$$\theta_i(x, d, \tau) = \psi_i(x, \tau); \quad (8-3-51)$$

$$-\lambda_q \frac{\partial \theta_1(h, y, \tau)}{\partial x} + \varphi_1(y, \tau) = 0; \quad (8-3-52)$$

$$a_m \left[ \frac{\partial \theta_2(h, y, \tau)}{\partial x} + \delta \frac{\partial \theta_1(h, y, \tau)}{\partial x} \right] + \varphi_2(y, \tau) = 0 \quad (l = 1, 2). \quad (8-3-53)$$

In this case we continue the functions  $\theta_l$  in the intervals  $(-h \leq x \leq 0)$  and  $(-d \leq y \leq 0)$  as odd functions and use transformation formula (8-3-27) and inversion formula (8-3-28). We then obtain system (8-3-29) in terms of transforms, where  $\sigma$  is defined by (8-3-30) and  $P_l(\tau)$  and  $Q_l(\tau)$  are

$$P_l(\tau) = (-1)^n \int_{-h}^{+h} \exp \left[ -i \left( m - \frac{1}{2} \right) \frac{\pi}{h} \xi \right] \left( in \frac{\pi}{d} \right) \psi_l(\xi, \tau) d\xi;$$

$$Q_l(\tau) = (-1)^m l \int_{-d}^{+d} \exp \left( -in \frac{\pi}{d} \eta \right) \varphi_l(\eta, \tau) d\eta.$$

By solving this system and inverting the solution, we obtain

$$\theta_l(x, y, \tau) = \sum_{i=1}^2 [A_i^l I_{i1} + B_i^l I_{i2} + E_i^l I_{i3} + a_m D_i^l I_{i2} + C_i^l M_{i1} + D_i^l M_{i2}],$$

where  $I_{li}$  is defined by (8-3-33) and where

$$I_{11}^* = \frac{1}{4hd} \int_0^{\tau} d\tau^* \left\{ \frac{\partial}{\partial y} \vartheta_2 \left[ \frac{d-y}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times \right.$$

$$\times \int_{-h}^{+h} \vartheta_2 \left[ \frac{x-\xi}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \psi_l(\xi, \tau^*) d\xi \Big\};$$

$$M_{11} = \frac{1}{4hd} \int_0^{\tau} d\tau^* \left\{ \vartheta_2 \left[ \frac{h-x}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times \right.$$

$$\times \int_{-d}^{+d} \vartheta_2 \left[ \frac{y-\eta}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \varphi_l(\eta, \tau^*) d\eta.$$

The constant coefficients are identical to those in subsection 3.

#### 10. Boundary conditions of the second kind

Let us solve the problem with boundary conditions (8-3-50), (8-3-52), (8-3-53), and

$$-\lambda_q \frac{\partial \theta_1(x, d, \tau)}{\partial y} + \psi_1(x, \tau) = 0;$$

$$a_m \left[ \frac{\partial \theta_2(x, d, \tau)}{\partial y} + \delta \frac{\partial \theta_1(x, d, \tau)}{\partial y} \right] + \psi_2(x, \tau) = 0.$$

For boundary conditions of the first and second kinds, transformation formula (8-3-36) and inversion formula (8-3-37) are used.

When we take the transforms with respect to the spatial coordinates, we obtain the system of ordinary differential equations (8-3-23), where  $\sigma$  is defined by (8-3-39) and  $P_l(\tau)$  is

$$P_l(\tau) = (-1)^m l \int_{-d}^{+d} \exp \left[ -i \left( n - \frac{1}{2} \right) \frac{\pi}{d} \eta \right] \varphi_l(\eta, \tau) d\eta +$$

$$+ (-1)^n l \int_{-h}^{+h} \exp \left[ -i \left( m - \frac{1}{2} \right) \frac{\pi}{h} \xi \right] \psi_l(\xi, \tau) d\xi.$$



By solving this system and inverting the solution, we obtain

$$\theta_i(x, y, \tau) = \sum_{i=1}^2 [A_i^I I_{i1} + B_i^I I_{i2} + C_i^I I_{i3}^* + D_i^I I_{i4}^*],$$

where  $I_{ii}$  is defined by (8-3-40) and where

$$\begin{aligned} I_{i1}^* = & \frac{1}{4hd} \int_0^{\tau} d\tau^* \left\{ \theta_2 \left[ \frac{h-x}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times \right. \\ & \times \int_{-d}^{+d} \theta_2 \left[ \frac{y-\eta}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \varphi_i(\eta, \tau^*) d\eta + \\ & \left. + \theta_2 \left[ \frac{d-y}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \int_{-h}^{+h} \theta_2 \left[ \frac{x-\xi}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \psi_i(\xi, \tau^*) d\xi \right\}. \end{aligned}$$

The constant coefficients are defined by formulas (8-3-25).

#### 11. Boundary conditions of the first kind

Let us solve system of heat-transfer and mass-transfer equations (8-1-1) and (8-1-2) for boundary conditions (8-3-50), (8-3-51), and

$$\theta_i(h, y, \tau) = \varphi_i(y, \tau).$$

For boundary conditions of the first kind formulas (8-3-4) and (8-3-5) are used. Then we obtain system of equations (8-3-38) in terms of transforms, where  $\sigma$  is defined by formula (8-3-11) and  $P_i(\tau)$  is

$$\begin{aligned} P_i(\tau) = & 2(-1)^n \int_{-d}^{+d} \exp \left( -in \frac{\pi}{d} \eta \right) \left( im \frac{\pi}{h} \right) \varphi_i(\eta, \tau) d\eta + \\ & + 2(-1)^n \int_{-h}^{+h} \exp \left( -im \frac{\pi}{h} \xi \right) \left( in \frac{\pi}{d} \right) \psi_i(\xi, \tau) d\xi. \end{aligned}$$

By solving this system and inverting the solution, we obtain

$$\theta_i(x, y, \tau) = \sum_{i=1}^2 [A_i^I I_{i1} + B_i^I I_{i2} + 2E_i^I I_{i3}^* + a_m D_i^I I_{i4}^*],$$

where  $I_{ii}$  is determined by (8-3-17) and where

$$\begin{aligned} I_{i1}^* = & \frac{1}{2hd} \left[ \int_0^{\tau} d\tau^* \left\{ \frac{\partial}{\partial x} \theta_2 \left[ \frac{h-x}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \times \right. \right. \\ & \times \int_{-d}^{+d} \theta_2 \left[ \frac{y-\eta}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \varphi_i(\eta, \tau^*) d\eta + \\ & \left. \left. + \frac{\partial}{\partial y} \theta_2 \left[ \frac{d-y}{2d}, i\pi v_i^2 \text{Lu Fo}_2 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \int_{-h}^{+h} \theta_2 \left[ \frac{x-\xi}{2h}, i\pi v_i^2 \text{Lu Fo}_1 \left( 1 - \frac{\tau^*}{\tau} \right) \right] \psi_i(\xi, \tau^*) d\xi \right\} \right]. \end{aligned}$$

The constant coefficients are identical to those in subsection 4.

#### 8-4. Uncoupled Transfer of Heat or Mass in a Rectangular Domain

The phenomena of heat transfer and mass transfer are often studied independently of one another. For uncoupled transfer solutions can be obtained either by taking limits, as in the previously considered solutions, or by solving some particular problem directly. In the latter case the use of integral transformations is once again the most effective method. We will now give several solutions for uncoupled transfer in a rectangular domain.

##### a) Uncoupled transfer in a parallelepiped

Consider a finite plate of dimensions  $2R_1 \times 2R_2 \times 2R_3$ , whose initial potential is  $\theta_0$ . At the initial moment the plate is placed in surroundings with a constant potential  $\theta_c > \theta_0$ . A source whose specific strength varies according to the exponential law  $w(\tau) = w_0 \exp(-k\tau)$  acts inside the body. The interrelation between the body and its surroundings satisfies boundary conditions of the third kind.

In this case we have

$$\frac{\partial \theta(x, y, z, \tau)}{\partial \tau} = a \nabla^2 \theta(x, y, z, \tau) + \frac{w(\tau)}{c\gamma} \quad (8-4-1)$$

$$(\tau > 0; -R_1 < x < +R_1; -R_2 < y < +R_2; -R_3 < z < +R_3);$$

$$\theta(x, y, z, 0) = \theta_0 = \text{const}; \quad (8-4-2)$$

$$\left. \begin{aligned} \mp \frac{\partial \theta(\pm R_1, y, z, \tau)}{\partial x} + \frac{a}{\lambda} [\theta_0 - \theta(\pm R_1, y, z, \tau)] &= 0; \\ \mp \frac{\partial \theta(x, \pm R_2, z, \tau)}{\partial y} + \frac{a}{\lambda} [\theta_0 - \theta(x, \pm R_2, z, \tau)] &= 0; \\ \mp \frac{\partial \theta(x, y, \pm R_3, \tau)}{\partial z} + \frac{a}{\lambda} [\theta_0 - \theta(x, y, \pm R_3, \tau)] &= 0. \end{aligned} \right\} \quad (8-4-3)$$

The distribution of potential  $\theta$  is symmetric with respect to the center of the plate.

To solve this problem, we represent the potential  $\theta$  as the sum of two functions:

$$\theta(x, y, z, \tau) = u(x, y, z, \tau) + v(x, y, z, \tau),$$

where  $u$  satisfies the homogeneous heat-conduction equation and conditions (8-4-2) and (8-4-3), while  $v$  satisfies the nonhomogeneous equation and homogeneous boundary and initial conditions [12]. The final solution can be represented in terms of dimensionless quantities as follows:

$$\begin{aligned} \frac{\theta(x, y, z, \tau) - \theta_0}{\theta_0 - \theta_0} &= 1 - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left\{ 1 - \frac{Po}{[(\mu_{n,1}^2 K_1^2 + \mu_{m,2}^2 K_2^2 + \mu_{l,3}^2 K_3^2) - Pd]Fo} \times \right. \\ &\times \left[ 1 - \exp[(\mu_{n,1}^2 K_1^2 + \mu_{m,2}^2 K_2^2 + \mu_{l,3}^2 K_3^2 - Pd)Fo] \right] \Big\} A_{n,1} A_{m,2} A_{l,3} \times \\ &\times \cos \mu_{n,1} \frac{x}{R_1} \cos \mu_{m,2} \frac{y}{R_2} \cos \mu_{l,3} \frac{z}{R_3} \exp[-(\mu_{n,1}^2 K_1^2 + \mu_{m,2}^2 K_2^2 + \mu_{l,3}^2 K_3^2)Fo]. \end{aligned} \quad (8-4-4)$$

Here

$$Po = \frac{w_0 \tau}{c\gamma(\theta_c - \theta_0)}; \quad Fo = \frac{a\tau}{R^2}; \quad Pd = \frac{k}{a} R^2; \quad Bi_i = \frac{\alpha}{\lambda} R_i;$$

$$K_i = \frac{R}{R_i}; \quad \frac{1}{R^2} = \frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2};$$

$$A_i = (-1)^{i+1} \frac{2Bi_i (Bi_i^2 + \mu^2)^{1/2}}{\mu (Bi_i^2 + Bi_i + \mu^2)} \quad (i = 1, 2, 3 \text{ or } k, m, l),$$

and  $\mu$  are the roots of the characteristic equation

$$\cot \mu = \frac{1}{Bi_i} \mu.$$

If the Pomerantsev number (Po) is zero, then we obtain the solution in the absence of a source:

$$\frac{\theta - \theta_0}{\theta_c - \theta_0} = 1 - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} A_{n,1} A_{m,2} A_{l,3} \cos \mu_{n,1} \frac{x}{R_1} \cos \mu_{m,2} \frac{x}{R_2} \cos \mu_{l,3} \frac{x}{R_3} \times \\ \times \exp [-(\mu_{n,1}^2 K_1^2 + \mu_{m,2}^2 K_2^2 + \mu_{l,3}^2 K_3^2) Fo]. \quad (8-4-5)$$

If  $Bi_i = \infty$ , we obtain the solution for boundary conditions of the first kind. In this case, in solutions (8-4-4) and (8-4-5) the coefficients  $A_i = (-1)^{n,m,l} + \frac{2}{\mu_{i,k}}$  and the roots  $\mu_k = (2k-1)\pi/2$  ( $k = m, n, l$ ).

If the potential of the surroundings is a linear function of time,  $\theta_c = \theta_0 + b\tau$ , then solution (8-4-5) has the form

$$\frac{\theta - \theta_0}{\theta_c - \theta_0} = Pd \left[ Fo - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{A_{n,1} A_{m,2} A_{l,3}}{\mu_{n,1}^2 K_1^2 + \mu_{m,2}^2 K_2^2 + \mu_{l,3}^2 K_3^2} \cos \mu_{n,1} \frac{x}{R_1} \cos \mu_{m,2} \frac{y}{R_2} \right] \times \\ \times \cos \mu_{l,3} \frac{z}{R_3} \{1 - \exp [-(\mu_{n,1}^2 K_1^2 + \mu_{m,2}^2 K_2^2 + \mu_{l,3}^2 K_3^2) Fo]\}. \quad (8-4-6)$$

If we set  $Bi_i = \infty$  in (8-4-6), then the solution will satisfy boundary conditions of the first kind. Several solutions for other laws of variation of the surface potential (for  $Bi = \infty$ ) are given in reference /10/.

#### b) Uncoupled transfer in a two-dimensional plate

Let us consider the two-dimensional problem for a rectangle on the sides of which a time-dependent transfer-potential distribution

$$\frac{\partial \theta}{\partial \tau} = a \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) \quad (\tau > 0; 0 \leq x \leq h; 0 \leq y \leq d) \quad (8-4-7)$$

is maintained in a specified way. The initial conditions are

$$\theta(x, y, 0) = f(x, y) \quad (8-4-8)$$

and the boundary conditions are

$$\left. \begin{aligned} \theta(0, y, \tau) &= \varphi(y, \tau); & \theta(h, y, \tau) &= 0; \\ \theta(x, 0, \tau) &= 0; & \theta(x, d, \tau) &= 0. \end{aligned} \right\} \quad (8-4-9)$$

This problem can be solved using the integral-value Fourier sine transformation

$$\theta_s = \theta_s(m, n, \tau) = \int_0^h \int_0^d \theta(x, y, \tau) \sin \frac{m\pi x}{h} \sin \frac{n\pi y}{d} dx dy$$

and the inversion formula

$$\theta(x, y, \tau) = \frac{4}{hd} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_s(m, n, \tau) \sin \frac{m\pi x}{h} \sin \frac{n\pi y}{d}$$

for the Laplace transformation. The solution of equations (8-4-7) through (8-4-9) is

$$\theta(x, y, \tau) = \frac{4}{hd} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \exp(-a\sigma\tau) \int_0^h \int_0^d f(x, y) \sin \frac{m\pi x}{h} \sin \frac{n\pi y}{d} dx dy + \right. \\ \left. + \frac{am\pi}{h} \exp(-a\sigma\tau) \int_0^d \varphi(y, \tau) \sin \frac{n\pi y}{d} dy \right\} \sin \frac{m\pi x}{h} \sin \frac{n\pi y}{d}, \quad (8-4-10)$$

where

$$\sigma = \pi^2 \left( \frac{m^2}{h^2} + \frac{n^2}{d^2} \right).$$

From (8-4-10) it is possible to obtain the particular solutions given by Sneddon /13/ and Grinberg /14/.

For a semi-infinite plate,  $d = \infty$ . In this case it can be shown that for the conditions

$$\theta(x, y, 0) = 0;$$

$$\theta(h, y, \tau) = \theta_c; \quad \theta(0, y, \tau) = \theta(x, 0, \tau) = \theta(x, d, \tau) = 0$$

solution (8-4-10) becomes

$$\begin{aligned} \frac{\theta(x, y, \tau)}{\theta_c} = & \frac{x}{h} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin \frac{m\pi x}{h} \exp\left(-\frac{m^2\pi^2}{h^2} a\tau\right) + \\ & + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin \frac{m\pi x}{h} \left\{ \exp\left(\frac{m\pi y}{h}\right) \operatorname{erfc}\left(\frac{y}{2\sqrt{a\tau}} + \frac{m\pi}{h} \sqrt{a\tau}\right) + \right. \\ & \left. + \exp\left(-m\pi \frac{y}{h}\right) \operatorname{erfc}\left(\frac{y}{2\sqrt{a\tau}} - \frac{m\pi}{h} \sqrt{a\tau}\right) - 2 \exp\left(-\frac{m^2\pi^2}{h^2} a\tau\right) \operatorname{erfc} \frac{y}{2\sqrt{a\tau}} \right\}. \end{aligned}$$

A detailed solution of the latter problem, using a double Laplace transformation, is given in /8/.

### 8-5. Uncoupled Transfer of Heat or Mass in a Cylindrical Domain

The differential equation describing uncoupled transfer in a circular-section three-dimensional cylindrical domain may be expressed in cylindrical coordinates as follows:

$$\frac{\partial \theta}{\partial \tau} = a \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} + \frac{1}{r^2} \cdot \frac{\partial^2 \theta}{\partial \varphi^2} \right) + \frac{1}{c_Y} w(r, z, \varphi, \tau). \quad (8-5-1)$$

If there is symmetry with respect to the  $z$  axis, then the operator  $\frac{\partial}{\partial \varphi}$  is identically zero and we obtain

$$\frac{\partial \theta}{\partial \tau} = a \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} \right) + \frac{1}{c_Y} w(r, z, \tau). \quad (8-5-2)$$

If in addition the cylinder is quite long, and if the initial and boundary conditions are such that parallel sections of the cylinder, normal to its axis, have the same transfer-potential distribution, then the operator  $\partial/\partial z$  is also identically zero. In this case we obtain an equation whose solution was derived in previous chapters.

#### a) Solution for an axially symmetric cylinder

Let us solve equation (8-5-2) with boundary conditions of the first, second, and third kinds. Such problems are encountered in the study of various phenomena, such as in the modeling of thermal processes in the active zone of a nuclear reactor, in the evaluation of the heat transfer for the heat-releasing elements of reactors, in problems of heat exchange between pipelines and the earth, and in problems of mass transfer (accompanied by chemical transformations) through a cylindrical porous medium.

1. First let us consider the solution for a finite cylinder of length  $l$  with boundary conditions of the second kind at the lateral surface. In addition, we assume that one of the end faces ( $z=0$ ) is insulated against transfer,

whereas the potential of the second face is a function of time and of the radial coordinate. Thus, we must solve equation (8-5-2) with the following boundary conditions:

$$\left. \begin{aligned} \frac{\partial \theta(0, z, \tau)}{\partial r} &\neq \infty; \quad \frac{\partial \theta(R, z, \tau)}{\partial r} = \frac{1}{\lambda} j(z, \tau); \\ \frac{\partial \theta(r, 0, \tau)}{\partial z} &= 0; \quad \theta(r, l, \tau) = \varphi(r, \tau); \\ \theta(r, z, 0) &= f(r, z). \end{aligned} \right\} \quad (8-5-3)$$

This problem can be solved by various methods, for instance, by finite Laplace integral transformations /15/, or by a combination of Fourier and Hankel integral transformations /16, 17/.

For simplicity in the calculations to follow, let us write equation (8-5-2) and conditions (8-5-3) in dimensionless form:

$$\frac{\partial \theta}{\partial Fo} = \frac{1}{X} \cdot \frac{\partial}{\partial X} \left( X \frac{\partial \theta}{\partial X} \right) + b^2 \frac{\partial^2 \theta}{\partial Z^2} + Po(X, Z, Fo); \quad (8-5-4)$$

$$\theta(X, Z, 0) = F(X, Z); \quad (8-5-5)$$

$$\frac{\partial \theta(1, Z, Fo)}{\partial X} = Ki(Z, Fo); \quad (8-5-6)$$

$$\frac{\partial \theta(X, 0, Fo)}{\partial Z} = 0; \quad \theta(X, \pi, Fo) = \Phi(X, Fo), \quad (8-5-7)$$

where  $\theta = (\theta - \theta_*)/\theta_*$  is the dimensionless potential ( $\theta_*$  being some initial value of the transfer potential, fixed for a given point of the cylinder);  $X = r/R$  and  $Z = \pi z/l$  are the dimensionless coordinates;

$$F(X, Z) = (f(r, z) - \theta_*)/\theta_*; \quad \Phi(X, Fo) = \frac{\varphi(r, \tau) - \theta_*}{\theta_*}; \quad b = \frac{\pi R}{l};$$

$Po(X, Z, Fo) = \frac{R^2}{\lambda \theta_*} w(r, z, \tau)$  is the Pomerantsev number; and  $Ki(Z, Fo) = \frac{R}{\lambda \theta_*} j(z, \tau)$  is the Kirpichev number.

Following Ermakov /16/, we apply a finite Hankel integral transformation with respect to the variable  $X$ :

$$\{\theta(X, Z, Fo)\}_H = \{\theta\}_H = \int_0^1 X J_0(\mu X) \theta(X, Z, Fo) dX \quad (8-5-8)$$

and use the inversion formula

$$\theta(X, Z, Fo) = 2 \sum_{\mu} \frac{J_0(\mu X)}{J_0^2(\mu)} \{\theta(X, Z, Fo)\}_H. \quad (8-5-9)$$

With respect to the variable  $Z$  we then apply the finite Fourier cosine transformation

$$\{\theta(X, Z, Fo)\}_{He} = \{\theta\}_{He} = \int_0^{\pi} \{\theta\}_H \cos\left(n + \frac{1}{2}\right) Z dZ \quad (8-5-10)$$

and use the inversion formula

$$\{\theta(X, Z, Fo)\}_H = \frac{2}{\pi} \sum_{n=0}^{\infty} \{\theta\}_{He} \cos\left(n + \frac{1}{2}\right) Z. \quad (8-5-11)$$

In formulas (8-5-8) and (8-5-9) the quantity  $\mu$  is the positive root of the characteristic equation

$$J_1(\mu) = 0.$$

By means of the above transformations it is possible to obtain the following solution:

$$\begin{aligned} \theta(X, Z, Fo) = & \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{J_0(\mu_m X)}{J_0^2(\mu_m)} \left\{ \sum_{n=0}^{\infty} \cos\left(n + \frac{1}{2}\right) Z \times \right. \\ & \times \left[ \exp\left(-\left[\mu_m^2 + b^2\left(n + \frac{1}{2}\right)^2\right] Fo\right) \{F(X, Z)\}_{He} + \right. \\ & + \int_0^{Fo} \exp\left(-\left[\mu_m^2 + b^2\left(n + \frac{1}{2}\right)^2\right] (Fo - Fo^*)\right) (J_0(\mu_m) \{Ki(Z, Fo^*)\}_o + \\ & \left. \left. + (-1)^n b^2\left(n + \frac{1}{2}\right) \{\Phi(X, Fo^*)\}_H + \{Po(X, Z, Fo^*)\}_{He} dFo^* \right] \right\}. \end{aligned} \quad (8-5-12)$$

If we set  $f(r, z) = \varphi(r, \tau) = \theta_o$  in (8-5-12) and take  $\theta_* = \theta_o$ , then (8-5-12) becomes

$$\begin{aligned} \frac{\theta - \theta_o}{\theta_o} = & \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{J_0(\mu_m X)}{J_0^2(\mu_m)} \left\{ \sum_{n=0}^{\infty} \cos \mu_n Z \int_0^{Fo} \exp\left[-(\mu_m^2 + b^2 \mu_n^2) (Fo - Fo^*)\right] \times \right. \\ & \times \left[ J_0(\mu_m) \int_0^1 Ki(Z, Fo^*) \cos \mu_n Z dZ + \right. \\ & \left. \left. + \int_0^1 \cos \mu_n Z \left[ \int_0^1 X J_0(\mu_m X) Po(X, Z, Fo^*) dX \right] dZ \right] dFo^* \right\}, \end{aligned}$$

where  $\mu_n = (2n+1)/2$  ( $n=0, 1, 2, \dots$ )

2. Let us now solve equation (8-5-4) when boundary conditions of the third kind are satisfied at the lateral surface of the cylinder. The limiting conditions of the problem are assumed to be

$$\frac{\partial \theta(R, z, \tau)}{\partial r} + \frac{\alpha}{\lambda} [t(R, z, \tau) - t_a] = 0;$$

$$\frac{\partial \theta(x, 0, \tau)}{\partial z} = 0; \quad \theta(x, l, \tau) = \theta_a;$$

$$\theta(x, z, 0) = \theta_o$$

or, in dimensionless form,

$$\theta(X, Z, 0) = \theta_o; \quad (8-5-13)$$

$$\frac{\partial \theta(1, Z, Fo)}{\partial X} + Bi \theta(1, Z, Fo) = 0; \quad (8-5-14)$$

$$\frac{\partial \theta(X, 0, Fo)}{\partial Z} = 0; \quad \theta(X, \pi, Fo) = 0. \quad (8-5-15)$$

In contrast to the previous problem, the potential  $\theta$  in equation (8-5-4) and in conditions (8-5-13) through (8-5-15) is here defined as

$$\theta(X, Z, Fo) = \frac{\theta - \theta_o}{\theta_o},$$

and in addition

$$\theta_0 = \frac{\theta_a - \theta_0}{\theta_0} \quad \text{and} \quad \text{Bi} = \frac{\alpha R}{\lambda}.$$

If we apply successively transformations (8-5-8) and (8-5-10) to equation (8-5-4) and to conditions (8-5-13) through (8-5-15), then we obtain an ordinary nonhomogeneous differential equation in terms of transforms. The solution of this equation is then inverted using inverse integral transformations (8-5-11) and (8-5-9), and after some simplifications we obtain the final solution /18/:

$$\begin{aligned} \theta(X, Z, \text{Fo}) = & \frac{2\text{Bi}}{\pi} \theta_0 \int_{\frac{Z}{2\pi}}^{\frac{\pi}{2}} \theta_1 \left( \frac{Z^*}{2\pi}, i \frac{b^2}{\pi} \text{Fo} \right) dZ^* \sum_{m=1}^{\infty} \frac{\exp(-\mu_m^2 \text{Fo})}{\mu_m^2 + \text{Bi}^2} \frac{J_0(\mu_m X)}{J_0(\mu_m)} + \\ & + \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\mu_m^2}{\mu_m^2 + \text{Bi}^2} \cdot \frac{J_0(\mu_m X)}{J_0^2(\mu_m)} \left\{ \sum_{n=0}^{\infty} \cos \mu_n Z \int_0^{\text{Fo}} \{ \text{Po}(X, Z, \text{Fo}^*) \}_{Hc} \times \right. \\ & \times \exp[-(\mu_m^2 + b^2 \mu_n^2)(\text{Fo} - \text{Fo}^*)] d\text{Fo}^* \Big\}, \end{aligned} \quad (8-5-16)$$

where

$$\{ \text{Po}(X, Z, \text{Fo}) \}_{Hc} = \int_0^{\frac{\pi}{2}} \int_0^1 X J_0(\mu_m X) \text{Po}(X, Z, \text{Fo}) dX \cos \mu_n Z dZ,$$

while  $\mu_m$  are the positive roots of the characteristic equation

$$\frac{J_0(\mu)}{J_0'(\mu)} = -\frac{1}{\text{Bi}} \mu \quad \text{and} \quad \mu_n = \frac{2n+1}{2} \quad (n=0, 1, 2, 3, \dots).$$

The following expansion of the theta function  $\theta_1$  has been introduced into (8-5-16):

$$\theta_1 \left( \frac{Z}{2\pi}, i \frac{b^2}{\pi} \text{Fo} \right) = 2 \sum_{n=0}^{\infty} (-1)^n \exp[-b^2 \mu_n^2 \text{Fo}] \sin \mu_n Z.$$

Let us consider some particular cases of solution (8-5-16). For certain high-intensity heat-exchange processes, the heat-source term (the Pomerantsev number) can be approximated by the following expression:

$$\text{Po}(X, Z, \text{Fo}) = \text{Po}_1 [1 - \exp(-\beta \text{Fo})] J_0(\mu_1 X) \sin Z, \quad (8-5-17)$$

where  $\mu_1$  is the smallest positive root of the zero-order Bessel function  $J_0(\mu_1) = 0$  (it is found that  $\mu_1 \approx 2.405$ ).

Now, taking into account (8-5-17) and Poisson's summation formula, and on condition that  $\mu_m^2 > \beta$ , we obtain equation (8-5-16) in the form /18/:

$$\begin{aligned} \theta(X, Z, \text{Fo}) = & \frac{2\text{Bi}}{\pi} \theta_0 \int_{\frac{Z}{2\pi}}^{\frac{\pi}{2}} \theta_1 \left( \frac{Z^*}{2\pi}, i \frac{b^2}{\pi} \text{Fo} \right) dZ^* \sum_{m=1}^{\infty} \frac{\exp(-\mu_m^2 \text{Fo})}{\mu_m^2 + \text{Bi}^2} \frac{J_0(\mu_m X)}{J_0(\mu_m)} + \\ & + 2\text{Po}_1 \mu_1 J_1(\mu_1) \sum_{m=1}^{\infty} \frac{\mu_m^2}{(\mu_m^2 + \text{Bi}^2)(\mu_m^2 - \mu_0^2)} \frac{J_0(\mu_m X)}{J_0(\mu_m)} \left\{ \frac{1}{\mu_m^2 + b^2} \left[ \frac{b}{\mu_m} \frac{\text{ch} \frac{\mu_m}{b}(\pi - Z)}{\text{sh} \frac{\mu_m}{b} \pi} - \right. \right. \\ & \left. \left. - 2 \frac{b}{\mu_m} \frac{\text{ch} \frac{\mu_m}{b}(2\pi - Z)}{\text{sh} \frac{\mu_m}{b} \pi} - \sin Z \right] - \frac{\exp(-\beta \text{Fo})}{\mu_m^2 + b^2 - \beta} \left[ \frac{b}{\sqrt{\mu_m^2 - \beta}} \frac{\text{ch} \frac{\sqrt{\mu_m^2 - \beta}}{b}(\pi - Z)}{\text{sh} \frac{\sqrt{\mu_m^2 - \beta}}{b} \pi} - \right. \right. \end{aligned}$$

$$\begin{aligned}
& -2 \frac{b}{\sqrt{\mu_m^2 - \beta}} \left[ \frac{\operatorname{ch} \frac{\sqrt{\mu_m^2 - \beta}}{b} (2\pi - Z)}{\operatorname{sh} 2 \frac{\sqrt{\mu_m^2 - \beta}}{b}} - \sin Z \right] + \\
& + \frac{2}{\pi} \frac{\beta}{b^2} \exp(-\mu_m^2 \operatorname{Fo}) \sum_{n=0}^{\infty} \frac{\exp[-b^2 \mu_n^2 \operatorname{Fo}] \cos \mu_n Z}{(\mu_n^2 - 1) \left( \mu_n^2 + \frac{\mu_m^2}{b} \right) \left( \mu_n^2 + \frac{\mu_m^2}{b^2} \right)}. \quad (8-5-18)
\end{aligned}$$

If  $\mu_m^2 < \beta$ , then everywhere in (8-5-18) it is possible to replace  $\sqrt{\mu_m^2 - \beta}$  by  $i\sqrt{\beta - \mu_m^2}$ .

Some other solutions for limiting conditions simpler than conditions (8-5-13) through (8-5-15) will now be given.

If transfer through the end faces of the cylinder can be neglected in comparison with transfer through the lateral surface, then (8-5-15) has the form

$$\frac{\partial \Theta(X, 0, \operatorname{Fo})}{\partial Z} = \frac{\partial \Theta(X, \pi, \operatorname{Fo})}{\partial Z} = 0$$

and a solution analogous to (8-5-16) can be obtained by means of an ordinary finite Fourier cosine transformation:

$$\begin{aligned}
\Theta(X, Z, \operatorname{Fo}) = & 2\operatorname{Bl} \Theta_0 \sum_{m=1}^{\infty} \frac{\exp(-\mu_m^2 \operatorname{Fo})}{\mu_m^2 + \operatorname{Bl}^2} \cdot \frac{J_0(\mu_m X)}{J_0^2(\mu_m)} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\mu_m^2}{\mu_m^2 + \operatorname{Bl}^2} \cdot \frac{J_0(\mu_m X)}{J_0^2(\mu_m)} \times \\
& \times \exp(-\mu_m^2 \operatorname{Fo}) \left[ \left( \int_0^{\operatorname{Fo}} \exp(\mu_m^2 \operatorname{Fo}^*) \left\{ \int_0^{\pi} \left( \int_0^1 X J_0(\mu_m X) \operatorname{Po}(X, Z, \operatorname{Fo}^*) dX \right) dZ \right\} d\operatorname{Fo}^* + \right. \right. \\
& + \sum_{n=1}^{\infty} \exp(-b^2 n^2 \operatorname{Fo}) \cos nZ \int_0^{\operatorname{Fo}} \exp \left[ (\mu_m^2 + b^2 n^2) \operatorname{Fo}^* \right] \left\{ \int_0^{\pi} \left( \int_0^1 X J_0(\mu_m X) \operatorname{Po}(X, Z, \operatorname{Fo}^*) dX \right) \right. \\
& \left. \left. \times \cos nZ dZ \right\} d\operatorname{Fo}^* \right].
\end{aligned}$$

If the dimensionless potential at the end faces is zero,  $\Theta(X, 0, \operatorname{Fo}) = \Theta(X, \pi, \operatorname{Fo}) = 0$ , then the problem can be solved using a finite Fourier sine transformation:

$$\begin{aligned}
\Theta(X, Z, \operatorname{Fo}) = & \frac{4\operatorname{Bl}}{\pi} \Theta_0 \int_0^Z \vartheta_s \left( \frac{\operatorname{Fo}^*}{\pi}, i \frac{4b^2}{\pi} \operatorname{Fo} \right) d\operatorname{Fo}^* \sum_{m=0}^{\infty} \frac{\exp(-\mu_m^2 \operatorname{Fo})}{\mu_m^2 + \operatorname{Bl}^2} \times \\
& \times \frac{J_0(\mu_m X)}{J_0^2(\mu_m)} + \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\mu_m^2}{\mu_m^2 + \operatorname{Bl}^2} \cdot \frac{J_0(\mu_m X)}{J_0^2(\mu_m)} \exp(-\mu_m^2 \operatorname{Fo}) \left[ \sum_{n=1}^{\infty} \exp(-b^2 n^2 \operatorname{Fo}) \times \right. \\
& \left. \sin(nZ) \int_0^{\operatorname{Fo}} \exp[(\mu_m^2 + b^2 n^2) \operatorname{Fo}^*] \left\{ \int_0^{\pi} \left( \int_0^1 X J_0(\mu_m X) \operatorname{Po}(X, Z, \operatorname{Fo}^*) dX \right) \sin nZ dZ \right\} d\operatorname{Fo}^* \right].
\end{aligned}$$

Some other solutions of this problem for source terms of the form of (8-5-17) are given by Perel'man /18/. Individual solutions of equation (8-5-2) or (8-5-4) in the absence of a source are given in references /8, 9, 10/. It should be noted that solutions when a source is absent can be obtained by superposing the solutions of particular problems. For example, for a finite cylinder this will be a superposition of the solutions for an infinite one-dimensional cylinder and plate.

The method for determining unsteady potential fields for a hollow, axially symmetric cylinder does not differ from the solution method for a solid cylinder. For the hollow cylinder, however, instead of finite Hankel integral transformation (8-5-8) it is necessary to use some other form of this transformation. For example, if boundary conditions of the third kind are specified at the internal and external lateral surfaces



of the cylinder, then the transformation

$$\{\theta(r, z, \tau)\}_H = \int_{R_1}^{R_2} \theta(r, z, \tau) U_0(\mu_m r / R_1) dr \quad (8-5-19)$$

must be used, where

$$U_0(\mu_m r / R_1) = \left[ J_0(\mu_m) + \frac{1}{\text{Bi}_1} \mu_m J_1(\mu_m) \right] Y_0(\mu_m r / R_1) - \left[ Y_0(\mu_m) + \frac{1}{\text{Bi}_1} Y_1(\mu_m) \right] J_0(\mu_m r / R_1);$$

$\mu_m$  are the roots of the equation

$$\frac{U_0(x \mu_m)}{U_1(x \mu_m)} = \frac{x \mu_m}{\text{Bi}_2} \quad (m = 1, 2, \dots) \quad (8-5-20)$$

and

$$U_1(\mu_m r / R_1) = \left[ J_0(\mu_m) + \frac{\mu_m}{\text{Bi}_1} J_1(\mu_m) \right] Y_1(\mu_m r / R_1) - \left[ Y_0(\mu_m) + \frac{1}{\text{Bi}_1} \mu_m Y_1(\mu_m) \right] Y_1(\mu_m r / R_1);$$

$$\text{Bi}_1 = \frac{a_1 R_1}{\lambda}; \quad \text{Bi}_2 = \frac{a_2 R_2}{\lambda}; \quad x = \frac{R_2}{R_1}.$$

The inversion formula for transformation (8-5-19) is

$$\theta(r, z, \tau) = \sum_{m=1}^{\infty} 2\{\theta\}_H U_0(\mu_m r / R_1) R_1^{-2} \left\{ x^2 U_0^2(x \mu_m) \left[ 1 + \left( \frac{\text{Bi}_2}{x \mu_m} \right)^2 \right] - \frac{4}{\pi^2 \text{Bi}_1^2} \left[ 1 + \left( \frac{\text{Bi}_1}{\mu_m} \right)^2 \right] \right\}^{-1}.$$

Only the final result for the problem with the following limiting conditions will be given here:

$$\begin{aligned} \theta(r, z, 0) &= \theta_0; \\ \frac{\partial \theta(R_1, z, \tau)}{\partial r} - \frac{a_1}{\lambda} \theta(R_1, z, \tau) &= 0; \quad \frac{\partial \theta(R_2, z, \tau)}{\partial r} + \frac{a_2}{\lambda} \theta(R_2, z, \tau) = 0; \\ \frac{\partial \theta(r, 0, \tau)}{\partial z} &= 0; \quad \frac{\partial \theta(r, l, \tau)}{\partial z} + \frac{a}{\lambda} \theta(r, l, \tau) = 0 \\ (R_1 < r < R_2, -l < z < l). \end{aligned}$$

This solution can be written as /19/:

$$\frac{\theta - \theta_0}{\theta_c - \theta_0} = 1 - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_m A_n U_0(\mu_m r / R_1) \cos \mu_n \frac{z}{l} \exp[-(\mu_m^2 + b^2 \mu_n^2) \text{Fo}_1], \quad (8-5-21)$$

where

$$\begin{aligned} M_m &= \frac{2[\text{Bi}_2 U_0(x \mu_m) + 2/\pi]}{x^2 U_0^2(x \mu_m) \left[ \mu_m^2 + \left( \frac{\text{Bi}_2}{x} \right)^2 \right] - \frac{4}{\pi^2 \text{Bi}_1^2} [\mu_m^2 + \text{Bi}_1^2]}; \\ A_n &= (-1)^{n+1} \frac{2\text{Bi}_1 \sqrt{\text{Bi}_1^2 + \mu_n^2}}{\mu_n^2 (\text{Bi}_1^2 + \text{Bi}_1 + \mu_n^2)}, \quad \text{Bi}_1 = \frac{a l}{\lambda}. \end{aligned}$$

For equal values of the coefficients of exchange at the external surface and the internal surface ( $a_1 = a_2 = a$ ), the coefficient  $M_m$  simplifies to

$$A_m = \frac{2\text{Bi}_1}{(\mu_m^2 + \text{Bi}_1^2) \left[ x U_0(x \mu_m) - \frac{2}{\pi \text{Bi}_1} \right]}.$$

The infinite sum in (8-5-21) converges rapidly, so that for practical calculations only one or two terms of the series need be retained. For convenience in carrying out the calculations, Plyat /19/ has plotted, for the condition  $a_1 = a_2$ , graphs of the first two roots  $\mu_m$  of equation (8-5-20), of the coefficients  $A_m$ , and of the function  $U_0(x, \mu_m)$ , all for ratios  $x = R_2/R_1$  from 1.5 to 4.0 and for Biot numbers  $\text{Bi}_1 = \frac{a R_1}{\lambda}$  from 1 to 10. These graphs are given in Figures (8-1) through (8-3).

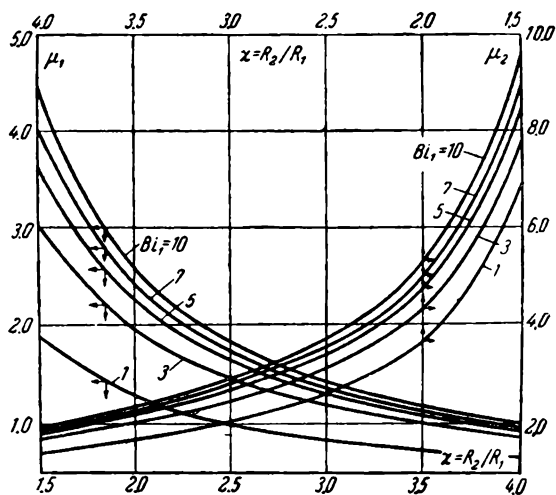


FIGURE 8-1. Characteristic-equation roots  $\mu_1$  and  $\mu_2$  as functions of the ratio  $\alpha = R_2/R_1$ , for various values of  $Bi_1$

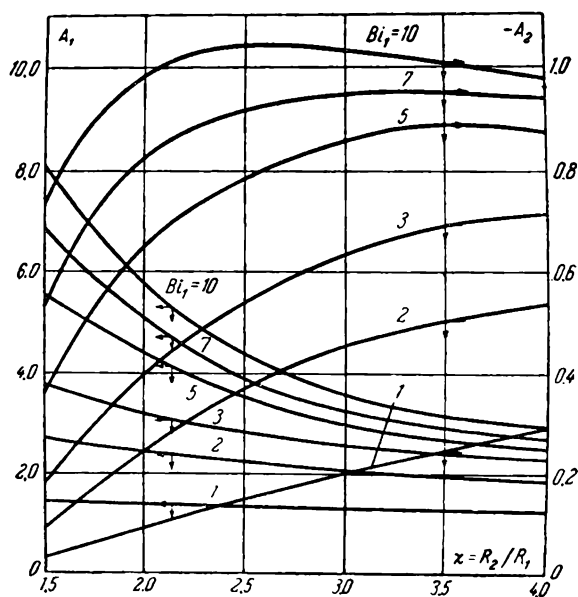


FIGURE 8-2. Expansion coefficients  $A_1$  and  $A_2$  as functions of the ratio  $\alpha = R_2/R_1$ , for various values of  $Bi_1$

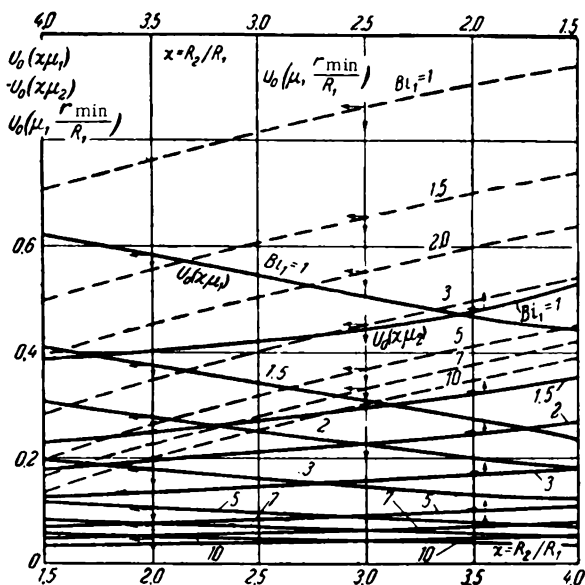


FIGURE 8-3. Quantities  $U_0(x\mu_1)$ ,  $U_0(x\mu_2)$ , and  $U_0(\mu, r_{min}/R_1)$  as functions of the ratio  $\kappa = R_2/R_1$ , for various values of  $Bi_1$

Other solutions for a hollow axially symmetric cylinder, for both constant and variable surrounding temperatures, have been given by Plyat /19/ and by Danilova /20/, Plyatsko /21/, Shvets and Fedorov /22/, and others.

#### b) Solution for an axially unsymmetric cylinder

We will conclude this section with a study of a general method for solving differential equation (8-5-1) for the axially unsymmetric case. Following Prudnikov and Gavrilova /23/, we will solve equation (8-5-1) with the very general limiting conditions

$$\theta(r, \varphi, z, 0) = f(r, \varphi, z);$$

$$\frac{\partial \theta(r, \varphi, 0, \tau)}{\partial z} = \psi(r, \varphi, \tau); \quad \theta(r, \varphi, l, \tau) = \chi(r, \varphi, \tau);$$

$$\lambda \frac{\partial \theta(R, \varphi, z, \tau)}{\partial r} + \theta(R, \varphi, z, \tau) = 0.$$

The solution will be sought in the form of a Fourier series:

$$\theta = \frac{\theta_0}{2} + \sum_{m=1}^{\infty} [\theta_m^{(1)} \sin m\varphi + \theta_m^{(2)} \cos m\varphi].$$

If we represent the functions  $w$ ,  $f$ ,  $\psi$ , and  $\chi$  analogously, then the problem at hand reduces to the solution of the differential equation

$$\frac{\partial \theta_m^{(i)}}{\partial \tau} = a \left( \frac{\partial^2 \theta_m^{(i)}}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_m^{(i)}}{\partial r} - \frac{m^2}{r^2} \theta_m^{(i)} + \frac{\partial^2 \theta_m^{(i)}}{\partial z^2} \right) + \frac{1}{c\tau} w_m^{(i)} \quad (8-5-22)$$

with the limiting conditions

$$\left. \begin{aligned} \theta_m^{(i)}(r, z, 0) &= f_m^{(i)}(r, z); \\ \frac{\partial \theta_m^{(i)}(r, 0, \tau)}{\partial z} &= \psi_m^{(i)}(r, \tau), \quad \theta_m^{(i)}(r, l, \tau) = \chi_m^{(i)}(r, \tau); \\ \lambda \frac{\partial \theta_m^{(i)}(R, z, \tau)}{\partial r} + \theta_m^{(i)}(R, z, \tau) &= 0 \quad (i = 1, 2). \end{aligned} \right\} \quad (8-5-23)$$

A further solution of the problem can be carried out in different ways. First, finite Hankel transformations developed specifically for this problem may be used, as is done by Elistratova /24/, Lykov, and others. Then, the solution may be obtained in the form of a series of eigenfunctions of the corresponding Sturm-Liouville problem, or else by expanding the functions entering into (8-5-23) in a Dini-Bessel series /9, 10, 23/:

$$\theta_m^{(i)} = \sum_{n=1}^{\infty} \theta_{mn}^{(i)} J_m(\mu_{mn} r / R),$$

where

$$\theta_{mn}^{(i)}(z, \tau) = \frac{2}{\pi R^2 [J'_m(\mu_{mn})]^2} \int_0^R \theta_m^{(i)} J_m(\mu_{mn} r / R) r dr;$$

and  $\mu_{mn}$  are the roots of the equation

$$\gamma x J'_m(x) + R J_m(x) = 0.$$

Equation (8-5-22) and conditions (8-5-23) then reduce to the new problem:

$$\begin{aligned} \frac{\partial \theta_{mn}^{(i)}}{\partial \tau} &= a \left[ \frac{\partial^2 \theta_{mn}^{(i)}}{\partial z^2} - \left( \frac{\mu_{mn}}{R} \right)^2 \theta_{mn}^{(i)} \right] + \frac{1}{c\gamma} w_{mn}^{(i)}; \\ \theta_{mn}^{(i)}(z, 0) &= f_{mn}^{(i)}(z); \quad \frac{\partial \theta_{mn}^{(i)}(0, \tau)}{\partial z} = \psi_{mn}^{(i)}(\tau); \quad \theta_{mn}^{(i)}(l, \tau) = \chi_{mn}^{(i)}(\tau). \end{aligned}$$

The solution of the latter problem does not involve any particular difficulty. The solution method is similar to the method for solving the usual transfer equations with boundary conditions of the second kind and can be obtained by a combined application of Fourier and Laplace transformations. As a result we obtain

$$\begin{aligned} \theta_{mn}^{(i)}(z, \tau) &= \exp \left[ -a \left( \frac{\mu_{mn}}{R} \right)^2 \right] \times \\ &\times \left\{ \frac{1}{2l} \int_0^l \left[ \vartheta_2 \left( \frac{z-\xi}{2l}, \frac{a\tau}{l^2} \right) + \vartheta_2 \left( \frac{z+\xi}{2l}, \frac{a\tau}{l^2} \right) \right] \right\} f_{mn}^{(i)}(\xi) d\xi + \\ &+ \frac{a}{l} \left\{ \int_0^\tau \vartheta_2 \left[ \frac{l-z}{2l}, \frac{a(\tau-\tau^*)}{l^2} \right] \chi_{mn}^{(i)}(\tau^*) d\tau^* - \right. \\ &\left. - \int_0^\tau \vartheta_2 \left[ \frac{z}{2l}, \frac{a(\tau-\tau^*)}{l^2} \right] \psi_{mn}^{(i)}(\tau^*) d\tau^* \right\} + \\ &+ \frac{1}{2l} \int_0^l \int_0^\tau \left\{ \vartheta_2 \left[ \frac{z-\xi}{2l}, \frac{a(\tau-\tau^*)}{l^2} \right] + \vartheta_2 \left[ \frac{z+\xi}{2l}, \frac{a(\tau-\tau^*)}{l^2} \right] \right\} \frac{1}{c\gamma} w_{mn}^{(i)}(\xi, \tau^*) d\xi d\tau^*, \end{aligned}$$

where

$$\theta_1(x, \tau) = 2 \sum_{n=0}^{\infty} \exp \left[ -\pi^2 \left( n + \frac{1}{2} \right)^2 \tau \right] \cos \pi (2n + 1) x.$$

Several specific problems for axially unsymmetric cylinders are considered by Carslaw and Jaeger /9, 10/, and so we will not discuss them here.

### 8-6. Unsteady Fields of Heat-Transfer and Mass-Transfer Potential in an Anisotropic Body

In the previous sections we considered solution methods for, and gave several solutions for, the system of heat-transfer and mass-transfer equations for an isotropic body (that is, a body in which the transfer of heat or mass is the same in all directions). In many processes, for some reason or other, the transfer phenomena may have different intensities in different directions (for example, during the thermal treatment and drying of certain porous materials, in thermoplasticity problems, etc.). Heat and mass transfer in an anisotropic dispersed medium can be represented in the simplest case by the system of equations:

$$\frac{\partial t}{\partial \tau} = \sum_{i=1}^3 C_i \frac{\partial^2 t}{\partial x_i^2} + \sum_{i=1}^3 D_i \frac{\partial^2 \theta}{\partial x_i^2}; \quad (8-6-1)$$

$$\frac{\partial \theta}{\partial \tau} = \sum_{i=1}^3 C^*_i \frac{\partial^2 t}{\partial x_i^2} + \sum_{i=1}^3 D^*_i \frac{\partial^2 \theta}{\partial x_i^2}, \quad (8-6-2)$$

where  $x_i$  are the Cartesian coordinates ( $i = 1, 2, 3$  and  $x_1 = x, x_2 = y, x_3 = z$ ), and

$$C_i = a_{qi} + \frac{\rho a_{\theta i} a_{m i}}{c_q}; \quad D_i = \frac{\rho a_{m i} c_m}{c_q}; \quad C^*_i = \frac{\delta_i a_{m i}}{c_m}; \quad D^*_i = a_{m i}.$$

The method for solving this type of problem does not differ in principle from the solution methods in the problems considered above. The first solution of system of equations (8-6-1) and (8-6-2) for a symmetric parallelepiped with dimensions  $2l_1 \times 2l_2 \times 2l_3$ , for limiting conditions of the first kind, was obtained by Gamayunov /25/. This solution was obtained by means of a combined application of a Laplace transformation and a Fourier cosine transformation. For the limiting conditions

$$t(x_1, x_2, x_3, 0) = f_1(x_1, x_2, x_3); \quad \theta(x_1, x_2, x_3, 0) = f_2(x_1, x_2, x_3);$$

$$t(l_i, \tau) = \varphi_i(\tau); \quad \theta(l_i, \tau) = \psi_i(\tau);$$

$$\frac{\partial t(0, \tau)}{\partial x_i} = \frac{\partial \theta(0, \tau)}{\partial x_i} = 0$$

the solution has the form

$$t(x_1, x_2, x_3, \tau) = \frac{1}{v_1 v_2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^2 (-1)^{j+1} [M_{kj}(u_{1j} + v_{1j}) - N_{kj}(u_{2j} + v_{2j})];$$

$$\theta(x_1, x_2, x_3, \tau) = \frac{1}{v_1 - v_2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^2 \sum_{j=1}^2 (-1)^{j+1} [N_{sj}(u_{sj} + v_{sj}) - M_{sj}(u_{sj} + v_{sj})],$$

where

$$M_{sj} = \frac{A_1}{L_1^2} \mu_k^2 + \frac{A_2}{L_2^2} \mu_m^2 + \frac{A_3}{L_3^2} \mu_n^2 - v_j \quad (j=1, 2);$$

$$M_{sj} = \left( \text{Fe}_1 + \frac{1}{\text{Lu}_1} \right) \frac{A_1}{L_1^2} \mu_k^2 + \left( \text{Fe}_2 + \frac{1}{\text{Lu}_2} \right) \frac{A_2}{L_2^2} \mu_m^2 + \left( \text{Fe}_3 + \frac{1}{\text{Lu}_3} \right) \frac{A_3}{L_3^2} \mu_n^2 - v_j;$$

$$N_{sj} = \frac{1}{\delta} \left( \frac{\text{Fe}_1 A_1}{\Delta_1 \cdot L_1^2} \mu_k^2 + \frac{\text{Fe}_2 A_2}{\Delta_2 \cdot L_2^2} \mu_m^2 + \frac{\text{Fe}_3 A_3}{\Delta_3 \cdot L_3^2} \mu_n^2 \right);$$

$$N_{sj} = \frac{1}{\delta} \left( \frac{A_1}{\Delta_1 \cdot L_1^2} \mu_k^2 + \frac{A_2}{\Delta_2 \cdot L_2^2} \mu_m^2 + \frac{A_3}{\Delta_3 \cdot L_3^2} \mu_n^2 \right);$$

$$A_i = \frac{a_m t}{a_m}; \quad L_i = \frac{l_i}{l}; \quad \Delta_i = \frac{\delta_i}{\delta};$$

$$a_m = \sqrt{a_{m1}^2 + a_{m2}^2 + a_{m3}^2}; \quad l = \sqrt{l_1^2 + l_2^2 + l_3^2}; \quad \delta = \sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2};$$

$$v_j = \frac{1}{2} \left\{ \frac{A_1 K_1}{L_1^2} \mu_k^2 + \frac{A_2 K_2}{L_2^2} \mu_m^2 + \frac{A_3 K_3}{L_3^2} \mu_n^2 + (-1)^{j+1} \times \right.$$

$$\times \left[ \left( \frac{A_1 K_1}{L_1^2} \mu_k^2 + \frac{A_2 K_2}{L_2^2} \mu_m^2 + \frac{A_3 K_3}{L_3^2} \mu_n^2 \right)^2 - \right.$$

$$- 4 \left[ \frac{A_1^2 \mu_k^2}{L_1^4 \text{Lu}_1} + \frac{A_2^2 \mu_m^2}{L_2^4 \text{Lu}_2} + \frac{A_3^2 \mu_n^2}{L_3^4 \text{Lu}_3} + \frac{A_1 A_2 \mu_k \mu_m}{L_1^2 L_2^2} \left( \frac{1}{\text{Lu}_1} + \frac{1}{\text{Lu}_2} \right) + \right. \\ \left. + \frac{A_1 A_3 \mu_k \mu_n}{L_1^2 L_3^2} \left( \frac{1}{\text{Lu}_1} + \frac{1}{\text{Lu}_3} \right) + \frac{A_2 A_3 \mu_m \mu_n}{L_2^2 L_3^2} \left( \frac{1}{\text{Lu}_2} + \frac{1}{\text{Lu}_3} \right) \right]^{1/2} \Bigg\};$$

$$K_i = 1 + \text{Fe}_i + 1/\text{Lu}_i; \quad \text{Fe}_i = s \text{ Ko Pn}_i; \quad \text{Lu}_i = a_{mi}/a_{qi};$$

$$u_{js} = 8 \int_0^1 \int_0^1 \int_0^1 f_s(x_1, x_2, x_3) \cos \mu_k \frac{x_1}{l_1} \cos \mu_m \frac{x_2}{l_2} \cos \mu_n \frac{x_3}{l_3} d \frac{x_1}{l_1} d \frac{x_2}{l_2} d \frac{x_3}{l_3} \times$$

$$\times \cos \mu_k \frac{x_1}{l_1} \cos \mu_m \frac{x_2}{l_2} \cos \mu_n \frac{x_3}{l_3} \exp(-v_j \text{Fo}_m) \quad (s=1, 2);$$

$$v_{js} = (-1)^{k+m+n+1} \frac{8}{\mu_k \mu_m \mu_n} \int_0^{\text{Fo}_m(\tau)} \left\{ \frac{A_1 \mu_k^2}{L_1^2} \left[ \left( \frac{1}{\text{Lu}_1} + \text{Fe}_1 \right) \varphi_1(\theta) + \frac{\text{Fe}_1}{\delta_1} \psi_1(\theta) \right] + \right.$$

$$+ \frac{A_2 \mu_m^2}{L_2^2} \left[ \left( \frac{1}{\text{Lu}_2} + \text{Fe}_2 \right) \varphi_2(\theta) + \frac{\text{Fe}_2}{\delta_2} \psi_2(\theta) \right] + \frac{A_3 \mu_n^2}{L_3^2} \left[ \left( \frac{1}{\text{Lu}_3} + \text{Fe}_3 \right) \varphi_3(\theta) + \right.$$

$$\left. + \frac{\text{Fe}_3}{\delta_3} \psi_3(\theta) \right] \Bigg\} \exp \left\{ -v_j [\text{Fo}_m(\tau) - \text{Fo}_m(\theta)] d \text{Fo}_m(\theta) \cos \mu_k \frac{x_1}{l_1} \times \right.$$

$$\times \cos \mu_m \frac{x_2}{l_2} \cos \mu_n \frac{x_3}{l_3} \Bigg\};$$

$$\mu_k = (2k-1)\pi/2; \quad \mu_m = (2m-1)\pi/2; \quad \mu_n = (2n-1)\pi/2$$

$$(k, m, n=1, 2, 3, \dots);$$

$$\text{Fo}_m = \frac{a_m \tau}{l^2}.$$

Some solutions for anisotropic uncoupled transfer have been considered by Mitskevich /26/, Carslaw /9/, and others.

## Chapter IX

### UNSTEADY POTENTIAL FIELDS FOR BOTH MACROSCOPIC AND MOLECULAR HEAT AND MASS TRANSFER

#### 9-1. Transfer-Potential Fields with Boundary Conditions of the Third Kind. Mass Exchange at Body Surface a Function of Mass-Transfer Potential

Modern engineering makes wide use of high temperatures and pressures to intensify heat and mass exchange. The mechanism of the transfer of matter and energy under such conditions undergoes a qualitative change, since in addition to the transfer resulting from the action of molecular forces a large (and in some cases a dominant) role is played by the transfer phenomena which accompany macroscopic seepage-type processes. The macroscopic-molecular mass transfer of a multiphase substance is of great interest, not only with respect to various technological processes but also in many applications of transfer theory to dispersed and conducting media.

Macroscopic mass transfer is caused when a stable total-pressure gradient  $\nabla p$  is set up in the material (or the given medium). For example, when a moist dispersed medium is heated intensely, a very marked vapor formation takes place, and this leads to the establishment of a stable pressure gradient in the vapor-gas mixture. This gradient is due to the fact that the time for relaxation of the excess pressure through the skeleton of the material is quite high, with the result that during this time the vapor necessary for the re-establishment of the initial state can form. In liquids and gases the establishment of an appreciable pressure differential may be due to some external field. The superposition of macroscopic transfer on to molecular processes leads to a rearrangement (modification) of the transfer mechanism and to a considerable intensification of the process accompanying it.

The derivation of a system of differential equations describing macroscopic-molecular heat and mass transfer was considered in Chapter II. If the zonal method for calculating the process in a one-dimensional body or medium is used, then this system has the following form, in terms of dimensionless quantities:

$$\frac{\partial T(X, Fo)}{\partial Fo} = \frac{\partial^2 T(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial T(X, Fo)}{\partial X} - \epsilon Ko \frac{\partial \Theta(X, Fo)}{\partial Fo}; \quad (9-1-1)$$

$$\begin{aligned} \frac{\partial \Theta(X, Fo)}{\partial Fo} = & Lu \left[ \frac{\partial^2 \Theta(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \Theta(X, Fo)}{\partial X} \right] - Lu Pn \left[ \frac{\partial^2 T(X, Fo)}{\partial X^2} + \right. \\ & \left. + \frac{\Gamma}{X} \frac{\partial T(X, Fo)}{\partial X} \right] - Lu_p \frac{Bu}{Ko} \left[ \frac{\partial^2 P(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial P(X, Fo)}{\partial X} \right]; \quad (9-1-2) \end{aligned}$$

$$\frac{\partial P(X, Fo)}{\partial Fo} = Lu_p \left[ \frac{\partial^2 P(X, Fo)}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial P(X, Fo)}{\partial X} \right] + \epsilon \frac{Ko}{Bu} \frac{\partial \Theta(X, Fo)}{\partial Fo}, \quad (9-1-3)$$

where  $Bu = \frac{\rho c_p \Delta T}{c_q \Delta T}$  is the Bulygin number;  $Lu_p = \frac{a_p}{a_q}$  is a parameter describing the coupling between the macroscopic heat and mass transfer; and  $P = (p - p_0)/p_0$  is the dimensionless potential for seepage mass transfer, for a constant initial potential distribution. If there is no macroscopic mass transfer in the body, the parameter  $Lu_p$  is zero and system of equations (9-1-1) through (9-1-3) simplifies to system (4-1-2) and (4-1-3).

Solutions of the system of macroscopic-molecular heat-transfer and mass-transfer equations have been given by Mikhailov /1 through 5/, Prudnikov /6, 7/, and Tsoi /8/. Experimental and analytical studies of the effects of individual factors on macroscopic-molecular heat and mass transfer have been carried out by Zhmakin /9/, Zuev /10/, Krasnikov /11, 12/, Lebedev /14, 15/, Krechetov /13/, Lykov /16 through 19/, Maksimov /19, 20/, Mikhailov /5, 21 through 23/, Shubin /24, 25/, and others.

Let us find the unsteady distribution of the dimensionless transfer potentials in a dispersed medium, for an infinite plate. Symmetry is assumed, and also constant initial distribution of the potentials over the cross section of the material. The limiting conditions are in this case

$$\frac{\partial T(1, Fo)}{\partial X} - Bi_q [1 - T(1, Fo)] + (1 - \epsilon) Ko Lu Bi_m [1 - \Theta(1, Fo)] = 0; \quad (9-1-4)$$

$$-\frac{\partial \Theta(1, Fo)}{\partial X} + Pn \frac{\partial T(1, Fo)}{\partial X} + \frac{Bu Lu_p}{Ko Lu} \frac{\partial P(1, Fo)}{\partial X} + Bi_m [1 - \Theta(1, Fo)] = 0; \quad (9-1-5)$$

$$P(1, Fo) = 0; \quad (9-1-6)$$

$$\frac{\partial T(0, Fo)}{\partial X} = \frac{\partial \Theta(0, Fo)}{\partial X} = \frac{\partial P(0, Fo)}{\partial X} = 0; \quad (9-1-7)$$

$$T(X, 0) = 0; \Theta(X, 0) = 0; P(X, 0) = 0. \quad (9-1-8)$$

The solution is derived using the following Laplace integral transformation:

$$F_L(X, s) = \int_0^\infty F(X, Fo) \exp(-s Fo) d Fo. \quad (9-1-9)$$

When transformation (9-1-9) is applied to system of equations (9-1-1) through (9-1-3), subject to initial conditions (9-1-8), we obtain

$$s T_L = T''_L - \epsilon Ko s \Theta_L; \quad (9-1-10)$$

$$s \Theta_L = Lu \Theta''_L - Lu Pn T''_L - Lu_p \frac{Bu}{Ko} P''_L; \quad (9-1-11)$$

$$s P_L = Lu_p P''_L + \epsilon \frac{Ko}{Bu} s \Theta_L. \quad (9-1-12)$$

Now, if  $\Theta_L$  is determined from equation (9-1-10) and substituted into (9-1-11) and (9-1-12), we obtain a system of two equations in  $T_L$  and  $P_L$ . By applying to this system a second Laplace transformation with respect to the dimensionless coordinate and solving it, we can find expressions for  $T_L$  and  $P_L$ . Finally,  $\Theta_L$  is determined by substituting  $T_L$ ,  $P_L$ , and their second derivatives into equation (9-1-10). After these transformations are carried out, we obtain

$$T_L = A_1 \exp(\sqrt{s v_s} X) + A_2 \exp(\sqrt{s v_s} X) + A_3 \exp(\sqrt{s v_s} X) + A_4 \exp(-\sqrt{s v_s} X) + A_5 \exp(-\sqrt{s v_s} X) + A_6 \exp(-\sqrt{s v_s} X);$$



$$\Theta_L = -\frac{1}{\epsilon K_0} [A_1 (1 - v_1^2) \exp(\sqrt{s} v_1 X) + A_2 (1 - v_2^2) \exp(\sqrt{s} v_2 X) + A_3 (1 - v_3^2) \exp(\sqrt{s} v_3 X) + \\ + A_4 (1 - v_1^2) \exp(-\sqrt{s} v_1 X) + A_5 (1 - v_2^2) \exp(-\sqrt{s} v_2 X) + A_6 (1 - v_3^2) \exp(-\sqrt{s} v_3 X)];$$

$$P_L = \frac{1}{\epsilon B_u} [A_1 v_1 \exp(\sqrt{s} v_1 X) + A_2 v_2 \exp(\sqrt{s} v_2 X) + A_3 v_3 \exp(\sqrt{s} v_3 X) + \\ + A_4 v_1 \exp(-\sqrt{s} v_1 X) + A_5 v_2 \exp(-\sqrt{s} v_2 X) + A_6 v_3 \exp(-\sqrt{s} v_3 X)].$$

Here

$$\sigma_j = (1 - \epsilon)(1 - v_j^2) - Lu(1 - v_j^2) v_j^2 - \epsilon K_0 P_n Lu v_j^2; \quad (9-1-13)$$

$$v_j^2 = \sqrt{\Delta_j + \beta_1/3} \quad (j = 1, 2, 3), \quad (9-1-14)$$

where the quantities  $\Delta_j$  in (9-1-14) are defined by the expression

$$\Delta = \sqrt[3]{-\frac{1}{2} \Pi_1 + \sqrt{\frac{1}{4} \Pi_2^2 + \frac{1}{27} \Pi_1^3}} + \sqrt[3]{-\frac{1}{2} \Pi_1 - \sqrt{\frac{1}{4} \Pi_2^2 + \frac{1}{27} \Pi_1^3}}.$$

At the same time, the eigenvalues of  $\Delta$  must satisfy the relation

$$\sqrt[3]{-\frac{1}{2} \Pi_1 + \sqrt{\frac{1}{4} \Pi_2^2 + \frac{1}{27} \Pi_1^3}} \cdot \sqrt[3]{-\frac{1}{2} \Pi_1 - \sqrt{\frac{1}{4} \Pi_2^2 + \frac{1}{27} \Pi_1^3}} = -\frac{1}{3} \Pi_1,$$

where

$$\Pi_1 = -\frac{\beta_1^2}{3} + \beta_3; \quad \Pi_2 = -\frac{2}{27} \beta_1^3 + \frac{1}{3} \beta_1 \beta_2 - \beta_3;$$

$$\beta_1 = 1 + (1 - \epsilon) \frac{1}{Lu} + \frac{1}{Lu_p} + \epsilon K_0 P_n; \quad \beta_2 = (1 - \epsilon) \frac{1}{Lu} + \left(1 + \frac{1}{Lu} + \epsilon K_0 P_n\right) \frac{1}{Lu_p}; \quad \beta_3 = 1/Lu Lu_p.$$

Values of  $v_j$  for various combinations of similarity criteria are listed in Table 9-1, while values of  $\sigma_j$  are given in Table 9-2.

The values of coefficients  $A_k$  ( $k=1, 2, \dots, 6$ ), which are constant with respect to  $X$ , are found from boundary conditions (9-1-4) through (9-1-6) and from symmetry conditions (9-1-7). In terms of transforms these conditions have the form

$$T'_L(1, s) - Bi_s \left[ \frac{1}{s} - T_L(1, s) \right] + (1 - \epsilon) K_0 Lu Bi_m \left[ \frac{1}{s} - \Theta_L(1, s) \right] = 0; \quad (9-1-15)$$

$$-\Theta'_L(1, s) + P_n T'_L(1, s) + \frac{Bu}{K_0} \cdot \frac{Lu_p}{Lu} P'_L(1, s) + Bi_m \left[ \frac{1}{s} - \Theta_L(1, s) \right] = 0; \quad (9-1-16)$$

$$P_L(1, s) = 0; \quad (9-1-17)$$

$$T'_L(0, s) = \Theta'_L(0, s) = P'_L(0, s) = 0. \quad (9-1-18)$$

If the transform solutions are to satisfy symmetry conditions (9-1-18), then we have

$$A_1 = A_4; \quad A_2 = A_5; \quad A_3 = A_6;$$

so that the solutions can be written as

$$T_L(X, s) = \sum_{j=1}^3 B_j \operatorname{ch} \sqrt{s} v_j X; \quad (9-1-19)$$

$$\Theta_L(X, s) = -\frac{1}{\epsilon K_0} \sum_{j=1}^3 B_j (1 - v_j^2) \operatorname{ch} \sqrt{s} v_j X; \quad (9-1-20)$$

$$P_L(X, s) = \frac{1}{\epsilon B_u} \sum_{j=1}^3 B_j v_j \operatorname{ch} \sqrt{s} v_j X. \quad (9-1-21)$$

TABLE 9-1

Values of  $\nu_j = \nu_j(Lu, Lu_p, \epsilon, Ko, Pn)$   $\epsilon = 0.25$ 

Lu	$\nu_j$	Ko Pn														
		Lu <sub>p</sub> = 100					Lu <sub>p</sub> = 500					Lu <sub>p</sub> = 1000				
		0.5	1.0	2.0	3.0	4.0	5.0	0.5	1.0	2.0	3.0	4.0	5.0	0.5	1.0	1.0
0.25	$\nu_1$	1.78303	1.83119	1.91927	1.99942	2.07380	2.14371	1.78372	1.83181	1.91980	1.99988	2.07421	2.14408	1.78381	1.83189	1.83189
	$\nu_2$	0.11554	0.11555	0.11557	0.11558	0.11560	0.11562	0.05165	0.05165	0.05165	0.05166	0.05165	0.05165	0.03651	0.03652	0.03652
	$\nu_3$	0.97080	0.94520	0.90169	0.86542	0.83426	0.80694	0.97091	0.94541	0.90205	0.86591	0.83485	0.80763	0.97092	0.94543	0.94543
0.50	$\nu_1$	1.33492	1.41350	1.53764	1.64059	1.73166	1.81480	1.33563	1.41407	1.53808	1.64095	1.73197	1.81507	1.33571	1.41414	1.41414
	$\nu_2$	0.11562	0.11563	0.11567	0.11570	0.11573	0.11577	0.05165	0.05165	0.05166	0.05166	0.05166	0.05166	0.03652	0.03652	0.03652
	$\nu_3$	0.91630	0.86524	0.79515	0.74504	0.70565	0.67313	0.91675	0.86587	0.79601	0.74607	0.70682	0.67442	0.91681	0.86595	0.86595
0.75	$\nu_1$	1.19170	1.28028	1.41382	1.52241	1.61775	1.70441	1.19216	1.28068	1.41413	1.52267	1.61798	1.70461	1.19222	1.28073	1.28073
	$\nu_2$	0.11569	0.11572	0.11577	0.11582	0.11587	0.11592	0.05166	0.05166	0.05167	0.05167	0.05167	0.05168	0.03652	0.03652	0.03652
	$\nu_3$	0.83753	0.77942	0.70549	0.65488	0.61601	0.58442	0.83849	0.78051	0.70679	0.65635	0.61764	0.58620	0.83861	0.78064	0.78064
1.00	$\nu_1$	1.13830	1.22440	1.35730	1.46651	1.56279	1.65048	1.13858	1.22468	1.35754	1.46671	1.56296	1.65064	1.13862	1.22471	1.22471
	$\nu_2$	0.11577	0.11580	0.11587	0.11594	0.11601	0.11608	0.05167	0.05167	0.05167	0.05168	0.05169	0.05169	0.03652	0.03652	0.03652
	$\nu_3$	0.75886	0.70529	0.63585	0.58815	0.55157	0.52195	0.76023	0.70675	0.63751	0.58999	0.55359	0.52413	0.76040	0.70693	0.70693
2.00	$\nu_1$	1.08756	1.16026	1.28336	1.38940	1.48478	1.57260	1.08764	1.16036	1.28347	1.38950	1.48488	1.57268	1.08765	1.16037	1.16037
	$\nu_2$	0.11608	0.11615	0.11630	0.11645	0.11660	0.11676	0.05169	0.05170	0.05171	0.05172	0.05173	0.05174	0.03653	0.03654	0.03654
	$\nu_3$	0.56012	0.52470	0.47377	0.43705	0.40843	0.38510	0.56246	0.52715	0.47648	0.44002	0.41166	0.38859	0.56274	0.52744	0.52744
3.00	$\nu_1$	1.07668	1.14404	1.26208	1.36594	1.46033	1.54772	1.07673	1.14411	1.26215	1.36601	1.46039	1.54777	1.07673	1.14411	1.14411
	$\nu_2$	0.11640	0.11652	0.11675	0.11700	0.11725	0.11752	0.05172	0.05173	0.05174	0.05174	0.05176	0.05180	0.03654	0.03654	0.03654
	$\nu_3$	0.46066	0.43311	0.39181	0.36126	0.33718	0.31742	0.46366	0.43628	0.39534	0.36516	0.34144	0.32205	0.46402	0.43665	0.43665
4.00	$\nu_1$	1.07205	1.13679	1.25210	1.35468	1.44844	1.53552	1.07208	1.13684	1.25215	1.35473	1.44848	1.53556	1.07208	1.13684	1.13684
	$\nu_2$	0.11675	0.11691	0.11725	0.11760	0.11798	0.11838	0.05174	0.05176	0.05178	0.05180	0.05183	0.05186	0.03655	0.03656	0.03656
	$\nu_3$	0.39948	0.37621	0.34058	0.31384	0.29259	0.27507	0.40308	0.38003	0.34487	0.31861	0.29784	0.28082	0.40349	0.38046	0.38046
5.00	$\nu_1$	1.06949	1.13269	1.24632	1.34809	1.44141	1.52828	1.06951	1.13273	1.24636	1.34812	1.44145	1.52832	1.06951	1.13273	1.13273
	$\nu_2$	0.11712	0.11733	0.11778	0.11827	0.11880	0.11937	0.05177	0.05179	0.05182	0.05185	0.05188	0.05191	0.03657	0.03657	0.03657
	$\nu_3$	0.35704	0.33650	0.30465	0.28049	0.26117	0.24514	0.36120	0.34094	0.30968	0.28613	0.26744	0.23209	0.36167	0.34143	0.34143

T A B L E 9 - 1 (continued)

$\epsilon = 0.50$

$\epsilon = 0.25$

Lu	y <sub>f</sub>	Ko Pn																	
		Lu <sub>p</sub> = 1 000						Lu <sub>p</sub> = 100						Lu <sub>p</sub> = 500					
		2.0	3.0	4.0	5.0	0.5	1.0	2.0	3.0	4.0	5.0	0.5	1.0	2.0	3.0				
0.25	y <sub>1</sub>	1.91987	1.99994	2.07426	2.14412	1.55499	1.66569	1.84640	1.99892	2.13489	2.25944	1.55689	1.66719	1.84749	1.99878				
	y <sub>2</sub>	0.03652	0.03652	0.03652	0.03652	0.14187	0.14197	0.14216	0.14235	0.14254	0.14274	0.06328	0.06329	0.06331	0.06332				
	y <sub>3</sub>	0.90210	0.86597	0.83493	0.80771	0.90657	0.84576	0.76197	0.70288	0.65721	0.62012	0.90778	0.84762	0.76471	0.70629				
0.50	y <sub>1</sub>	1.53813	1.64099	1.73201	1.81510	1.27929	1.41303	1.61717	1.78430	1.93130	2.06486	1.28048	1.41308	1.61786	1.78484				
	y <sub>2</sub>	0.03652	0.03652	0.03652	0.03652	0.14234	0.14254	0.14293	0.14335	0.14377	0.14421	0.06332	0.06334	0.06337	0.06341				
	y <sub>3</sub>	0.79612	0.74620	0.70696	0.67458	0.77663	0.70216	0.61181	0.55291	0.50931	0.47491	0.77997	0.70615	0.61685	0.55885				
0.75	y <sub>1</sub>	1.41417	1.52270	1.61800	1.70463	1.20746	1.33912	1.54453	1.71414	1.86369	1.99967	1.20816	1.33975	1.54501	1.71452				
	y <sub>2</sub>	0.03652	0.03652	0.03653	0.03653	0.14283	0.14313	0.14376	0.14443	0.14513	0.14587	0.06336	0.06339	0.06344	0.06349				
	y <sub>3</sub>	0.70695	0.65653	0.61784	0.58641	0.66956	0.60244	0.52002	0.46640	0.42690	0.39585	0.67454	0.60805	0.52585	0.47440				
1.00	y <sub>1</sub>	1.35757	1.46673	1.56299	1.65066	1.17857	1.30600	1.50977	1.67978	1.83021	1.96719	1.17903	1.30645	1.51013	1.68007				
	y <sub>2</sub>	0.03653	0.03653	0.03653	0.03653	0.14333	0.14376	0.14465	0.14561	0.14665	0.14778	0.06341	0.06344	0.06350	0.06357				
	y <sub>3</sub>	0.63771	0.59021	0.55383	0.52439	0.59198	0.53264	0.45789	0.40883	0.37257	0.34399	0.59822	0.53959	0.46633	0.41872				
2.00	y <sub>1</sub>	1.28348	1.38951	1.48489	1.57269	1.14389	1.26191	1.46018	1.62949	1.78062	1.91875	1.14408	1.26212	1.46036	1.62965				
	y <sub>2</sub>	0.03654	0.03654	0.03655	0.03655	0.14557	0.14660	0.14896	0.15182	0.15551	0.16074	0.06357	0.06364	0.06378	0.06392				
	y <sub>3</sub>	0.47680	0.44036	0.41203	0.38898	0.42464	0.38221	0.32510	0.28582	0.25535	0.22927	0.43479	0.39371	0.33953	0.30359				
3.00	y <sub>1</sub>	1.26216	1.36602	1.46040	1.54778	1.13440	1.24876	1.44442	1.61314			1.13451	1.24889	1.44454	1.61324				
	y <sub>2</sub>	0.03655	0.03656	0.03656	0.03657	0.14827	0.15025	0.15541	0.16431			0.06374	0.06385	0.06406	0.06428				
	y <sub>3</sub>	0.39575	0.36560	0.34191	0.32255	0.34325	0.30772	0.25719	0.21782			0.35705	0.32381	0.27902	0.24897				
4.00	y <sub>1</sub>	1.25216	1.35474	1.44849	1.53557	1.13000	1.24247	1.43671				1.13008	1.24257	1.43680	1.60512				
	y <sub>2</sub>	0.03656	0.03657	0.03658	0.03659	0.15168	0.15531	0.16959				0.06392	0.06406	0.06436	0.06468				
	y <sub>3</sub>	0.34535	0.31912	0.29840	0.28141	0.29172	0.25911	0.20521				0.30958	0.28091	0.24181	0.21539				
5.00	y <sub>1</sub>	1.24637	1.34813	1.44146	1.52832	1.12746	1.23879					1.12752	1.23887	1.43220	1.60028				
	y <sub>2</sub>	0.03658	0.03659	0.03660	0.03661	0.15629	0.16385					0.06410	0.06428	0.06468	0.06510				
	y <sub>3</sub>	0.31022	0.28672	0.26808	0.25277	0.25379	0.22033					0.27674	0.25113	0.21591	0.19199				



TABLE 9-1 (continued)

$\mu = 0.75$

Lu	$\nu_j$	Ko Pn											
		Lu <sub>p</sub> = 500						Lu <sub>p</sub> = 1 000					
		0.5	1.0	2.0	3.0	4.0	5.0	0.5	1.0	2.0	3.0	4.0	5.0
0.25	$\nu_1$	1.35122	1.52214	1.78457	1.99970	2.18866	2.35999	1.35162	1.52244	1.78477	1.99985	2.18878	2.36009
	$\nu_2$	0.08982	0.08992	0.09014	0.09036	0.09058	0.09081	0.06338	0.06341	0.06349	0.06356	0.06371	0.06371
	$\nu_3$	0.73697	0.65345	0.55803	0.49501	0.45116	0.41735	0.73831	0.65510	0.55816	0.49755	0.45407	0.42061
0.50	$\nu_1$	1.24587	1.41391	1.67992	1.90024	2.09415	2.26991	1.24604	1.41406	1.68003	1.90033	2.09421	2.26996
	$\nu_2$	0.09021	0.09043	0.09088	0.09136	0.09186	0.09238	0.06351	0.06359	0.06374	0.06389	0.06405	0.06421
	$\nu_3$	0.56274	0.49465	0.41424	0.36431	0.32879	0.30162	0.56511	0.49738	0.41765	0.36834	0.33342	0.30683
0.75	$\nu_1$	1.21784	1.38135	1.64616	1.86744	2.06267	2.23977	1.21795	1.38145	1.64624	1.86750	2.06272	2.23980
	$\nu_2$	0.09061	0.09096	0.09168	0.09246	0.09330	0.09421	0.06365	0.06376	0.06400	0.06424	0.06448	0.06474
	$\nu_3$	0.46795	0.41099	0.34215	0.29907	0.26833	0.24472	0.47104	0.41455	0.34660	0.30439	0.27453	0.25182
1.00	$\nu_1$	1.20522	1.36588	1.62957	1.85114	2.04696	2.22469	1.20530	1.36595	1.62963	1.85119	2.04700	2.22472
	$\nu_2$	0.09103	0.09152	0.09255	0.09369	0.09497	0.09643	0.06379	0.06394	0.06426	0.06460	0.06494	0.06531
	$\nu_3$	0.40761	0.35777	0.29653	0.25785	0.23004	0.20846	0.41131	0.36206	0.30197	0.26445	0.23787	0.21764
2.00	$\nu_1$	1.18797	1.34372	1.60509	1.82685	2.02344		1.18800	1.34375	1.60511	1.82687	2.02346	2.20208
	$\nu_2$	0.09290	0.09409	0.09595	0.10090	0.10763		0.06437	0.06471	0.06544	0.06624	0.06714	0.06816
	$\nu_3$	0.28652	0.25012	0.20322	0.17155	0.14521		0.29241	0.25716	0.21290	0.18478	0.16459	0.14897
3.00	$\nu_1$	1.18264	1.33662	1.59704				1.18267	1.33665	1.59706	1.81881	2.01563	2.19454
	$\nu_2$	0.09517	0.09749	0.10476				0.06500	0.06556	0.06682	0.06834	0.07026	0.07287
	$\nu_3$	0.22940	0.19815	0.15433				0.23749	0.20834	0.17107	0.14688	0.12892	0.11417
4.00	$\nu_1$	1.18006	1.33313					1.18008	1.33315	1.59306	1.81490	2.01172	
	$\nu_2$	0.09806	0.10258					0.06652	0.06683	0.06714	0.06759	0.07594	
	$\nu_3$	0.19324	0.16351					0.20397	0.17829	0.14483	0.12220	0.10349	
5.00	$\nu_1$	1.17853	1.33105					1.17855	1.33107	1.59066	1.81239		
	$\nu_2$	1.10207	0.11502					0.06645	0.06762	0.07075	0.07652		
	$\nu_3$	0.16625	0.13063					0.18059	0.15711	0.12566	0.10197		

TABLE 9-2  
Values of  $\sigma_j = \sigma_j(Lu, Lu_p, \epsilon, Ko Pn)$

$\epsilon = 0.25$		Ko Pn									
Lu	$\sigma_j$	$Lu_p = 100$									
		0.5	1.0	2.0	3.0	4.0	5.0				
0.25	$\sigma_1$	-0.17190 · 10 <sup>-2</sup>	-0.17597 · 10 <sup>-2</sup>	-0.18263 · 10 <sup>-2</sup>	-0.18793 · 10 <sup>-2</sup>	-0.19232 · 10 <sup>-2</sup>	-0.19693 · 10 <sup>-2</sup>				
	$\sigma_2$	0.73628	0.73586	0.73502	0.73418	0.73334	0.73250				
	$\sigma_3$	0.15432 · 10 <sup>-3</sup>	0.30167 · 10 <sup>-3</sup>	0.58201 · 10 <sup>-3</sup>	0.84933 · 10 <sup>-3</sup>	0.11079 · 10 <sup>-2</sup>	0.13602 · 10 <sup>-2</sup>				
0.50	$\sigma_1$	-0.11033 · 10 <sup>-2</sup>	-0.12550 · 10 <sup>-2</sup>	-0.14488 · 10 <sup>-2</sup>	-0.15770 · 10 <sup>-2</sup>	-0.16719 · 10 <sup>-2</sup>	-0.17462 · 10 <sup>-2</sup>				
	$\sigma_2$	0.73254	0.73170	0.73002	0.72834	0.72665	0.72496				
	$\sigma_3$	0.48332 · 10 <sup>-3</sup>	0.86076 · 10 <sup>-3</sup>	0.14774 · 10 <sup>-3</sup>	0.20405 · 10 <sup>-2</sup>	0.25723 · 10 <sup>-2</sup>	0.30858 · 10 <sup>-2</sup>				
0.75	$\sigma_1$	-0.74487 · 10 <sup>-3</sup>	-0.98077 · 10 <sup>-3</sup>	-0.12556 · 10 <sup>-2</sup>	-0.14275 · 10 <sup>-2</sup>	-0.15507 · 10 <sup>-2</sup>	-0.16451 · 10 <sup>-2</sup>				
	$\sigma_2$	0.72880	0.72754	0.72500	0.72247	0.71993	0.71738				
	$\sigma_3$	0.10793 · 10 <sup>-2</sup>	0.16423 · 10 <sup>-2</sup>	0.25747 · 10 <sup>-2</sup>	0.34088 · 10 <sup>-2</sup>	0.41989 · 10 <sup>-2</sup>	0.49650 · 10 <sup>-2</sup>				
1.00	$\sigma_1$	-0.57503 · 10 <sup>-3</sup>	-0.83799 · 10 <sup>-3</sup>	-0.11492 · 10 <sup>-2</sup>	-0.13438 · 10 <sup>-2</sup>	-0.14824 · 10 <sup>-2</sup>	-0.15881 · 10 <sup>-2</sup>				
	$\sigma_2$	0.72505	0.72336	0.71997	0.716576	0.71317	0.70976				
	$\sigma_3$	0.18738 · 10 <sup>-2</sup>	0.25777 · 10 <sup>-2</sup>	0.37769 · 10 <sup>-2</sup>	0.48679 · 10 <sup>-2</sup>	0.59116 · 10 <sup>-2</sup>	0.69311 · 10 <sup>-2</sup>				
2.00	$\sigma_1$	-0.38964 · 10 <sup>-2</sup>	-0.64773 · 10 <sup>-2</sup>	-0.98810 · 10 <sup>-2</sup>	-0.12112 · 10 <sup>-1</sup>	-0.13722 · 10 <sup>-1</sup>	-0.14951 · 10 <sup>-1</sup>				
	$\sigma_2$	0.70994	0.70652	0.69965	0.69274	0.68579	0.67880				
	$\sigma_3$	0.56485 · 10 <sup>-2</sup>	0.68286 · 10 <sup>-2</sup>	0.90406 · 10 <sup>-2</sup>	0.11173 · 10 <sup>-1</sup>	0.13283 · 10 <sup>-1</sup>	0.15395 · 10 <sup>-1</sup>				
3.00	$\sigma_1$	-0.34641 · 10 <sup>-2</sup>	-0.59445 · 10 <sup>-2</sup>	-0.93637 · 10 <sup>-2</sup>	-0.11663 · 10 <sup>-1</sup>	-0.13339 · 10 <sup>-1</sup>	-0.14624 · 10 <sup>-1</sup>				
	$\sigma_2$	0.69465	0.68946	0.67899	0.66843	0.65776	0.64699				
	$\sigma_3$	0.97399 · 10 <sup>-2</sup>	0.11437 · 10 <sup>-1</sup>	0.14745 · 10 <sup>-1</sup>	0.18038 · 10 <sup>-1</sup>	0.213693 · 10 <sup>-1</sup>	0.24770 · 10 <sup>-1</sup>				
4.00	$\sigma_1$	-0.32758 · 10 <sup>-2</sup>	-0.56987 · 10 <sup>-2</sup>	-0.91117 · 10 <sup>-2</sup>	-0.11439 · 10 <sup>-1</sup>	-0.13146 · 10 <sup>-1</sup>	-0.14458 · 10 <sup>-1</sup>				
	$\sigma_2$	0.67918	0.67215	0.65796	0.64358	0.62898	0.61415				
	$\sigma_3$	0.14046 · 10 <sup>-1</sup>	0.16317 · 10 <sup>-1</sup>	0.20850 · 10 <sup>-1</sup>	0.25468 · 10 <sup>-1</sup>	0.30234 · 10 <sup>-1</sup>	0.35193 · 10 <sup>-1</sup>				
5.00	$\sigma_1$	-0.31708 · 10 <sup>-2</sup>	-0.55577 · 10 <sup>-2</sup>	-0.89631 · 10 <sup>-2</sup>	-0.11306 · 10 <sup>-1</sup>	-0.13030 · 10 <sup>-1</sup>	-0.14358 · 10 <sup>-1</sup>				
	$\sigma_2$	0.66350	0.65458	0.63651	0.61809	0.59928	0.58003				
	$\sigma_3$	0.18567 · 10 <sup>-1</sup>	0.21475 · 10 <sup>-1</sup>	0.27388 · 10 <sup>-1</sup>	0.33539 · 10 <sup>-1</sup>	0.40019 · 10 <sup>-1</sup>	0.46905 · 10 <sup>-1</sup>				

TABLE 9-2 (continued)

$\epsilon = 0.25$		Ko Pn						
Lu		$Lu_p = 500$						
	$\sigma_j$	0.5	1.0	2.0	3.0	4.0	5.0	
0.25	$\sigma_1$	-0.34306·10 <sup>-3</sup>	-0.35120·10 <sup>-3</sup>	-0.36453·10 <sup>-3</sup>	-0.375172·10 <sup>-3</sup>	-0.38396·10 <sup>-3</sup>	-0.39141·10 <sup>-3</sup>	
	$\sigma_2$	0.74725	0.74717	0.74700	0.74683	0.74667	0.74650	
	$\sigma_3$	0.30480·10 <sup>-4</sup>	0.59552·10 <sup>-4</sup>	0.11477·10 <sup>-3</sup>	0.16730·10 <sup>-3</sup>	0.21801·10 <sup>-3</sup>	0.26740·10 <sup>-3</sup>	
0.50	$\sigma_1$	-0.21996·10 <sup>-3</sup>	-0.25020·10 <sup>-3</sup>	-0.28889·10 <sup>-3</sup>	-0.31455·10 <sup>-3</sup>	-0.33354·10 <sup>-3</sup>	-0.34844·10 <sup>-3</sup>	
	$\sigma_2$	0.74650	0.74633	0.74600	0.74567	0.74533	0.74500	
	$\sigma_3$	0.95160·10 <sup>-4</sup>	0.16735·10 <sup>-3</sup>	0.29001·10 <sup>-3</sup>	0.39971·10 <sup>-3</sup>	0.50283·10 <sup>-3</sup>	0.60193·10 <sup>-3</sup>	
0.75	$\sigma_1$	-0.14841·10 <sup>-3</sup>	-0.19538·10 <sup>-3</sup>	-0.25022·10 <sup>-3</sup>	-0.28459·10 <sup>-3</sup>	-0.30924·10 <sup>-3</sup>	-0.32815·10 <sup>-3</sup>	
	$\sigma_2$	0.74575	0.74550	0.74500	0.74450	0.74400	0.74349	
	$\sigma_3$	0.21177·10 <sup>-3</sup>	0.32181·10 <sup>-3</sup>	0.50291·10 <sup>-3</sup>	0.66372·10 <sup>-3</sup>	0.81497·10 <sup>-3</sup>	0.96065·10 <sup>-3</sup>	
1.00	$\sigma_1$	-0.11448·10 <sup>-3</sup>	-0.16685·10 <sup>-3</sup>	-0.22894·10 <sup>-3</sup>	-0.26782·10 <sup>-3</sup>	-0.29556·10 <sup>-3</sup>	-0.31672·10 <sup>-3</sup>	
	$\sigma_2$	0.745002	0.74467	0.74400	0.74333	0.74266	0.74199	
	$\sigma_3$	0.36639·10 <sup>-3</sup>	0.50302·10 <sup>-3</sup>	0.73388·10 <sup>-3</sup>	0.94183·10 <sup>-3</sup>	0.11389·10 <sup>-2</sup>	0.13298·10 <sup>-2</sup>	
2.00	$\sigma_1$	-0.77465·10 <sup>-4</sup>	-0.12884·10 <sup>-3</sup>	-0.19671·10 <sup>-3</sup>	-0.24128·10 <sup>-3</sup>	-0.27347·10 <sup>-3</sup>	-0.29808·10 <sup>-3</sup>	
	$\sigma_2$	0.741998	0.74133	0.73999	0.73864	0.73730	0.73596	
	$\sigma_3$	0.10873·10 <sup>-3</sup>	0.13087·10 <sup>-3</sup>	0.17175·10 <sup>-3</sup>	0.21042·10 <sup>-3</sup>	0.24798·10 <sup>-3</sup>	0.28490·10 <sup>-3</sup>	
3.00	$\sigma_1$	-0.68839·10 <sup>-4</sup>	-0.11820·10 <sup>-3</sup>	-0.18636·10 <sup>-3</sup>	-0.23229·10 <sup>-3</sup>	-0.26581·10 <sup>-3</sup>	-0.29153·10 <sup>-3</sup>	
	$\sigma_2$	0.73899	0.73798	0.73596	0.73394	0.73192	0.72990	
	$\sigma_3$	0.18429·10 <sup>-3</sup>	0.21494·10 <sup>-3</sup>	0.27341·10 <sup>-3</sup>	0.32993·10 <sup>-3</sup>	0.38551·10 <sup>-3</sup>	0.44059·10 <sup>-3</sup>	
4.00	$\sigma_1$	-0.65084·10 <sup>-4</sup>	-0.11330·10 <sup>-3</sup>	-0.18133·10 <sup>-3</sup>	-0.22781·10 <sup>-3</sup>	-0.26193·10 <sup>-3</sup>	-0.28819·10 <sup>-3</sup>	
	$\sigma_2$	0.73597	0.73462	0.73193	0.72923	0.72652	0.72381	
	$\sigma_3$	0.26096·10 <sup>-3</sup>	0.30037·10 <sup>-3</sup>	0.37673·10 <sup>-3</sup>	0.451461·10 <sup>-3</sup>	0.52548·10 <sup>-3</sup>	0.59924·10 <sup>-3</sup>	
5.00	$\sigma_1$	-0.62991·10 <sup>-4</sup>	-0.110483·10 <sup>-3</sup>	-0.17636·10 <sup>-3</sup>	-0.22513·10 <sup>-3</sup>	-0.25960·10 <sup>-3</sup>	-0.28617·10 <sup>-3</sup>	
	$\sigma_2$	0.73295	0.73126	0.72788	0.72450	0.72110	0.71770	
	$\sigma_3$	0.33843·10 <sup>-3</sup>	0.38679·10 <sup>-3</sup>	0.48142·10 <sup>-3</sup>	0.57477·10 <sup>-3</sup>	0.66772·10 <sup>-3</sup>	0.76075·10 <sup>-3</sup>	

TABLE 9-2 (continued)

$\epsilon = 0.25$		Ko Pn									
Lu	$\sigma_j$	$Lu_p = 1.000$									
		0.5	1.0	2.0	3.0	4.0	5.0				
0.25	$\sigma_1$	-0.17148·10 <sup>-8</sup>	-0.17555·10 <sup>-8</sup>	-0.18222·10 <sup>-8</sup>	-0.18754·10 <sup>-8</sup>	-0.19194·10 <sup>-8</sup>	-0.19566·10 <sup>-8</sup>				
	$\sigma_2$	0.74862	0.74858	0.74850	0.74842	0.74833	0.74825				
	$\sigma_3$	0.15219·10 <sup>-6</sup>	0.29730·10 <sup>-6</sup>	0.57287·10 <sup>-6</sup>	0.83500·10 <sup>-6</sup>	0.10879·10 <sup>-5</sup>	0.133419·10 <sup>-5</sup>				
0.50	$\sigma_1$	-0.10994·10 <sup>-8</sup>	-0.12505·10 <sup>-8</sup>	-0.14439·10 <sup>-8</sup>	-0.15722·10 <sup>-8</sup>	-0.16672·10 <sup>-8</sup>	-0.17417·10 <sup>-8</sup>				
	$\sigma_2$	0.74825	0.74817	0.74800	0.74783	0.74767	0.74750				
	$\sigma_3$	0.47489·10 <sup>-6</sup>	0.83507·10 <sup>-6</sup>	0.14468·10 <sup>-5</sup>	0.19935·10 <sup>-5</sup>	0.25071·10 <sup>-5</sup>	0.30005·10 <sup>-5</sup>				
0.75	$\sigma_1$	-0.74169·10 <sup>-6</sup>	-0.97644·10 <sup>-6</sup>	-0.12505·10 <sup>-5</sup>	-0.14224·10 <sup>-5</sup>	-0.154564·10 <sup>-5</sup>	-0.16402·10 <sup>-5</sup>				
	$\sigma_2$	0.74787	0.74775	0.74750	0.74725	0.74700	0.74675				
	$\sigma_3$	0.10563·10 <sup>-8</sup>	0.16050·10 <sup>-8</sup>	0.25073·10 <sup>-8</sup>	0.33077·10 <sup>-8</sup>	0.40600·10 <sup>-8</sup>	0.47840·10 <sup>-8</sup>				
1.00	$\sigma_1$	-0.57210·10 <sup>-6</sup>	-0.83379·10 <sup>-6</sup>	-0.11441·10 <sup>-5</sup>	-0.13385·10 <sup>-5</sup>	-0.14772·10 <sup>-5</sup>	-0.15830·10 <sup>-5</sup>				
	$\sigma_2$	0.74750	0.74733	0.74700	0.74666	0.74633	0.74600				
	$\sigma_3$	0.18269·10 <sup>-8</sup>	0.25076·10 <sup>-8</sup>	0.36565·10 <sup>-8</sup>	0.46902·10 <sup>-8</sup>	0.56690·10 <sup>-8</sup>	0.66157·10 <sup>-8</sup>				
2.00	$\sigma_1$	-0.38703·10 <sup>-6</sup>	-0.64377·10 <sup>-6</sup>	-0.98298·10 <sup>-6</sup>	-0.12058·10 <sup>-5</sup>	-0.13668·10 <sup>-5</sup>	-0.14898·10 <sup>-5</sup>				
	$\sigma_2$	0.74600	0.74566	0.74500	0.74433	0.74365	0.74299				
	$\sigma_3$	0.54117·10 <sup>-8</sup>	0.65100·10 <sup>-8</sup>	0.85346·10 <sup>-8</sup>	0.10446·10 <sup>-7</sup>	0.12298·10 <sup>-7</sup>	0.14116·10 <sup>-7</sup>				
3.00	$\sigma_1$	-0.34391·10 <sup>-6</sup>	-0.590592·10 <sup>-6</sup>	-0.93126·10 <sup>-6</sup>	-0.11608·10 <sup>-5</sup>	-0.13284·10 <sup>-5</sup>	-0.14570·10 <sup>-5</sup>				
	$\sigma_2$	0.74450	0.74399	0.74299	0.74199	0.74098	0.73997				
	$\sigma_3$	0.91537·10 <sup>-8</sup>	0.10668·10 <sup>-7</sup>	0.13549·10 <sup>-7</sup>	0.16526·10 <sup>-7</sup>	0.19049·10 <sup>-7</sup>	0.21739·10 <sup>-7</sup>				
4.00	$\sigma_1$	-0.32515·10 <sup>-6</sup>	-0.56606·10 <sup>-6</sup>	-0.90607·10 <sup>-6</sup>	-0.11384·10 <sup>-5</sup>	-0.13090·10 <sup>-5</sup>	-0.14403·10 <sup>-5</sup>				
	$\sigma_2$	0.74299	0.74232	0.74098	0.73964	0.73830	0.73695				
	$\sigma_3$	0.12935·10 <sup>-7</sup>	0.14874·10 <sup>-7</sup>	0.18618·10 <sup>-7</sup>	0.22267·10 <sup>-7</sup>	0.25867·10 <sup>-7</sup>	0.29440·10 <sup>-7</sup>				
5.00	$\sigma_1$	-0.31468·10 <sup>-6</sup>	-0.55200·10 <sup>-6</sup>	-0.89120·10 <sup>-6</sup>	-0.11250·10 <sup>-5</sup>	-0.12973·10 <sup>-5</sup>	-0.14302·10 <sup>-5</sup>				
	$\sigma_2$	0.74149	0.74065	0.73897	0.73729	0.73561	0.73393				
	$\sigma_3$	0.16741·10 <sup>-7</sup>	0.19109·10 <sup>-7</sup>	0.23725·10 <sup>-7</sup>	0.28255·10 <sup>-7</sup>	0.32744·10 <sup>-7</sup>	0.37212·10 <sup>-7</sup>				



TABLE 9-2 (continued)

$\epsilon = 0.60$		Ko Pn							
Lu		Lu p = 100							
	$\sigma_j$	0.5	1.0	2.0	3.0	4.0	5.0		
0.25	$\sigma_1$	-0.2943 $\cdot 10^{-2}$	-0.32095 $\cdot 10^{-2}$	-0.35438 $\cdot 10^{-2}$	-0.37581 $\cdot 10^{-2}$	-0.39115 $\cdot 10^{-2}$	-0.40285 $\cdot 10^{-2}$		
	$\sigma_2$	0.48375	0.48247	0.47989	0.47731	0.47470	0.47209		
	$\sigma_3$	0.10970 $\cdot 10^{-2}$	0.20181 $\cdot 10^{-2}$	0.36751 $\cdot 10^{-2}$	0.52265 $\cdot 10^{-2}$	0.67319 $\cdot 10^{-2}$	0.82159 $\cdot 10^{-2}$		
0.50	$\sigma_1$	-0.19568 $\cdot 10^{-2}$	-0.25084 $\cdot 10^{-2}$	-0.31000 $\cdot 10^{-2}$	-0.34403 $\cdot 10^{-2}$	-0.36693 $\cdot 10^{-2}$	-0.38363 $\cdot 10^{-2}$		
	$\sigma_2$	0.47741	0.47481	0.46956	0.46425	0.45887	0.45342		
	$\sigma_3$	0.33451 $\cdot 10^{-2}$	0.52477 $\cdot 10^{-2}$	0.85873 $\cdot 10^{-2}$	0.11739 $\cdot 10^{-1}$	0.14648 $\cdot 10^{-1}$	0.17966 $\cdot 10^{-1}$		
0.75	$\sigma_1$	-0.15814 $\cdot 10^{-2}$	-0.22242 $\cdot 10^{-2}$	-0.29163 $\cdot 10^{-2}$	-0.33096 $\cdot 10^{-2}$	-0.35707 $\cdot 10^{-2}$	-0.37590 $\cdot 10^{-2}$		
	$\sigma_2$	0.47099	0.46702	0.45898	0.45078	0.44241	0.43384		
	$\sigma_3$	0.62935 $\cdot 10^{-2}$	0.90254 $\cdot 10^{-2}$	0.14008 $\cdot 10^{-1}$	0.18852 $\cdot 10^{-1}$	0.23738 $\cdot 10^{-1}$	0.28743 $\cdot 10^{-1}$		
1.00	$\sigma_1$	-0.14105 $\cdot 10^{-2}$	-0.20807 $\cdot 10^{-2}$	-0.28188 $\cdot 10^{-2}$	-0.32395 $\cdot 10^{-2}$	-0.35178 $\cdot 10^{-2}$	-0.37176 $\cdot 10^{-2}$		
	$\sigma_2$	0.46447	0.45909	0.44813	0.43684	0.42519	0.41322		
	$\sigma_3$	0.95401 $\cdot 10^{-2}$	0.13085 $\cdot 10^{-1}$	0.19791 $\cdot 10^{-1}$	0.26499 $\cdot 10^{-1}$	0.33428 $\cdot 10^{-1}$	0.40695 $\cdot 10^{-1}$		
2.00	$\sigma_1$	-0.11879 $\cdot 10^{-2}$	-0.18719 $\cdot 10^{-2}$	-0.26674 $\cdot 10^{-2}$	-0.31287 $\cdot 10^{-2}$	-0.34338 $\cdot 10^{-2}$	-0.36518 $\cdot 10^{-2}$		
	$\sigma_2$	0.43732	0.42570	0.401139	0.37428	0.34397	0.30756		
	$\sigma_3$	0.24062 $\cdot 10^{-1}$	0.31373 $\cdot 10^{-1}$	0.46729 $\cdot 10^{-1}$	0.64046 $\cdot 10^{-1}$	0.84664 $\cdot 10^{-1}$	0.11129		
3.00	$\sigma_1$	-0.11233 $\cdot 10^{-2}$	-0.18052 $\cdot 10^{-2}$	-0.26160 $\cdot 10^{-2}$	-0.30904 $\cdot 10^{-2}$	-3.90642	-7.80976		
	$\sigma_2$	0.40801	0.38866	0.34476	0.28619	-5.75895	6.83892		
	$\sigma_3$	0.40910 $\cdot 10^{-1}$	0.53448 $\cdot 10^{-1}$	0.83158 $\cdot 10^{-1}$	0.12719	29.21763	48.29151		
4.00	$\sigma_1$	-0.10928 $\cdot 10^{-2}$	-0.17726 $\cdot 10^{-2}$	-0.25902 $\cdot 10^{-2}$	-4.13085	-8.25910	-12.38736		
	$\sigma_2$	0.37558	0.34554	0.25883	-6.37634	-7.20213	7.20213		
	$\sigma_3$	0.60911 $\cdot 10^{-1}$	0.81632 $\cdot 10^{-1}$	0.14916	19.94338	39.24560	64.77004		
5.00	$\sigma_1$	-0.10750 $\cdot 10^{-2}$	-0.17533 $\cdot 10^{-2}$	-3.89827	-7.67479	-11.51131	-15.34784		
	$\sigma_2$	0.33810	0.28883	-4.69241	-7.92896	-9.81789	7.67756		
	$\sigma_3$	0.85976 $\cdot 10^{-1}$	0.12342	8.65505	25.07450	49.27172	81.24672		

TABLE 9-2 (continued)

$s = 0.50$		Ko Pn									
Lu		$L_{4.5} = 500$									
		0.5	1.0	2.0	3.0	4.0	5.0				
0.25	$\sigma_1$	$-0.58793 \cdot 10^{-3}$	$-0.64069 \cdot 10^{-3}$	$-0.70743 \cdot 10^{-3}$	$-0.75032 \cdot 10^{-3}$	$-0.78108 \cdot 10^{-3}$	$-0.80453 \cdot 10^{-3}$				
	$\sigma_2$	$0.49675$	$0.49660$	$0.49599$	$0.49549$	$0.49489$	$0.49448$				
	$\sigma_3$	$0.21400 \cdot 10^{-3}$	$0.39296 \cdot 10^{-3}$	$0.71249 \cdot 10^{-3}$	$0.10087 \cdot 10^{-3}$	$0.12932 \cdot 10^{-3}$	$0.157104 \cdot 10^{-3}$				
0.50	$\sigma_1$	$-0.39058 \cdot 10^{-3}$	$-0.50033 \cdot 10^{-3}$	$-0.61842 \cdot 10^{-3}$	$-0.68652 \cdot 10^{-3}$	$-0.73241 \cdot 10^{-3}$	$-0.76590 \cdot 10^{-3}$				
	$\sigma_2$	$0.49550$	$0.49499$	$0.49398$	$0.49297$	$0.49196$	$0.49094$				
	$\sigma_3$	$0.64589 \cdot 10^{-3}$	$0.10094 \cdot 10^{-3}$	$0.16367 \cdot 10^{-3}$	$0.22161 \cdot 10^{-3}$	$0.27753 \cdot 10^{-3}$	$0.33244 \cdot 10^{-3}$				
0.75	$\sigma_1$	$-0.31533 \cdot 10^{-3}$	$-0.44336 \cdot 10^{-3}$	$-0.58156 \cdot 10^{-3}$	$-0.66026 \cdot 10^{-3}$	$-0.71260 \cdot 10^{-3}$	$-0.75036 \cdot 10^{-3}$				
	$\sigma_2$	$0.49424$	$0.49348$	$0.49196$	$0.49044$	$0.48891$	$0.48738$				
	$\sigma_3$	$0.12031 \cdot 10^{-3}$	$0.17139 \cdot 10^{-3}$	$0.26214 \cdot 10^{-3}$	$0.34742 \cdot 10^{-3}$	$0.43054 \cdot 10^{-3}$	$0.51271 \cdot 10^{-3}$				
1.00	$\sigma_1$	$-0.28104 \cdot 10^{-3}$	$-0.41460 \cdot 10^{-3}$	$-0.56199 \cdot 10^{-3}$	$-0.64618 \cdot 10^{-3}$	$-0.70196 \cdot 10^{-3}$	$-0.74203 \cdot 10^{-3}$				
	$\sigma_2$	$0.49298$	$0.49197$	$0.48993$	$0.48789$	$0.48584$	$0.48378$				
	$\sigma_3$	$0.18044 \cdot 10^{-3}$	$0.24514 \cdot 10^{-3}$	$0.36319 \cdot 10^{-3}$	$0.47579 \cdot 10^{-3}$	$0.58644 \cdot 10^{-3}$	$0.69642 \cdot 10^{-3}$				
2.00	$\sigma_1$	$-0.23637 \cdot 10^{-3}$	$-0.37270 \cdot 10^{-3}$	$-0.53160 \cdot 10^{-3}$	$-0.62392 \cdot 10^{-3}$	$-0.68507 \cdot 10^{-3}$	$-0.72879 \cdot 10^{-3}$				
	$\sigma_2$	$0.48791$	$0.48586$	$0.48173$	$0.47756$	$0.47336$	$0.46911$				
	$\sigma_3$	$0.43355 \cdot 10^{-3}$	$0.55225 \cdot 10^{-3}$	$0.78099 \cdot 10^{-3}$	$0.10068 \cdot 10^{-3}$	$0.12335 \cdot 10^{-3}$	$0.14625 \cdot 10^{-3}$				
3.00	$\sigma_1$	$-0.22342 \cdot 10^{-3}$	$-0.35932 \cdot 10^{-3}$	$-0.52127 \cdot 10^{-3}$	$-0.61623 \cdot 10^{-3}$	$-0.67921 \cdot 10^{-3}$	$-0.72418 \cdot 10^{-3}$				
	$\sigma_2$	$0.48278$	$0.47967$	$0.47337$	$0.46699$	$0.46051$	$0.45392$				
	$\sigma_3$	$0.68533 \cdot 10^{-3}$	$0.87029 \cdot 10^{-3}$	$0.12158 \cdot 10^{-3}$	$0.15637 \cdot 10^{-3}$	$0.19183 \cdot 10^{-3}$	$0.22815 \cdot 10^{-3}$				
4.00	$\sigma_1$	$-0.21730 \cdot 10^{-3}$	$-0.35278 \cdot 10^{-3}$	$-0.51609 \cdot 10^{-3}$	$-0.61233 \cdot 10^{-3}$	$-0.67623 \cdot 10^{-3}$	$-0.72184 \cdot 10^{-3}$				
	$\sigma_2$	$0.47760$	$0.47339$	$0.46486$	$0.45615$	$0.44724$	$0.43812$				
	$\sigma_3$	$0.96355 \cdot 10^{-3}$	$0.11976 \cdot 10^{-3}$	$0.16573 \cdot 10^{-3}$	$9.21481 \cdot 10^{-4}$	$0.26452 \cdot 10^{-3}$	$0.31620 \cdot 10^{-3}$				
5.00	$\sigma_1$	$-0.21374 \cdot 10^{-3}$	$-0.34890 \cdot 10^{-3}$	$-0.51297 \cdot 10^{-3}$	$-0.60997 \cdot 10^{-3}$	$-0.67442 \cdot 10^{-3}$	$-0.72042 \cdot 10^{-3}$				
	$\sigma_2$	$0.47235$	$0.46703$	$0.45516$	$0.44500$	$0.43549$	$0.42558$				
	$\sigma_3$	$0.12381 \cdot 10^{-3}$	$0.15342 \cdot 10^{-3}$	$0.21367 \cdot 10^{-3}$	$0.27630 \cdot 10^{-3}$	$0.34205 \cdot 10^{-3}$	$0.41155 \cdot 10^{-3}$				

TABLE 9-2 (continued)

* = 0.60		Ko Pn					
Lu		Lu <sub>p</sub> = 1 000					
	$\sigma_f$	0.5	1.0	2.0	3.0	4.0	5.0
0.25	$\sigma_1$	-0.29391 · 10 <sup>-3</sup>	-0.32027 · 10 <sup>-3</sup>	-0.35363 · 10 <sup>-3</sup>	-0.37508 · 10 <sup>-3</sup>	-0.39046 · 10 <sup>-3</sup>	-0.40219 · 10 <sup>-3</sup>
	$\sigma_2$	0.49837	0.49825	0.49800	0.49775	0.49750	0.49725
	$\sigma_3$	0.10667 · 10 <sup>-3</sup>	0.19584 · 10 <sup>-3</sup>	0.354898 · 10 <sup>-3</sup>	0.50216 · 10 <sup>-3</sup>	0.64350 · 10 <sup>-3</sup>	0.78130 · 10 <sup>-3</sup>
0.50	$\sigma_1$	-0.19524 · 10 <sup>-3</sup>	-0.25008 · 10 <sup>-3</sup>	-0.30911 · 10 <sup>-3</sup>	-0.34316 · 10 <sup>-3</sup>	-0.36611 · 10 <sup>-3</sup>	-0.38287 · 10 <sup>-3</sup>
	$\sigma_2$	0.49775	0.49750	0.49699	0.49649	0.49599	0.49549
	$\sigma_3$	0.32156 · 10 <sup>-3</sup>	0.50235 · 10 <sup>-3</sup>	0.813658 · 10 <sup>-3</sup>	0.11005 · 10 <sup>-2</sup>	0.13767 · 10 <sup>-2</sup>	0.16473 · 10 <sup>-2</sup>
0.75	$\sigma_1$	-0.15761 · 10 <sup>-3</sup>	-0.22159 · 10 <sup>-3</sup>	-0.29067 · 10 <sup>-3</sup>	-0.33003 · 10 <sup>-3</sup>	-0.35620 · 10 <sup>-3</sup>	-0.37509 · 10 <sup>-3</sup>
	$\sigma_2$	0.49712	0.49674	0.49599	0.49523	0.49448	0.49372
	$\sigma_3$	0.59827 · 10 <sup>-3</sup>	0.85170 · 10 <sup>-3</sup>	0.13005 · 10 <sup>-2</sup>	0.17208 · 10 <sup>-2</sup>	0.21291 · 10 <sup>-2</sup>	0.25312 · 10 <sup>-2</sup>
1.00	$\sigma_1$	-0.14045 · 10 <sup>-3</sup>	-0.20720 · 10 <sup>-3</sup>	-0.28088 · 10 <sup>-3</sup>	-0.32298 · 10 <sup>-3</sup>	-0.35088 · 10 <sup>-3</sup>	-0.37092 · 10 <sup>-3</sup>
	$\sigma_2$	0.49649	0.49599	0.49498	0.49397	0.49296	0.49195
	$\sigma_3$	0.89626 · 10 <sup>-3</sup>	0.12163 · 10 <sup>-2</sup>	0.17981 · 10 <sup>-2</sup>	0.23505 · 10 <sup>-2</sup>	0.28907 · 10 <sup>-2</sup>	0.34252 · 10 <sup>-2</sup>
2.00	$\sigma_1$	-0.11811 · 10 <sup>-3</sup>	-0.18624 · 10 <sup>-3</sup>	-0.26568 · 10 <sup>-3</sup>	-0.31185 · 10 <sup>-3</sup>	-0.34243 · 10 <sup>-3</sup>	-0.36430 · 10 <sup>-3</sup>
	$\sigma_2$	0.49398	0.49296	0.49093	0.48889	0.48685	0.48479
	$\sigma_3$	0.21425 · 10 <sup>-3</sup>	0.27230 · 10 <sup>-3</sup>	0.38334 · 10 <sup>-3</sup>	0.49193 · 10 <sup>-3</sup>	0.59984 · 10 <sup>-3</sup>	0.70778 · 10 <sup>-3</sup>
3.00	$\sigma_1$	-0.11163 · 10 <sup>-3</sup>	-0.17955 · 10 <sup>-3</sup>	-0.26051 · 10 <sup>-3</sup>	-0.30799 · 10 <sup>-3</sup>	-0.33950 · 10 <sup>-3</sup>	-0.36199 · 10 <sup>-3</sup>
	$\sigma_2$	0.49145	0.48992	0.48685	0.48376	0.48065	0.47752
	$\sigma_3$	0.34177 · 10 <sup>-3</sup>	0.42629 · 10 <sup>-3</sup>	0.59133 · 10 <sup>-3</sup>	0.75504 · 10 <sup>-3</sup>	0.91924 · 10 <sup>-3</sup>	-0.10847 · 10 <sup>-1</sup>
4.00	$\sigma_1$	-0.10857 · 10 <sup>-3</sup>	-0.17628 · 10 <sup>-3</sup>	-0.25792 · 10 <sup>-3</sup>	-0.30604 · 10 <sup>-3</sup>	-0.33801 · 10 <sup>-3</sup>	-0.36081 · 10 <sup>-3</sup>
	$\sigma_2$	0.48890	0.48685	0.48273	0.47857	0.47437	0.47013
	$\sigma_3$	0.47097 · 10 <sup>-3</sup>	0.58259 · 10 <sup>-3</sup>	0.80316 · 10 <sup>-3</sup>	0.10241 · 10 <sup>-1</sup>	0.12474 · 10 <sup>-1</sup>	0.14737 · 10 <sup>-1</sup>
5.00	$\sigma_1$	-0.10679 · 10 <sup>-3</sup>	-0.17434 · 10 <sup>-3</sup>	-0.25636 · 10 <sup>-3</sup>	-0.30486 · 10 <sup>-3</sup>	-0.33710 · 10 <sup>-3</sup>	-0.36010 · 10 <sup>-3</sup>
	$\sigma_2$	0.48634	0.48377	0.47732	0.47332	0.46799	0.46261
	$\sigma_3$	0.60167 · 10 <sup>-3</sup>	0.74103 · 10 <sup>-3</sup>	0.10188 · 10 <sup>-1</sup>	0.12993 · 10 <sup>-1</sup>	0.15847 · 10 <sup>-1</sup>	0.18758 · 10 <sup>-1</sup>

TABLE 9-2 (continued)

$\sigma = 0.75$		Ko Pr							
Lu		Lu <sub>p</sub> = 190							
	$\sigma_f$	0.5	1.0	2.0	3.0	4.0	5.0		
0.25	$\sigma_1$	-0.33915 · 10 <sup>-2</sup>	-0.42712 · 10 <sup>-2</sup>	-0.51589 · 10 <sup>-2</sup>	-0.56363 · 10 <sup>-2</sup>	-0.59454 · 10 <sup>-2</sup>	-0.61636 · 10 <sup>-2</sup>		
	$\sigma_2$	0.22659	0.22126	0.21215	0.20228	0.19140	0.17902		
	$\sigma_3$	0.68926 · 10 <sup>-2</sup>	0.11160 · 10 <sup>-1</sup>	0.19280 · 10 <sup>-1</sup>	0.27752 · 10 <sup>-1</sup>	0.37074 · 10 <sup>-1</sup>	0.47792 · 10 <sup>-1</sup>		
0.50	$\sigma_1$	-0.26745 · 10 <sup>-2</sup>	-0.37607 · 10 <sup>-2</sup>	-0.48569 · 10 <sup>-2</sup>	-0.54366 · 10 <sup>-2</sup>	-1.35859	-2.71175		
	$\sigma_2$	0.20953	0.19344	0.17358	0.13640	31.11945	265.07462		
	$\sigma_3$	0.18821 · 10 <sup>-1</sup>	0.28124 · 10 <sup>-1</sup>	0.49527 · 10 <sup>-1</sup>	0.85336 · 10 <sup>-1</sup>	3.01084	5.32625		
0.75	$\sigma_1$	-0.24526 · 10 <sup>-2</sup>	-0.35837 · 10 <sup>-2</sup>	-1.07567	-2.14776	-3.21985	-4.29193		
	$\sigma_2$	0.19093	0.17157	-1.25520	-0.11535	32.50295	295.49006		
	$\sigma_3$	0.33784 · 10 <sup>-1</sup>	0.51458 · 10 <sup>-1</sup>	0.50302	2.18447	4.80343	8.35989		
1.00	$\sigma_1$	-0.23477 · 10 <sup>-2</sup>	-0.54651	-1.63483	-2.72316	-3.81148	-4.89981		
	$\sigma_2$	0.16758	-0.66894	-1.63978	-0.62014	36.00043	339.22972		
	$\sigma_3$	0.53615 · 10 <sup>-1</sup>	-0.30110	0.74640	3.04390	6.59140	11.38890		
2.00	$\sigma_1$	0.10808	-0.98024	-3.15689	-5.33354	-7.51019	-9.68684		
	$\sigma_2$	-0.76184	-1.36474	-3.18821	-2.28938	54.13478	541.16919		
	$\sigma_3$	-0.74664	-0.55331	1.70835	6.47002	13.73169	23.49335		
3.00	$\sigma_1$	0.21851	-1.41398	-4.67895	-7.94392	-11.20890	-14.47387		
	$\sigma_2$	-1.18521	-2.06559	-4.74134	-3.84627	73.67698	752.50210		
	$\sigma_3$	-1.14182	-0.81015	2.66568	9.89151	20.86735	35.59318		
4.00	$\sigma_1$	0.32893	-1.84771	-6.20101	-10.55431	-14.90760	-19.26090		
	$\sigma_2$	-1.60974	-2.76767	-6.29574	-5.37708	93.52969	965.94382		
	$\sigma_3$	-1.53815	-1.06815	3.62185	13.31185	28.00185	47.69185		
5.00	$\sigma_1$	0.43936	-2.28145	-7.72307	-13.16469	-18.60631	-24.04793		
	$\sigma_2$	-2.03472	-3.47026	-7.85065	-6.89986	113.50079	1180.19396		
	$\sigma_3$	-1.93494	-1.32661	4.57755	16.73172	35.13589	59.79005		

TABLE 9-2 (continued)

$\epsilon = 0.7\%$		Ko Pn						
Lu	$\sigma_j$	$Lw_p = 500$						
		0.5	1.0	2.0	3.0	4.0	5.0	
0.25	$\sigma_1$	$-0.67919 \cdot 10^{-3}$	$-0.85332 \cdot 10^{-3}$	$-0.10296 \cdot 10^{-2}$	$-0.11254 \cdot 10^{-2}$	$-0.11873 \cdot 10^{-2}$	$-0.12311 \cdot 10^{-2}$	
	$\sigma_2$	0.24523	0.24446	0.24291	0.24134	0.23976	0.23816	
	$\sigma_3$	0.12665 $\cdot 10^{-2}$	0.20224 $\cdot 10^{-2}$	0.33735 $\cdot 10^{-2}$	0.46596 $\cdot 10^{-2}$	0.59277 $\cdot 10^{-2}$	0.71941 $\cdot 10^{-2}$	
0.50	$\sigma_1$	$-0.53431 \cdot 10^{-3}$	$-0.75043 \cdot 10^{-3}$	$-0.96917 \cdot 10^{-3}$	$-0.10852 \cdot 10^{-2}$	$-0.11585 \cdot 10^{-2}$	$-0.12003 \cdot 10^{-2}$	
	$\sigma_2$	0.24240	0.24083	0.23764	0.23438	0.23105	0.22764	
	$\sigma_3$	0.32572 $\cdot 10^{-2}$	0.46686 $\cdot 10^{-2}$	0.73268 $\cdot 10^{-2}$	0.99517 $\cdot 10^{-2}$	0.12609 $\cdot 10^{-1}$	0.15325 $\cdot 10^{-1}$	
0.75	$\sigma_1$	$-0.48929 \cdot 10^{-3}$	$-0.71464 \cdot 10^{-3}$	$-0.94716 \cdot 10^{-3}$	$-0.10705 \cdot 10^{-2}$	$-0.11480 \cdot 10^{-2}$	$-0.12015 \cdot 10^{-2}$	
	$\sigma_2$	0.23953	0.23712	0.23219	0.22708	0.22176	0.21622	
	$\sigma_3$	0.53994 $\cdot 10^{-2}$	0.74686 $\cdot 10^{-2}$	0.11510 $\cdot 10^{-1}$	0.15619 $\cdot 10^{-1}$	0.19885 $\cdot 10^{-1}$	0.24359 $\cdot 10^{-1}$	
1.00	$\sigma_1$	$-0.46799 \cdot 10^{-3}$	$-0.69572 \cdot 10^{-3}$	$-0.93584 \cdot 10^{-3}$	$-0.10629 \cdot 10^{-2}$	$-0.11425 \cdot 10^{-2}$	$-0.11974 \cdot 10^{-2}$	
	$\sigma_2$	0.23660	0.23332	0.22652	0.21935	0.21175	0.20359	
	$\sigma_3$	0.76200 $\cdot 10^{-2}$	0.10381 $\cdot 10^{-1}$	0.15921 $\cdot 10^{-1}$	0.21714 $\cdot 10^{-1}$	0.27900 $\cdot 10^{-1}$	0.34611 $\cdot 10^{-1}$	
2.00	$\sigma_1$	$-0.43775 \cdot 10^{-3}$	$-0.66998 \cdot 10^{-3}$	$-0.91848 \cdot 10^{-3}$	$-0.10512 \cdot 10^{-2}$	$-0.11342 \cdot 10^{-2}$	$-6.14263$	
	$\sigma_2$	0.22425	0.21696	0.20083	0.18148	0.15470	547.23587	
	$\sigma_3$	0.17190 $\cdot 10^{-1}$	0.23218 $\cdot 10^{-1}$	0.36593 $\cdot 10^{-1}$	0.53073 $\cdot 10^{-1}$	0.76939 $\cdot 10^{-1}$	23.44867	
3.00	$\sigma_1$	$-0.42815 \cdot 10^{-3}$	$-0.66114 \cdot 10^{-3}$	$-0.91260 \cdot 10^{-3}$	$-5.73963$	$-11.47835$	$-17.21707$	
	$\sigma_2$	0.21062	0.19800	0.16531	$-3.77256$	74.79127	760.79456	
	$\sigma_3$	0.28070 $\cdot 10^{-1}$	0.38673 $\cdot 10^{-1}$	0.67114 $\cdot 10^{-1}$	9.85237	20.81420	35.52604	
4.00	$\sigma_1$	$-0.42344 \cdot 10^{-3}$	$-0.65573 \cdot 10^{-3}$	$-5.33238$	$-10.66410$	$-15.99582$	$-21.32754$	
	$\sigma_2$	0.19508	0.17415	$-6.29602$	$-5.28500$	94.94503	976.49827	
	$\sigma_3$	0.40856 $\cdot 10^{-1}$	0.59017 $\cdot 10^{-1}$	3.58825	13.25958	27.93092	47.60225	
5.00	$\sigma_1$	$-0.42065 \cdot 10^{-3}$	$-0.65409 \cdot 10^{-3}$	$-6.64455$	$-13.28844$	$-19.93233$	$-26.57622$	
	$\sigma_2$	0.176305	0.131809	$-7.85102$	$-6.78522$	115.21929	1193.02416	
	$\sigma_3$	0.56886 $\cdot 10^{-1}$	0.97864 $\cdot 10^{-1}$	4.53550	16.66533	35.04717	59.67800	

TABLE 9-2 (continued)

$\kappa = 0.75$			Ko Pn									
Lu	$\sigma_f$	$\sigma_j$	$Lu_p = 1000$									
			0.5	1.0	2.0	3.0	4.0	5.0				
0.25	$\sigma_1$	$\sigma_2$	$-0.33965 \cdot 10^{-3}$	$-0.42660 \cdot 10^{-3}$	$-0.51471 \cdot 10^{-3}$	$-0.56261 \cdot 10^{-3}$	$-0.59357 \cdot 10^{-3}$	$-0.61546 \cdot 10^{-3}$				
	$\sigma_2$	$\sigma_3$	0.24762	0.24724	0.24648	0.24571	0.24494	0.24417				
	$\sigma_3$	$\sigma_4$	0.62702 $\cdot 10^{-3}$	0.99996 $\cdot 10^{-3}$	0.16627 $\cdot 10^{-3}$	0.22888 $\cdot 10^{-3}$	0.29017 $\cdot 10^{-3}$	0.36092 $\cdot 10^{-3}$				
0.50	$\sigma_1$	$\sigma_2$	$-0.26712 \cdot 10^{-3}$	$-0.37511 \cdot 10^{-3}$	$-0.48445 \cdot 10^{-3}$	$-0.54246 \cdot 10^{-3}$	$-0.57912 \cdot 10^{-3}$	$-0.60456 \cdot 10^{-3}$				
	$\sigma_2$	$\sigma_3$	0.24623	0.24546	0.24391	0.24235	0.24078	0.23919				
	$\sigma_3$	$\sigma_4$	0.16036 $\cdot 10^{-3}$	0.22910 $\cdot 10^{-3}$	0.35702 $\cdot 10^{-3}$	0.48134 $\cdot 10^{-3}$	0.60510 $\cdot 10^{-3}$	0.72937 $\cdot 10^{-3}$				
0.75	$\sigma_1$	$\sigma_2$	$-0.24457 \cdot 10^{-3}$	$-0.35719 \cdot 10^{-3}$	$-0.47343 \cdot 10^{-3}$	$-0.53510 \cdot 10^{-3}$	$-0.57386 \cdot 10^{-3}$	$-0.60062 \cdot 10^{-3}$				
	$\sigma_2$	$\sigma_3$	0.24482	0.24366	0.24131	0.23892	0.23650	0.23403				
	$\sigma_3$	$\sigma_4$	0.26422 $\cdot 10^{-3}$	0.36355 $\cdot 10^{-3}$	0.55391 $\cdot 10^{-3}$	0.74246 $\cdot 10^{-3}$	0.93254 $\cdot 10^{-3}$	0.11255 $\cdot 10^{-1}$				
1.00	$\sigma_1$	$\sigma_2$	$-0.23390 \cdot 10^{-3}$	$-0.34822 \cdot 10^{-3}$	$-0.46776 \cdot 10^{-3}$	$-0.53130 \cdot 10^{-3}$	$-0.57115 \cdot 10^{-3}$	$-0.59858 \cdot 10^{-3}$				
	$\sigma_2$	$\sigma_3$	0.24340	0.24184	0.23866	0.23541	0.23209	0.22869				
	$\sigma_3$	$\sigma_4$	0.37051 $\cdot 10^{-3}$	0.50096 $\cdot 10^{-3}$	0.75560 $\cdot 10^{-3}$	0.10119 $\cdot 10^{-1}$	0.12730 $\cdot 10^{-1}$	0.15409 $\cdot 10^{-1}$				
2.00	$\sigma_1$	$\sigma_2$	$-0.21875 \cdot 10^{-3}$	$-0.33483 \cdot 10^{-3}$	$-0.45907 \cdot 10^{-3}$	$-0.52544 \cdot 10^{-3}$	$-0.56695 \cdot 10^{-3}$	$-0.59546 \cdot 10^{-3}$				
	$\sigma_2$	$\sigma_3$	0.23760	0.23433	0.22756	0.22042	0.21285	0.20474				
	$\sigma_3$	$\sigma_4$	0.81167 $\cdot 10^{-3}$	0.10754 $\cdot 10^{-1}$	0.16154 $\cdot 10^{-1}$	0.21855 $\cdot 10^{-1}$	0.27967 $\cdot 10^{-1}$	0.34604 $\cdot 10^{-1}$				
3.00	$\sigma_1$	$\sigma_2$	$-0.21394 \cdot 10^{-3}$	$-0.33040 \cdot 10^{-3}$	$-0.45612 \cdot 10^{-3}$	$-0.52344 \cdot 10^{-3}$	$-0.56552 \cdot 10^{-3}$	$-0.59440 \cdot 10^{-3}$				
	$\sigma_2$	$\sigma_3$	0.23157	0.22641	0.21545	0.20336	0.18961	0.17309				
	$\sigma_3$	$\sigma_4$	0.12773 $\cdot 10^{-1}$	0.16918 $\cdot 10^{-1}$	0.25757 $\cdot 10^{-1}$	0.35670 $\cdot 10^{-1}$	0.47212 $\cdot 10^{-1}$	0.61508 $\cdot 10^{-1}$				
4.00	$\sigma_1$	$\sigma_2$	$-0.21158 \cdot 10^{-3}$	$-0.32819 \cdot 10^{-3}$	$-0.45464 \cdot 10^{-3}$	$-0.52243 \cdot 10^{-3}$	$-0.56479 \cdot 10^{-3}$	$-0.59416 \cdot 10^{-3}$				
	$\sigma_2$	$\sigma_3$	0.22526	0.21799	0.20195	0.18275	0.16541	0.14963				
	$\sigma_3$	$\sigma_4$	0.17703 $\cdot 10^{-1}$	0.23567 $\cdot 10^{-1}$	0.36758 $\cdot 10^{-1}$	0.53023 $\cdot 10^{-1}$	0.76405 $\cdot 10^{-1}$	0.10616				
5.00	$\sigma_1$	$\sigma_2$	$-0.21018 \cdot 10^{-3}$	$-0.32687 \cdot 10^{-3}$	$-0.45375 \cdot 10^{-3}$	$-0.52182 \cdot 10^{-3}$	$-0.56182 \cdot 10^{-3}$	$-0.59162$				
	$\sigma_2$	$\sigma_3$	0.21864	0.20895	0.18630	0.15355	0.13370	0.11543				
	$\sigma_3$	$\sigma_4$	0.22951 $\cdot 10^{-1}$	0.30884 $\cdot 10^{-1}$	0.49904 $\cdot 10^{-1}$	0.78975 $\cdot 10^{-1}$	0.11543	0.03608				

where  $B_i$  ( $i = 1, 2, 3$ ) are new constants with respect to the coordinates. The substitution of solutions (9-1-19) through (9-1-21) into conditions (9-1-15) through (9-1-17) results in the system of algebraic equations

$$\left. \begin{aligned} \sum_{j=1}^3 B_j P_j + \frac{\epsilon K_0}{s} &= 0; \\ \sum_{j=1}^3 B_j Q_j + \frac{\epsilon K_0 K_1 - 1}{s} &= 0; \\ \sum_{j=1}^3 B_j X_j &= 0, \end{aligned} \right\} \quad (9-1-22)$$

whose solution gives

$$\begin{aligned} B_1 &= \frac{(\epsilon K_0 K_1 - 1) \left( P_1 - \frac{X_1}{X_3} P_3 \right) - \epsilon K_0 \left( Q_1 - \frac{X_1}{X_3} Q_3 \right)}{s \left[ \left( P_1 - \frac{X_1}{X_3} P_3 \right) \left( Q_2 - \frac{X_2}{X_3} Q_3 \right) - \left( P_2 - \frac{X_2}{X_3} P_3 \right) \left( Q_1 - \frac{X_1}{X_3} Q_3 \right) \right]}; \\ B_2 &= - \frac{(\epsilon K_0 K_1 - 1) \left( P_1 - \frac{X_1}{X_3} P_3 \right) - \epsilon K_0 \left( Q_1 - \frac{X_1}{X_3} Q_3 \right)}{s \left[ \left( P_1 - \frac{X_1}{X_3} P_3 \right) \left( Q_2 - \frac{X_2}{X_3} Q_3 \right) - \left( P_2 - \frac{X_2}{X_3} P_3 \right) \left( Q_1 - \frac{X_1}{X_3} Q_3 \right) \right]}; \\ B_3 &= \frac{(\epsilon K_0 K_1 - 1) \left( P_1 \frac{X_2}{X_3} - P_2 \frac{X_1}{X_3} \right) - \epsilon K_0 \left( Q_1 \frac{X_2}{X_3} - Q_2 \frac{X_1}{X_3} \right)}{s \left[ \left( P_1 - \frac{X_1}{X_3} P_3 \right) \left( Q_2 - \frac{X_2}{X_3} Q_3 \right) - \left( P_2 - \frac{X_2}{X_3} P_3 \right) \left( Q_1 - \frac{X_1}{X_3} Q_3 \right) \right]}, \end{aligned}$$

where

$$\begin{aligned} P_j &= (1 - \nu_j^2) \operatorname{ch} \sqrt{s} \nu_j + \frac{(1 - \nu_j^2) + \epsilon K_0 P_n + \frac{Lu_p}{Lu} \sigma_j}{B l_m} \sqrt{s} \nu_j \operatorname{sh} \sqrt{s} \nu_j; \\ Q_j &= [1 + (1 - \nu_j^2) K_1] \operatorname{ch} \sqrt{s} \nu_j + \frac{1}{B l_q} \sqrt{s} \nu_j \operatorname{sh} \sqrt{s} \nu_j; \\ X_j &= \sigma_j \operatorname{ch} \sqrt{s} \nu_j. \end{aligned}$$

The inverse transforms may be found using the expansion theorem. Since the roots of the denominator of the two generalized polynomials  $\Phi(s)/\psi(s)$  are a)  $s = 0$  and b)  $s = s_n$ , where  $s_n$  are the roots of the equation

$$\varphi(s) = \left( P_1 - \frac{X_1}{X_3} P_3 \right) \left( Q_2 - \frac{X_2}{X_3} Q_3 \right) - \left( P_2 - \frac{X_2}{X_3} P_3 \right) \left( Q_1 - \frac{X_1}{X_3} Q_3 \right) = 0, \quad (9-1-23)$$

therefore the following inversion formula is used:

$$L^{-1} \left[ \frac{\Phi(s)}{\psi(s)} \right] = \frac{\Phi(0)}{\psi'(0)} + \sum_{n=1}^{\infty} \frac{\Phi(s_n)}{\psi'(s_n)} \exp(s_n Fo). \quad (9-1-24)$$

Here  $\psi(s) = s \varphi(s)$ , whereas  $\Phi(s)$  represents the corresponding expressions which remain in the numerators of equations (9-1-19) through (9-1-21).

When we determine  $\psi'(0)$ ,  $\Phi(0)$ ,  $\psi'(s_n)$ ,  $\Phi(s_n)$  for the dimensionless transfer-potential fields and substitute the results into (9-1-24), then after some simple transformations we obtain the final solution

$$T(X, Fo) = 1 - \sum_{n=1}^{\infty} \sum_{j=1}^3 C_{nj} \cos \nu_j \mu_n X \exp(-\mu_n^2 Fo); \quad (9-1-25)$$

$$\Theta(X, Fo) = 1 + \frac{1}{\epsilon K_0} \sum_{n=1}^{\infty} \sum_{j=1}^3 C_{nj} (1 - \nu_j^2) \cos \nu_j \mu_n X \exp(-\mu_n^2 Fo); \quad (9-1-26)$$

$$P(X, Fo) = -\frac{1}{\varepsilon \text{Bi}_q} \sum_{n=1}^{\infty} \sum_{j=1}^3 C_{nj} \sigma_j \cos \nu_j \mu_n X \exp(-\mu_n^2 Fo). \quad (9-1-27)$$

The coefficients  $C_{nj}$  are defined by the expressions

$$C_{n1} = \frac{2}{\mu_n \psi_n} \left[ (1 - \varepsilon \text{Ko} K_1) \left( P_{n2} - \frac{\chi_{n2}}{\chi_{n3}} P_{n3} \right) + \varepsilon \text{Ko} \left( Q_{n2} - \frac{\chi_{n2}}{\chi_{n3}} Q_{n3} \right) \right]; \quad (9-1-28)$$

$$C_{n2} = -\frac{2}{\mu_n \psi_n} \left[ (1 - \varepsilon \text{Ko} K_1) \left( P_{n1} - \frac{\chi_{n1}}{\chi_{n3}} P_{n3} \right) + \varepsilon \text{Ko} \left( Q_{n1} - \frac{\chi_{n1}}{\chi_{n3}} Q_{n3} \right) \right]; \quad (9-1-29)$$

$$C_{n3} = \frac{2}{\mu_n \psi_n} \left[ (1 - \varepsilon \text{Ko} K_1) \left( P_{n1} \frac{\chi_{n2}}{\chi_{n3}} - P_{n2} \frac{\chi_{n1}}{\chi_{n3}} \right) + \varepsilon \text{Ko} \left( Q_{n1} \frac{\chi_{n2}}{\chi_{n3}} - Q_{n2} \frac{\chi_{n1}}{\chi_{n3}} \right) \right]; \quad (9-1-30)$$

where

$$\begin{aligned} \psi_n = & \left( P_{n2} - \frac{\chi_{n2}}{\chi_{n3}} P_{n3} \right) \left[ \nu_1 A_{n1} - \frac{\chi_{n1}}{\chi_{n3}} (\nu_3 A_{n3} + b_{n1} Q_{n3}) \right] + \\ & + \left( Q_{n1} - \frac{\chi_{n1}}{\chi_{n3}} Q_{n3} \right) \left[ \nu_2 B_{n2} - \frac{\chi_{n2}}{\chi_{n3}} (\nu_3 B_{n3} + b_{n2} P_{n3}) \right] - \\ & - \left( P_{n1} - \frac{\chi_{n1}}{\chi_{n3}} P_{n3} \right) \left[ \nu_3 A_{n3} - \frac{\chi_{n3}}{\chi_{n3}} (\nu_3 A_{n3} + b_{n3} Q_{n3}) \right] - \\ & - \left( Q_{n3} - \frac{\chi_{n3}}{\chi_{n3}} Q_{n3} \right) \left[ \nu_1 B_{n1} - \frac{\chi_{n1}}{\chi_{n3}} (\nu_3 B_{n3} + b_{n1} P_{n3}) \right]; \end{aligned} \quad (9-1-31)$$

$$A_{nj} = \left[ 1 + \frac{1}{\text{Bi}_q} + (1 - \nu_j^2) K_1 \right] \sin \nu_j \mu_n + \frac{1}{\text{Bi}_q} \nu_j \mu_n \cos \nu_j \mu_n;$$

$$B_{nj} = (1 - \nu_j^2) \sin \nu_j \mu_n + \frac{(1 - \nu_j^2) + \varepsilon \text{Ko} Pn + \frac{\text{Lu}_p}{\text{Lu}} \sigma_j}{\text{Bi}_m} (\sin \nu_j \mu_n + \nu_j \mu_n \cos \nu_j \mu_n);$$

$$b_{nj} = \nu_j \tan \nu_j \mu_n - \nu_3 \tan \nu_3 \mu_n;$$

$$P_{nj} = (1 - \nu_j^2) \cos \nu_j \mu_n - \frac{(1 - \nu_j^2) + \varepsilon \text{Ko} Pn + \frac{\text{Lu}_p}{\text{Lu}} \sigma_j}{\text{Bi}_m} \nu_j \mu_n \sin \nu_j \mu_n;$$

$$Q_{nj} = [1 + (1 - \nu_j^2) K_1] \cos \nu_j \mu_n - \frac{1}{\text{Bi}_q} \nu_j \mu_n \sin \nu_j \mu_n;$$

$$\chi_{nj} = \sigma_j \cos \nu_j \mu_n.$$

The characteristic-equation roots  $\mu_n = i \sqrt{s_n}$  are determined by solving the equation

$$\left( P_{n1} - \frac{\chi_{n1}}{\chi_{n3}} P_{n3} \right) \left( Q_{n2} - \frac{\chi_{n2}}{\chi_{n3}} Q_{n3} \right) - \left( P_{n2} - \frac{\chi_{n2}}{\chi_{n3}} P_{n3} \right) \left( Q_{n1} - \frac{\chi_{n1}}{\chi_{n3}} Q_{n3} \right) = 0. \quad (9-1-32)$$

Figures 9-1 through 9-5 show the characteristic-equation roots  $\mu_n$  as functions of  $\text{Bi}_q$  for various values of various dimensionless parameters. The values of  $\mu_n$  were obtained by means of graphoanalytical solution of equation (9-1-32).

Expressions for the average values of the dimensionless transfer potentials (9-1-25) through (9-1-27) are obtained using the familiar formula. These average potentials are

$$\bar{T}(Fo) = 1 - \sum_{n=1}^{\infty} \sum_{j=1}^3 D_{nj} \exp(-\mu_n^2 Fo); \quad (9-1-33)$$

$$\bar{\theta}(Fo) = 1 + \frac{1}{\varepsilon \text{Ko}} \sum_{n=1}^{\infty} \sum_{j=1}^3 D_{nj} (1 - \nu_j^2) \exp(-\mu_n^2 Fo); \quad (9-1-34)$$



$$\bar{P}(Fo) = -\frac{1}{\pi Bu} \sum_{n=1}^{\infty} \sum_{j=1}^3 D_{nj} \sigma_j \exp(-\mu_n^2 Fo); \quad (9-1-35)$$

where

$$D_{nj} = C_{nj} \frac{\sin \nu_j \mu_n}{\nu_j \mu_n}.$$

In addition, the rates of variation of the respective potentials, as well as the amounts of heat and mass consumed during the heat and mass exchange process, can be obtained easily using the above solutions.

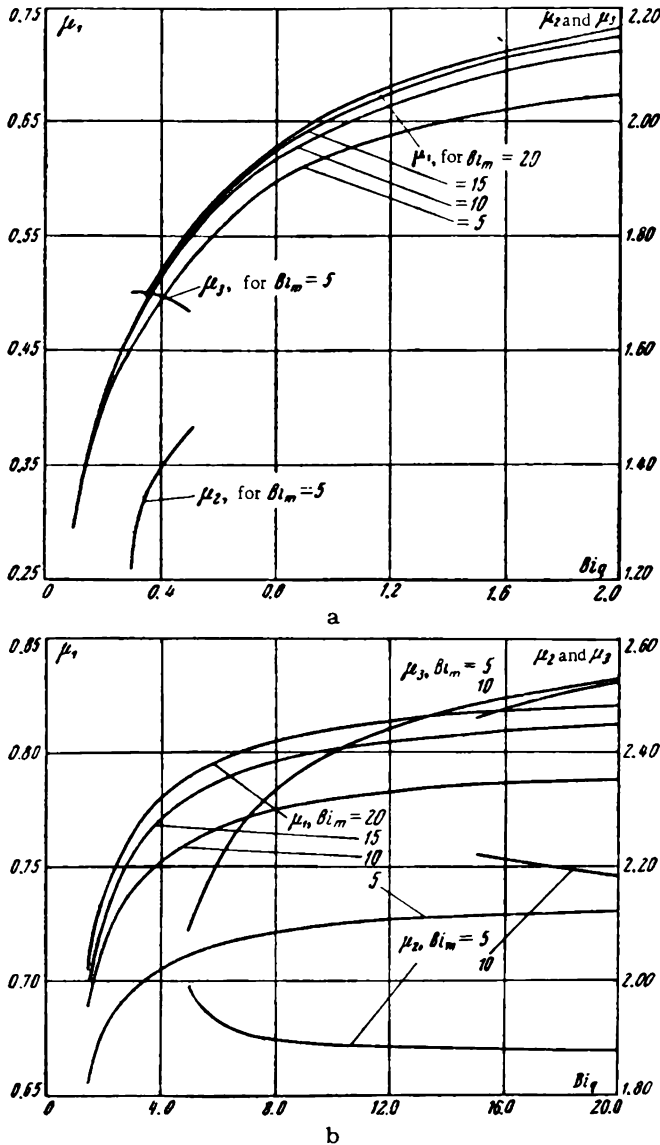


FIGURE 9-1. Characteristic-equation roots [as functions of  $Bi_q$ ], for various values of  $Bi_m$ , for an infinite plate.

a)  $Bi_q$  from 0.1 to 2.0; b)  $Bi_q$  from 1.0 to 20

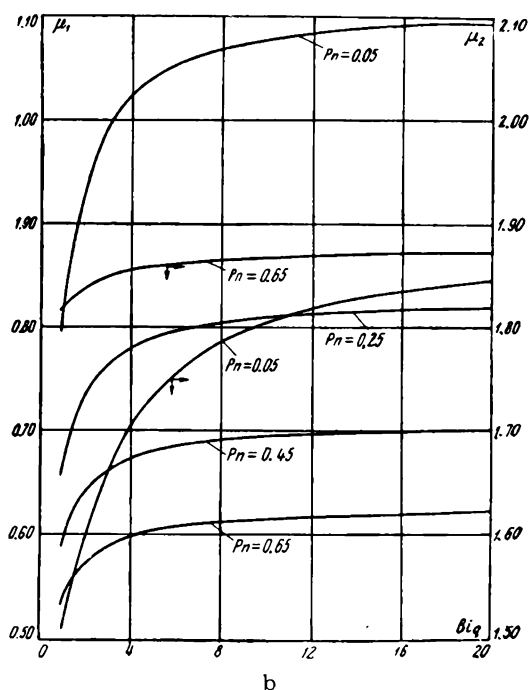
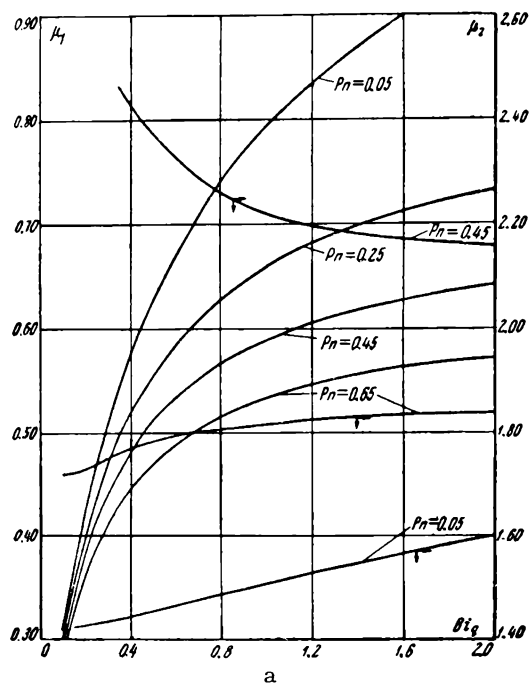


FIGURE 9-2. Characteristic-equation roots [as functions of  $Bi_q$ ], for various values of  $Pn$ , for an infinite plate  
a)  $Bi_q$  from 0.1 to 2.0; b)  $Bi_q$  from 1.0 to 20

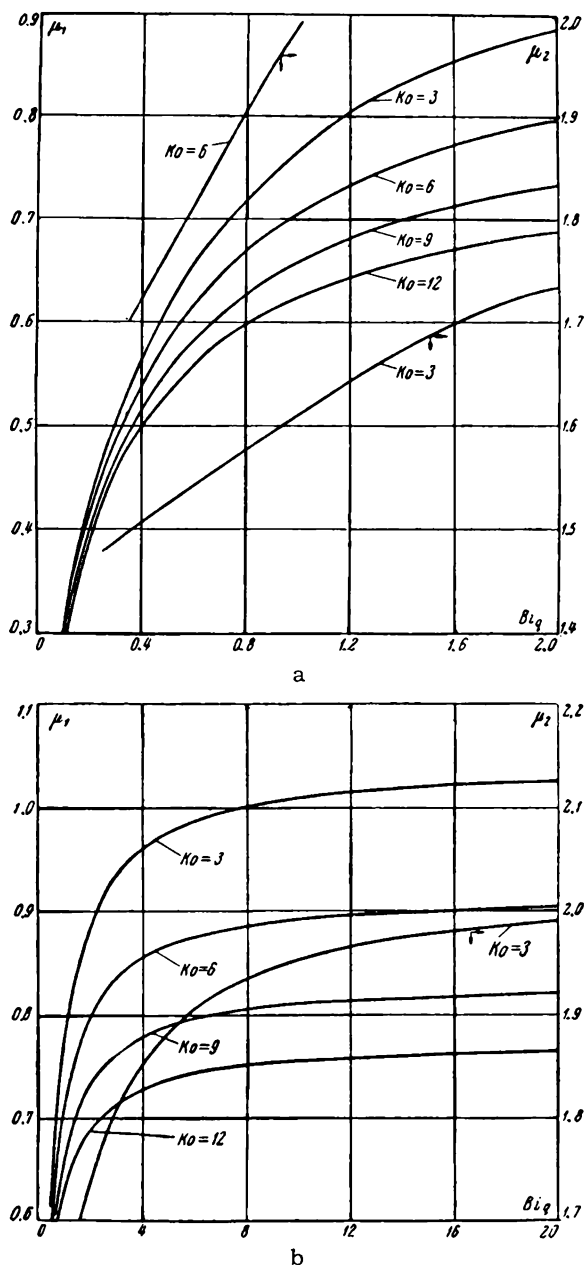


FIGURE 9-3. Characteristic-equation roots [as functions of  $Bi_q$ ], for various values of  $Ko$ , for an infinite plate

a)  $Bi_q$  from 0.1 to 2.0; b)  $Bi_q$  from 1.0 to 20

The results of numerical calculations on the basis of formulas (9-1-25) through (9-1-27) and (9-1-23) through (9-1-35) show that the infinite sums in the solutions converge quite rapidly. Thus, above some value of the Fourier number it is possible to drop all series terms after the first two

or three terms. On the average, above  $Fo = 0.7$  the calculation error when only the first two or three terms are retained does not exceed 1 or 2%.

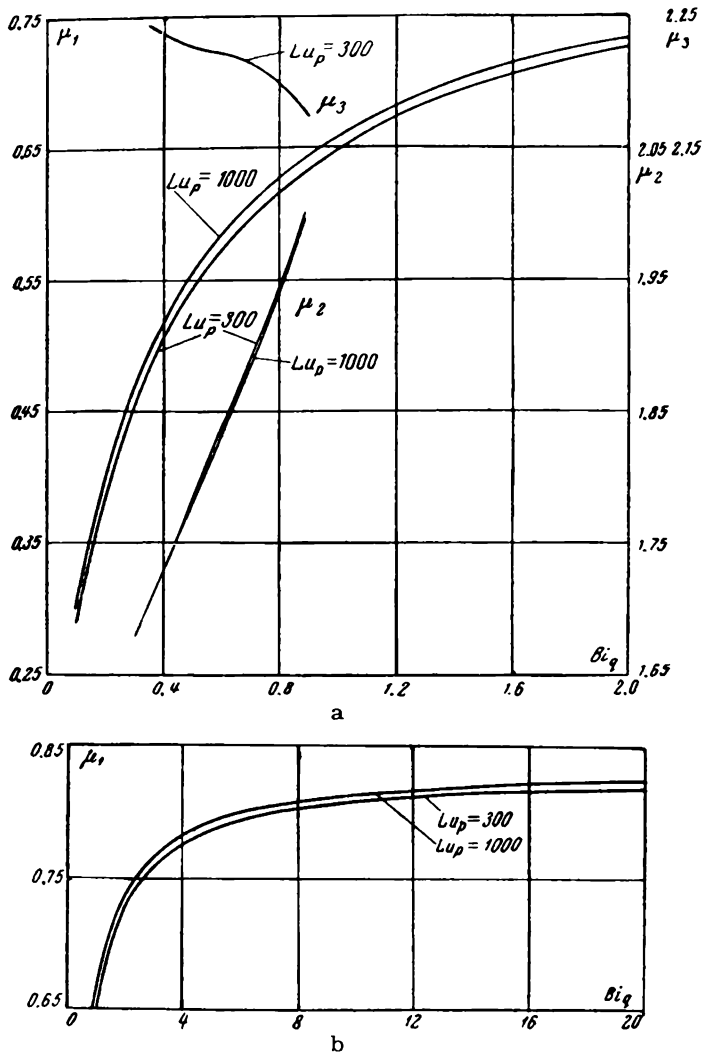


FIGURE 9-4. Characteristic-equation roots [as functions of  $Bi_q$ ], for various values of  $Lu_p$ , for an infinite plate  
a)  $Bi_q$  from 0.1 to 2.0; 2)  $Bi_q$  from 1.0 to 20

Consequently, for practical calculations it is convenient to simplify solutions (9-1-33) through (9-1-35) to

$$\bar{T} \approx 1 - \sum_{n=1}^2 D_{Tn} \exp(-\mu_n^2 Fo); \quad (9-1-36)$$

$$\bar{\theta} \approx 1 + \sum_{n=1}^2 D_{\theta n} \exp(-\mu_n^2 Fo); \quad (9-1-37)$$

$$\bar{P} \approx \sum_{n=1}^2 D_{Pn} \exp(-\mu_n^2 Fo), \quad (9-1-38)$$

where

$$D_{Tn} = \sum_{j=1}^3 D_{nj}; \quad D_{\theta n} = \frac{1}{\epsilon K_0} \sum_{j=1}^3 D_{nj} (1 - v_j^2); \quad D_{Pn} = -\frac{1}{\epsilon Bu} \sum_{j=1}^3 D_{nj} \sigma_j.$$

Values of coefficients  $C_{nj}$  and  $D_{nj}$ , as well as of  $D_{Tn}$ ,  $D_{\theta n}$ , and  $D_{Pn}$ , for various combinations of heat-transfer and mass-transfer parameters are listed in Tables 9-3 through 9-7.

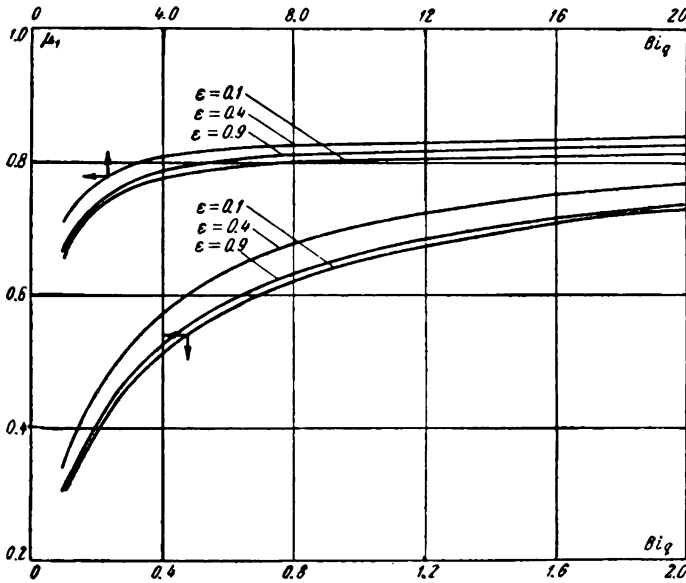


FIGURE 9-5. Characteristic-equation roots [as functions of  $Bi_q$ ], for various values of  $\epsilon$ , for an infinite plate

In a similar manner it is possible to find the solutions for a cylinder and a sphere. The generalized solution for bodies of regular geometrical form is

$$T(X, Fo) = 1 - \sum_{n=1}^{\infty} \sum_{j=1}^3 C_{nj} \Phi_{nr}(\nu_j \mu_n X) \exp(-\mu_n^2 Fo);$$

$$\theta(X, Fo) = 1 + \frac{1}{\epsilon K_0} \sum_{n=1}^{\infty} \sum_{j=1}^3 C_{nj} (1 - v_j^2) \Phi_{nr}(\nu_j \mu_n X) \exp(-\mu_n^2 Fo);$$

$$P(X, Fo) = -\frac{1}{\epsilon Bu} \sum_{n=1}^{\infty} \sum_{j=1}^3 C_{nj} \sigma_j \Phi_{nr}(\nu_j \mu_n X) \exp(-\mu_n^2 Fo),$$

where  $C_{nj}$ ,  $\mu_n$ , and  $\nu_n$  are determined respectively by equations (9-1-28) through (9-1-32), and  $\sigma_j$  and  $\nu_j$  are determined by (9-1-13) and (9-1-14).

TABLE 9 - 3

Values of the constant coefficients for an infinite plate, for various values of  $Bl_q$   
 $(Lu = 0.3; Lu_p = 500; Bl_m = 20; \epsilon = 0.7; Ko = 9; Pr = 0.25; Bu = 1 \cdot 10^{-4})$

n	$Bl_q$	$C_{n,j}$			$D_{n,j}$			$DT_n$	$De_n$	$DP_n$
		$j = 1$	$j = 2$	$j = 3$	$j = 1$	$j = 2$	$j = 3$			
1	5	3.053	-0.00632	0.7861	2.118	-0.00631	0.7617	2.873	-0.6792	2.420
	10	3.052	-0.00176	0.2965	2.073	-0.00176	0.2868	2.358	-0.7149	2.657
	15	3.052	-0.00018	0.1252	2.057	-0.00018	0.1210	2.178	-0.7271	2.746
	20	3.054	0.00064	0.0367	2.051	0.00064	0.0355	2.087	-0.7344	2.790
1		3.051	0.00144	-0.0507	2.041	0.00144	-0.0490	1.993	-0.7400	2.834
2		-1.586	0.00454	-0.9590	0.342	0.00451	-0.6992	-0.352	-0.2000	1.910
3		0.641	0.00119	-0.4161	-0.137	0.00118	-0.2894	-0.425	-0.0175	0.6403

TABLE 9 - 4

Values of the constant coefficients for an infinite plate, for various values of  $Bl_m$   
 $(Lu = 0.3; Lu_p = 500; Bl_q = 20; \epsilon = 0.7; Ko = 9; Pr = 0.25; Bu = 1 \cdot 10^{-4})$

n	$Bl_m$	$C_{n,j}$			$D_{n,j}$			$DT_n$	$De_n$	$DP_n$
		$j = 1$	$j = 2$	$j = 3$	$j = 1$	$j = 2$	$j = 3$			
1		3.022	0.008495	-0.552	2.218	0.008489	-0.537	1.689	-0.856	2.437
2	5	-0.671	0.003650	-1.221	0.049	0.009512	-1.016	-0.958	-0.129	0.831
3		-0.165	-0.000284	0.184	0.036	-0.000282	0.130	0.166	0.002	-0.434
1		3.032	0.004320	-0.248	2.120	0.004317	-0.240	1.884	-0.789	2.646
2	8	-0.816	0.008902	-1.337	0.123	0.008860	-1.067	-0.935	-0.161	1.459
3		-0.033	-0.000014	0.049	-0.007	-0.000014	0.035	0.028	0.006	-0.168
1		3.049	0.001445	-0.026	2.066	0.001444	-0.026	2.042	-0.746	2.761
2	15	-1.977	0.006335	-1.114	0.421	0.006296	-0.823	-0.395	-0.242	1.860
3		0.957	0.000597	-0.578	-0.208	0.000592	-0.411	-0.619	0.029	1.341
1	20	3.054	0.000640	0.037	2.051	0.000639	0.036	2.087	-0.734	2.790

TABLE 9-5

Values of the constant coefficients for an infinite plate, for various values of  $Ko$   
 $(Lu = 0.3; Lu_p = 500; Bl_q = 20; Bl_m = 20; \epsilon = 0.7; Pn = 0.25; Bu = 1 \cdot 10^{-9})$

$n$	$Ko$	$C_{nj}$			$D_{nj}$			$D_{Tn}$	$D_{\theta n}$	$D_{Pn}$
		$j=1$	$j=2$	$j=3$	$j=1$	$j=2$	$j=3$			
1	3	2.199	0.00846	-0.0677	1.407	0.000846	-0.0608	1.436	-0.7501	1.313
2		-0.113	0.001013	-0.836	-0.012	0.001009	-0.5911	-0.602	-0.1376	0.805
1	6	2.721	0.000711	0.0042	1.832	0.000710	0.0040	1.835	-0.7303	1.979
2		-0.317	0.002889	-1.383	0.047	0.002872	-0.9797	-0.930	-0.1651	2.226
3		-0.550	0.000460	0.003	0.119	0.000457	0.0020	0.122	-0.0173	-0.050
1	9	3.054	0.000640	0.037	2.051	0.000639	0.0355	2.087	-0.7344	2.790
1	12	3.291	-0.000281	0.091	2.192	-0.000281	0.0889	2.281	-0.7330	3.037
2		-1.557	-0.001517	0.472	0.338	-0.001568	0.3708	0.707	-0.0816	-0.974
3		0.570	0.008058	-2.460	-0.087	0.00799	-1.7500	-1.829	-0.1246	6.859

TABLE 9-6

Values of the constant coefficients for an infinite plate, for various values of  $Pn$   
 $(Lu = 0.3; Lu_p = 500; Bl_q = 20; Bl_m = 20; \epsilon = 0.7; Ko = 9; Bu = 1 \cdot 10^{-9})$

$n$	$Pn$	$C_{nj}$			$D_{nj}$			$D_{Tn}$	$D_{\theta n}$	$D_{Pn}$
		$j=1$	$j=2$	$j=3$	$j=1$	$j=2$	$j=3$			
1	0.6500	1.576	-0.001663	0.082	1.039	-0.001662	0.081	1.119	-0.8017	2.017
2		-0.537	-0.001889	0.119	0.116	-0.001881	0.107	0.222	-0.0771	-0.159
1	0.4500	2.063	-0.000884	0.082	1.367	-0.000884	0.081	1.447	-0.7751	1.919
2		-0.756	-0.001420	0.186	0.164	-0.001413	0.158	0.320	-0.0747	-0.323
1	0.3333	2.528	-0.000217	0.073	1.684	-0.000216	0.071	1.755	-0.7506	2.317
2		-1.138	-0.001152	0.345	0.247	-0.001145	0.271	0.517	-0.0795	-0.733
3		0.366	0.005173	-1.579	-0.056	0.005131	-1.123	-1.174	-0.1067	4.400
1	0.2500	3.054	0.000640	0.037	2.051	0.000639	0.036	2.087	-0.7344	2.790
1	0.1667	3.927	0.001025	0.006	2.644	0.001024	0.006	2.651	-0.7022	2.836
2		-0.582	0.005297	-2.532	0.087	0.005266	-1.794	-1.702	-0.2016	4.074
3		-0.673	0.000567	0.010	0.146	0.000563	0.006	0.153	-0.0380	-0.077
1	0.0500	7.849	0.003174	-0.475	5.386	0.003170	-0.422	4.967	-0.6598	3.904
2		-0.848	0.004937	-5.725	-0.226	0.004917	-4.038	-4.259	-0.2473	5.847

TABLE 9-7

Values of the constant coefficients for an infinite plate, for various values of  $Lu_p$   
( $Lu = 0.3$ ;  $Ko = 9$ ;  $Bi_g = 20$ ;  $Bi_m = 20$ ;  $\epsilon = 0.7$ ;  $Pn = 0.25$ ;  $Bu = 1 \cdot 10^{-2}$ )

n	$Lu_p$	$C_{nj}$			$D_{nj}$			$D_{Tn}$	$D_{\theta n}$	$D_{Pn}$
		$j=1$	$j=2$	$j=3$	$j=1$	$j=2$	$j=3$			
I	300	3.031	0.001588	0.03014	2.038	0.001587	0.02913	2.069	-0.7298	4.931
	500	3.054	0.000640	0.03672	2.051	0.000693	0.03549	2.087	-0.7344	2.790
	700	3.073	0.000272	0.04202	2.063	0.000272	0.00041	2.104	-0.7381	1.609
	1000	3.040	0.000189	0.05231	2.033	0.000189	0.05053	2.084	-0.7267	1.371

For an infinite plate ( $\Gamma=0$ )  $\Phi_{n0} = \cos v_j \mu_n X$ ; the rest of the quantities involved were defined above.

For an infinite cylinder ( $\Gamma=1$ )

$$\Phi_{n1} = J_0(v_j \mu_n X);$$

$$A_{nj} = [1 + (1 - v_j^2) K_1] J_1(v_j \mu_n) + \frac{1}{Bi_g} v_j \mu_n J_0(v_j \mu_n);$$

$$B_{nj} = (1 - v_j^2) J_1(v_j \mu_n) + \Omega_j v_j \mu_n J_0(v_j \mu_n);$$

$$P_{nj} = (1 - v_j^2) J_0(v_j \mu_n) - \Omega_j v_j \mu_n J_1(v_j \mu_n);$$

$$Q_{nj} = [1 + (1 - v_j^2) K_1] J_0(v_j \mu_n) - \frac{1}{Bi_g} v_j \mu_n J_1(v_j \mu_n);$$

$$b_{nj} = v_j \frac{J_1(v_j \mu_n)}{J_0(v_j \mu_n)} - v_j \frac{J_1(v_j \mu_n)}{J_0(v_j \mu_n)};$$

$$\chi_{nj} = \sigma_j J_0(v_j \mu_n); \Omega_j = \frac{(1 - v_j^2) + \epsilon Ko Pn + \sigma_j Lu_p / Lu}{Bi_m}.$$

Values of the characteristic-equation roots  $\mu_n$  for a cylinder, calculated using the "Ural"\* computer, are listed in Tables 9-8 and 9-9 and plotted in Figures 9-6 through 9-10. Values of the constant coefficients  $C_{nj}$  for various combinations of dimensionless numbers are given in Tables 9-10 through 9-14. The average dimensionless transfer potentials are defined by equations (9-1-33) through (9-1-35), where

$$D_{nj} = 2C_{nj} \frac{J_1(v_j \mu_n)}{v_j \mu_n}.$$

Values of the latter coefficients are listed in Tables 9-15 through 9-19. Above Fourier numbers of 0.5 to 0.7, the average transfer potentials can also be calculated using the simplified formulas (9-1-36) through (9-1-38).

\* The calculation was carried out by Engineer I. V. Romanina at the Institut matematiki i vychislitel'noi tekhniki [the Institute of Mathematics and Computer Technology] of the Academy of Sciences of the Belorussian SSR.



TABLE 9-8

Infinite cylinder  
Roots of characteristic equation (9-1-32), for various values of  $Lu$  and  $Bi_g$   
( $\epsilon = 0.5$ ;  $Ko = 9$ ;  $Pn = 0.25$ ;  $Lu_p = 500$ ;  $Bi_m = 20$ )

$Bi_g$	$Lu$									
	0.25		0.50		0.75		1.00		2.00	
	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$	$\mu_1$	$\mu_2$
5	1.17911	4.23073	1.30429	3.26053	1.36872	3.27597	1.39342	3.32508	1.42999	3.40386
10	1.20380	4.31903	1.35390	3.22764	1.42747	3.33675	1.45827	3.40032	1.50450	3.49884
15	1.21200	4.35534	1.37138	3.21753	1.44835	3.36075	1.48159	3.43179	1.53173	3.54067
20	1.21608	2.83222	1.38032	3.21264	1.45904	3.37364	1.49356	3.44915	1.54585	3.56427
30	1.22015	2.81111	1.38936	3.19999	1.46989	3.38722	1.50579	3.46778	1.56028	3.58999

TABLE 9-9

Infinite cylinder  
Roots of characteristic equation (9-1-32), for various values of  $Bi_m$ ,  $Lu_p$ ,  $\epsilon$ ,  $Ko$ , and  $Pn^*$

Dimensionless number		$\mu_1$	$\mu_2$	Dimensionless number		$\mu_1$	$\mu_2$
$Bi_m$	5	1.309	2.940	$Ko$	3	1.693	3.065
	10	1.365	3.166		6	1.513	3.228
	15	1.383	3.212		9	1.380	3.213
	20	1.380	3.213		12	1.301	3.004
	30	1.401	3.241				
$Lu_p$	100	1.381	3.213	$Pn$	0.05	1.804	4.070
	250	1.380	3.213		0.25	1.380	3.213
	500	1.380	3.213		0.45	1.202	2.771
	750	1.380	3.213		0.65	1.079	2.484
	1000	1.380	3.213	$\epsilon$	0.25	1.457	3.177
			0.50		1.380	3.213	
			0.75		1.341	3.088	

\* The table gives virtual variations of the dimensionless parameters, for which the following initial combination of values were assumed:  $Lu = 0.5$ ;  $Lu_p = 500$ ;  $\epsilon = 0.5$ ;  $Ko = 9$ ;  $Pn = 0.25$ ;  $Bi_g = 20$ ;  $Bi_m = 20$ .

TABLE 9-10

Values of constant coefficients  $C_{n1}$  for an infinite cylinder, for various values of  $Lu$  and  $Bi_g$   
( $\epsilon = 0.5$ ;  $Ko = 9$ ;  $Pn = 0.25$ ;  $Lu_p = 500$ ;  $Bi_m = 20$ )

$Lu$	$Bi_g$	$C_{n1}$		$C_{n2}$		$C_{n3}$	
		$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
0.25	5	2.62797	1.15266	$-0.42476 \cdot 10^{-3}$	$-0.14928 \cdot 10^{-4}$	0.94541	-1.19531
	10	2.63963	1.24584	$-0.12021 \cdot 10^{-3}$	$-0.43646 \cdot 10^{-4}$	0.40029	-0.63305
	15	2.64147	1.26910	$-0.18916 \cdot 10^{-4}$	$0.10519 \cdot 10^{-4}$	0.21673	-0.36518
	20	2.64199	-2.63306	$-0.31585 \cdot 10^{-4}$	$-0.90923 \cdot 10^{-4}$	-0.00140	0.02875
	30	2.64223	-2.08983	$-0.81965 \cdot 10^{-4}$	$0.73608 \cdot 10^{-4}$	0.03289	0.29246
0.50	5	3.47300	-4.94228	$-0.26575 \cdot 10^{-3}$	$-0.59136 \cdot 10^{-3}$	1.04812	6.76122
	10	3.47855	-3.60229	$-0.10966 \cdot 10^{-3}$	$-0.22274 \cdot 10^{-3}$	0.48204	2.61155
	15	3.47766	-3.22568	$-0.54855 \cdot 10^{-3}$	$-0.11347 \cdot 10^{-3}$	0.28000	1.43911
	20	3.47660	-3.04799	$-0.26994 \cdot 10^{-3}$	$-0.60954 \cdot 10^{-3}$	0.17656	0.88483
	30	-1.38682	-2.85445	$-0.20326 \cdot 10^{-3}$	$-0.51005 \cdot 10^{-4}$	0.64336	0.31456

TABLE 9-10 (continued)

Lu	Bl <sub>q</sub>	C <sub>n1</sub>		C <sub>n2</sub>		C <sub>n3</sub>	
		n=1	n=2	n=1	n=2	n=1	n=2
0.75	5	3.74087	-3.10534	-0.20349·10 <sup>-3</sup>	-0.22081·10 <sup>-3</sup>	0.92621	2.06338
	10	3.75156	-2.94123	-0.95091·10 <sup>-3</sup>	-0.11357·10 <sup>-3</sup>	0.45118	1.13519
	15	3.75136	-2.87214	-0.56912·10 <sup>-3</sup>	-0.70994·10 <sup>-3</sup>	0.28134	0.74426
	20	3.75035	-2.84200	-0.37482·10 <sup>-3</sup>	-0.48237·10 <sup>-3</sup>	0.19436	0.52995
	30	3.74867	-2.79429	-0.17834·10 <sup>-3</sup>	-0.24021·10 <sup>-3</sup>	0.10604	0.29709
1.00	5	3.88640	-3.06108	-0.28310·10 <sup>-3</sup>	-0.26670·10 <sup>-3</sup>	0.88399	1.44559
	10	3.88961	-2.93843	-0.13710·10 <sup>-3</sup>	-0.14517·10 <sup>-3</sup>	0.43947	0.82839
	15	3.88634	-2.88413	-0.85356·10 <sup>-3</sup>	-0.94865·10 <sup>-3</sup>	0.27983	0.55933
	20	3.88364	-2.85384	-0.58933·10 <sup>-3</sup>	-0.67367·10 <sup>-3</sup>	0.19788	0.40862
	30	3.88012	-2.82122	-0.32207·10 <sup>-3</sup>	-0.38080·10 <sup>-3</sup>	0.11466	0.24514
2.00	5	4.09207	-3.07515	-0.60582·10 <sup>-3</sup>	-0.49853·10 <sup>-3</sup>	0.80547	0.87743
	10	4.07759	-2.97214	-0.30711·10 <sup>-3</sup>	-0.28856·10 <sup>-3</sup>	0.41235	0.52058
	15	4.06692	-2.92036	-0.20038·10 <sup>-3</sup>	-0.19812·10 <sup>-3</sup>	0.27076	0.36226
	20	4.06008	-2.88961	-0.14581·10 <sup>-3</sup>	-0.14807·10 <sup>-3</sup>	0.19813	0.27335
	30	4.05209	-2.85486	-0.90434·10 <sup>-3</sup>	-0.94446·10 <sup>-3</sup>	0.12424	0.17706

TABLE 9-11

Values of constant coefficients  $C_{nj}$  for an infinite cylinder, for various values of  $Lu_p$   
( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $Pn = 0.25$ ;  $Ko = 9$ ;  $Bl_q = 20$ ;  $Bl_m = 20$ )

Lu <sub>p</sub>	C <sub>n1</sub>		C <sub>n2</sub>		C <sub>n3</sub>	
	n=1	n=2	n=1	n=2	n=1	n=2
100	3.48178	-3.03283	0.15961·10 <sup>-2</sup>	-0.35804·10 <sup>-2</sup>	0.17731	0.85172
250	3.47789	-3.04412	-0.56212·10 <sup>-3</sup>	-0.12672·10 <sup>-2</sup>	0.17669	0.87635
500	3.47660	-3.04799	-0.26994·10 <sup>-3</sup>	-0.60954·10 <sup>-3</sup>	0.17656	0.88483
750	3.47617	-3.04924	-0.17754·10 <sup>-3</sup>	-0.40117·10 <sup>-3</sup>	0.17650	0.88760
1000	3.47596	-3.04992	-0.13226·10 <sup>-3</sup>	-0.29901·10 <sup>-3</sup>	0.17648	0.88907

TABLE 9-12

Values of constant coefficients  $C_{nj}$  for an infinite cylinder, for various values of  $Bl_m$   
( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $Pn = 0.25$ ;  $Ko = 9$ ;  $Lu_p = 500$ ;  $Bl_q = 20$ )

Bl <sub>m</sub>	C <sub>n1</sub>		C <sub>n2</sub>		C <sub>n3</sub>	
	n=1	n=2	n=1	n=2	n=1	n=2
5	3.48282	-2.77702	0.48460·10 <sup>-3</sup>	0.86574·10 <sup>-3</sup>	-0.13691	-0.74889
10	3.48389	-3.35006	0.48056·10 <sup>-4</sup>	-0.65621·10 <sup>-4</sup>	0.07679	0.51237
15	3.47206	-3.03111	-0.92769·10 <sup>-4</sup>	-0.25501·10 <sup>-3</sup>	0.14658	0.79579
20	3.47660	-3.04799	-0.26994·10 <sup>-3</sup>	-0.60954·10 <sup>-3</sup>	0.17656	0.88483
30	3.45372	-2.63746	-0.22908·10 <sup>-3</sup>	-0.32891·10 <sup>-3</sup>	0.21453	0.89199

TABLE 9-13

Values of constant coefficients  $C_{nj}$  for an infinite cylinder, for various values of  $Ko$   
( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $Pn = 0.25$ ;  $Lu_p = 500$ ;  $Bl_q = 20$ ;  $Bl_m = 20$ )

Ko	C <sub>n1</sub>		C <sub>n2</sub>		C <sub>n3</sub>	
	n=1	n=2	n=1	n=2	n=1	n=2
3	2.47043	-0.11424	0.23405·10 <sup>-4</sup>	0.69290·10 <sup>-4</sup>	0.08468	-0.79181
6	3.06457	-2.24926	-0.48291·10 <sup>-4</sup>	0.34959·10 <sup>-3</sup>	0.13908	-0.40344
9	3.47660	-3.04799	-0.26994·10 <sup>-3</sup>	-0.60954·10 <sup>-3</sup>	0.17656	0.88483
12	3.76296	-2.75144	-0.30277·10 <sup>-3</sup>	-0.38610·10 <sup>-3</sup>	0.21560	-0.53821

TABLE 9-14

Values of constant coefficients  $C_{n1}$  for an infinite cylinder, for various values of  $Pn$   
( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $Ko = 9$ ;  $Lu_p = 500$ ;  $Bi_q = 20$ ;  $Bi_m = 20$ )

$Pn$	$C_{n1}$		$C_{n2}$		$C_{n3}$	
	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
0.05	8.32360	-0.69299	$0.13364 \cdot 10^{-2}$	$0.41055 \cdot 10^{-2}$	0.21575	-6.45129
0.25	3.47660	-3.04799	$-0.26994 \cdot 10^{-2}$	$-0.60954 \cdot 10^{-2}$	0.17656	0.88483
0.45	2.40914	-1.65495	$-0.31505 \cdot 10^{-2}$	$-0.34154 \cdot 10^{-2}$	0.15132	0.26007
0.65	1.88352	-1.26189	$-0.41800 \cdot 10^{-2}$	$-0.42111 \cdot 10^{-2}$	0.13137	0.17528

TABLE 9-15

Values of constant coefficients  $D_{n1}$  for an infinite cylinder, for various values of  $Lu$  and  $Bi_q$

$Lu$	$Bi_q$	$D_{n1}$		$D_{n2}$		$D_{n3}$	
		$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
0.25	5	1.30389	0.06716	$-0.42462 \cdot 10^{-2}$	$-0.14861 \cdot 10^{-2}$	0.85624	-0.20691
	10	1.26542	0.07745	$-0.12016 \cdot 10^{-2}$	$-0.43442 \cdot 10^{-2}$	0.36099	-0.09694
	15	1.25158	0.08023	$-0.18909 \cdot 10^{-2}$	$0.10469 \cdot 10^{-2}$	0.19517	-0.05298
	20	1.24450	0.34090	$0.31573 \cdot 10^{-2}$	$-0.90740 \cdot 10^{-2}$	-0.00126	0.01535
	30	1.23732	0.27254	$0.81934 \cdot 10^{-2}$	$0.73462 \cdot 10^{-2}$	0.02958	0.15778
0.50	5	1.79540	0.62898	$-0.26552 \cdot 10^{-2}$	$-0.58820 \cdot 10^{-2}$	0.96982	4.00049
	10	1.69523	0.46461	$-0.10956 \cdot 10^{-2}$	$-0.22158 \cdot 10^{-2}$	0.44332	1.56326
	15	1.65844	0.41749	$-0.54803 \cdot 10^{-2}$	$-0.11288 \cdot 10^{-2}$	0.25693	0.86450
	20	1.63933	0.39514	$-0.26968 \cdot 10^{-2}$	$-0.60638 \cdot 10^{-2}$	0.16183	0.53244
	30	-0.64643	0.37149	$-0.20306 \cdot 10^{-2}$	$-0.50743 \cdot 10^{-2}$	-0.00112	0.19012
0.75	5	1.92278	0.40970	$-0.20340 \cdot 10^{-2}$	$-0.22022 \cdot 10^{-2}$	0.87035	1.41638
	10	1.80233	0.38367	$-0.95042 \cdot 10^{-2}$	$-0.11325 \cdot 10^{-2}$	0.42164	0.76753
	15	1.75741	0.37208	$-0.56882 \cdot 10^{-2}$	$-0.70793 \cdot 10^{-2}$	0.26238	0.50016
	20	1.73399	0.36661	$-0.37462 \cdot 10^{-2}$	$-0.48100 \cdot 10^{-2}$	0.18108	0.35497
	30	1.70994	0.35869	$-0.17824 \cdot 10^{-2}$	$-0.23952 \cdot 10^{-2}$	0.09869	0.19830
1.00	5	2.01057	0.40467	$-0.28296 \cdot 10^{-2}$	$-0.26596 \cdot 10^{-2}$	0.84055	1.07148
	10	1.87128	0.38434	$-0.13703 \cdot 10^{-2}$	$-0.14475 \cdot 10^{-2}$	0.41585	0.60517
	15	1.81899	0.37412	$-0.85309 \cdot 10^{-2}$	$-0.94585 \cdot 10^{-2}$	0.26431	0.40609
	20	1.79170	0.36815	$-0.58900 \cdot 10^{-2}$	$-0.67166 \cdot 10^{-2}$	0.18674	0.29565
	30	1.76353	0.36152	$-0.32188 \cdot 10^{-2}$	$-0.37965 \cdot 10^{-2}$	0.10810	0.17671
2.00	5	2.13881	0.40674	$-0.60551 \cdot 10^{-2}$	$-0.49707 \cdot 10^{-2}$	0.78315	0.74552
	10	1.96702	0.38994	$-0.30694 \cdot 10^{-2}$	$-0.28768 \cdot 10^{-2}$	0.39971	0.43813
	15	1.90190	0.37942	$-0.20026 \cdot 10^{-2}$	$-0.19750 \cdot 10^{-2}$	0.26217	0.30358
	20	1.86766	0.37276	$-0.14572 \cdot 10^{-2}$	$-0.14759 \cdot 10^{-2}$	0.19172	0.22852
	30	1.83232	0.36494	$-0.90378 \cdot 10^{-2}$	$-0.94139 \cdot 10^{-2}$	0.12015	0.14762

TABLE 9-16

Values of constant coefficients  $D_{n1}$  for an infinite cylinder, for various values of  $Lu_p$   
( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $Pn = 0.25$ ;  $Ko = 9$ ;  $Bi_q = 20$ ;  $Bi_m = 20$ )

$Lu_p$	$D_{n1}$		$D_{n2}$		$D_{n3}$	
	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
100	1.64199	0.39330	$-0.15883 \cdot 10^{-2}$	$-0.34867 \cdot 10^{-2}$	0.16276	0.51750
250	1.64001	0.39467	$-0.56104 \cdot 10^{-2}$	$-0.12540 \cdot 10^{-2}$	0.16201	0.52858
500	1.63933	0.39514	$-0.26098 \cdot 10^{-2}$	$-0.60638 \cdot 10^{-2}$	0.16183	0.53244
750	1.63912	0.39529	$-0.17743 \cdot 10^{-2}$	$-0.39979 \cdot 10^{-2}$	0.16176	0.53370
1 000	1.63901	0.39537	$-0.13220 \cdot 10^{-2}$	$-0.29824 \cdot 10^{-2}$	0.16173	0.53438

TABLE 9-17

Values of constant coefficients  $D_{nj}$  for an infinite cylinder, for various values of  $Bi_m$   
( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $Pn = 0.25$ ;  $Ko = 9$ ;  $Lu_p = 500$ ;  $Bi_q = 20$ )

$Bi_m$	$D_{n1}$		$D_{n2}$		$D_{n3}$	
	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
5	1.78968	0.35541	$0.48439 \cdot 10^{-3}$	$0.86386 \cdot 10^{-3}$	-0.12658	-0.49232
10	1.67548	0.43966	$0.48033 \cdot 10^{-3}$	$-0.65456 \cdot 10^{-3}$	0.07051	0.31295
15	1.63186	0.39305	$-0.92725 \cdot 10^{-3}$	$-0.25436 \cdot 10^{-3}$	0.13428	0.47846
20	1.63933	0.39514	$-0.26968 \cdot 10^{-3}$	$-0.60638 \cdot 10^{-3}$	0.16183	0.53244
30	1.58583	0.33841	$-0.22897 \cdot 10^{-3}$	$-0.32805 \cdot 10^{-3}$	0.19608	0.53080

TABLE 9-18

Values of constant coefficients  $D_{nj}$  for an infinite cylinder, for various values of  $Ko$   
( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $Pn = 0.25$ ;  $Lu_p = 500$ ;  $Bi_q = 20$ ;  $Bi_m = 20$ )

$Ko$	$D_{n1}$		$D_{n2}$		$D_{n3}$	
	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
3	1.16933	0.00656	$0.23389 \cdot 10^{-4}$	$0.69127 \cdot 10^{-4}$	0.06914	-0.38144
6	1.43500	0.29000	$-0.48264 \cdot 10^{-4}$	$0.34868 \cdot 10^{-4}$	0.12264	-0.21574
9	1.63933	0.39514	$-0.26968 \cdot 10^{-3}$	$-0.60638 \cdot 10^{-3}$	0.16183	0.53244
12	1.73721	0.35509	$-0.30264 \cdot 10^{-3}$	$-0.38523 \cdot 10^{-3}$	0.20164	0.36925

TABLE 9-19

Values of constant coefficients  $D_{nj}$  for an infinite cylinder, for various values of  $Pn$   
( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $Ko = 9$ ;  $Lu_p = 500$ ;  $Bi_q = 20$ ;  $Bi_m = 20$ )

$Pn$	$D_{n1}$		$D_{n2}$		$D_{n3}$	
	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
0.05	3.96306	-0.03437	$0.13353 \cdot 10^{-3}$	$0.40971 \cdot 10^{-3}$	0.16540	-3.12637
0.25	1.63933	0.39514	$-0.26968 \cdot 10^{-3}$	$-0.60638 \cdot 10^{-3}$	0.16183	0.53244
0.45	1.10444	0.21309	$-0.31493 \cdot 10^{-3}$	$-0.34088 \cdot 10^{-3}$	0.14419	0.19925
0.65	0.85581	0.16178	$-0.41787 \cdot 10^{-3}$	$-0.42046 \cdot 10^{-3}$	0.12737	0.14823

For a sphere ( $\Gamma=2$ )

$$\Phi_{n3} = -\frac{\sin v_j \mu_n \chi}{\chi};$$

$$A_{nj} = [1 + (1 - v_j^2) K_1] \cos v_j \mu_n - \frac{1}{Bi_q} v_j \mu_n \sin v_j \mu_n;$$

$$B_{nj} = (1 - v_j^2) \cos v_j \mu_n - \Omega_j v_j \mu_n \sin v_j \mu_n;$$

$$P_{nj} = (1 - v_j^2) \sin v_j \mu_n + \Omega_j (v_j \mu_n \cos v_j \mu_n - \sin v_j \mu_n);$$

$$Q_{nj} = \left[ 1 + (1 - v_j^2) K_1 - \frac{1}{Bi_q} \right] \sin v_j \mu_n + \frac{1}{Bi_q} v_j \mu_n \cos v_j \mu_n;$$

$$b_{nj} = v_j \cot v_j \mu_n - v_0 \cot v_0 \mu_n;$$

$$\chi_{nj} = \sigma_j \sin v_j \mu_n.$$

Values of the characteristic-equation roots  $\mu_n$  for a sphere, calculated using the "Ural" computer, are listed in Tables 9-20 and 9-21 and plotted in Figures 9-11 through 9-15. The average dimensionless transfer potentials are defined by equations (9-1-33) through (9-1-35), where

$$D_{nj} = 3C_{nj} \frac{v_j \mu_n \cos v_j \mu_n - \sin v_j \mu_n}{v_j^2 \mu_n^2}.$$

Values of constant coefficients  $C_{nj}$  and  $D_{nj}$  for the first two characteristic-equation roots are given in Tables 9-22 through 9-31.

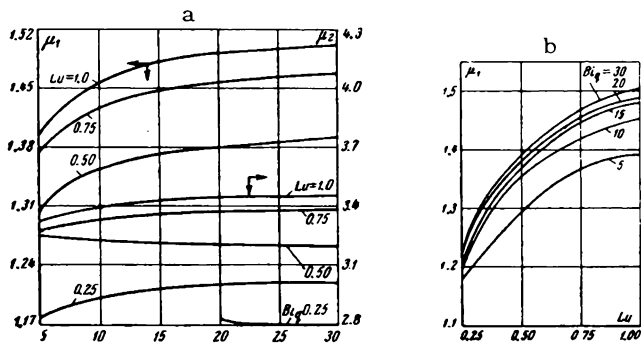


FIGURE 9-6. a) characteristic-equation roots for an infinite cylinder, as functions of  $Bi_1$ , for various values of  $Lu$ ; b) characteristic-equation roots for an infinite cylinder, as functions of  $Lu$ , for various values of  $Bi_1$

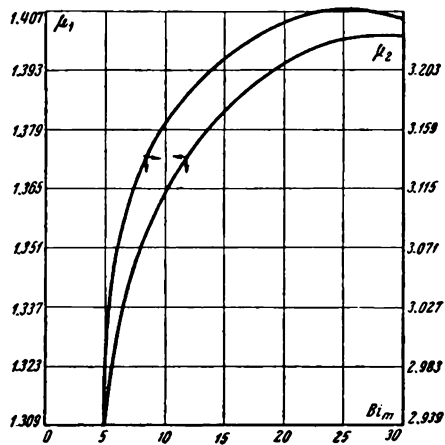


FIGURE 9-7. Characteristic-equation roots for an infinite cylinder, as functions of  $Bi_1$

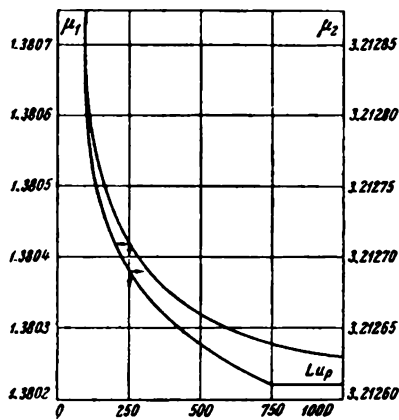


FIGURE 9-8. Characteristic-equation roots for an infinite cylinder, as functions of  $Lu$

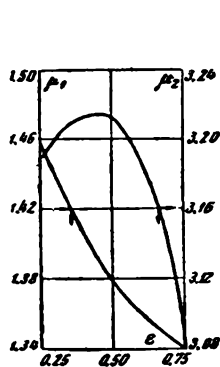


FIGURE 9-9. Characteristic-equation roots for an infinite cylinder, as functions of  $\epsilon$

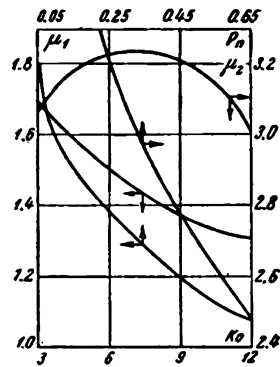


FIGURE 9-10. Characteristic-equation roots for an infinite cylinder, as functions of  $Ko$  or  $Pn$

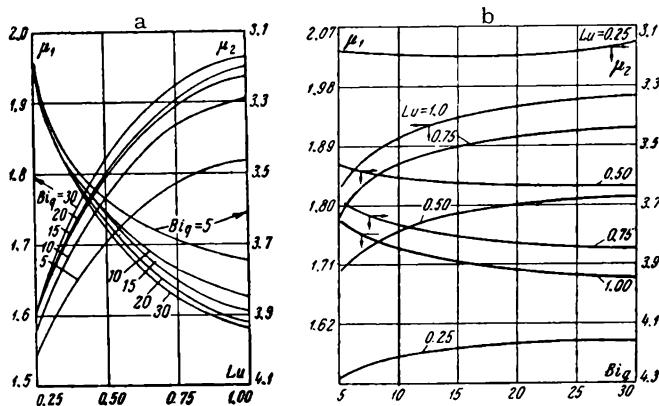


FIGURE 9-11. Characteristic-equation roots for a sphere

a) as functions of  $Lu$ , for various values of  $Bi_q$ ; b) as functions of  $Bi_q$ , for various values of  $Lu$

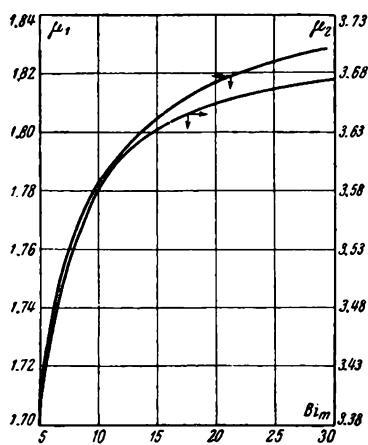


FIGURE 9-12. Characteristic-equation roots for a sphere, as functions of  $Bi_m$

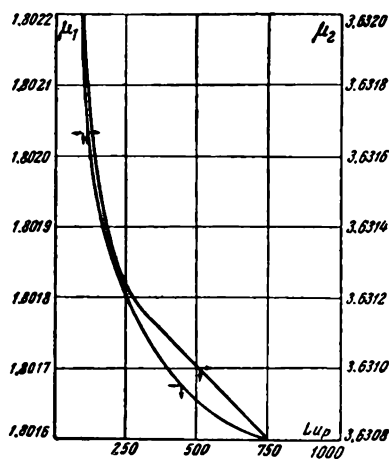


FIGURE 9-13. Characteristic-equation roots for a sphere, as functions of  $Lu_p$

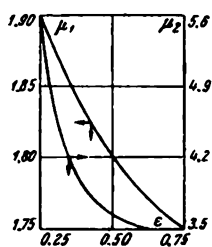


FIGURE 9-14. Characteristic-equation roots for a sphere, as functions of  $\epsilon$

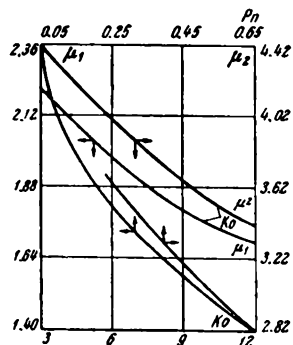


FIGURE 9-15. Characteristic-equation roots for a sphere, as functions of  $Ko$  or  $Pn$

TABLE 9-20

Sphere

Roots of characteristic equation (9-1-32), for various values of  $Lu$  and  $Bi_g$   
 ( $\epsilon = 0.5$ ;  $Ko = 9$ ;  $Pn = 0.25$ ;  $Lu_p = 500$ ;  $Bi_m = 20$ )

$Lu$	$Bi_g$	$\mu_1$	$\mu_2$	$\mu_3$
0.25	5	1.5371	3.1790	3.6256
	10	1.5707	3.1877	3.8489
	15	1.5817	3.1896	3.9186
	20	1.5870	3.152	3.3826
	30	1.5926	3.149	3.9818
0.50	5	1.6984	3.5610	6.3729
	10	1.7661	3.6060	7.0806
	15	1.7897	3.6224	7.1711
	20	1.8017	3.6309	7.2194
	30	1.8138	3.6396	4.9790
0.75	5	1.7824	3.6794	6.9708
	10	1.8626	3.7727	7.3356
	15	1.8908	3.8091	7.4943
	20	1.9051	3.8287	7.5758
	30	1.9196	3.8490	7.6767
1.00	5	1.8142	3.7460	6.0707
	10	1.9028	3.8512	6.0376
	15	1.9342	3.8942	6.0184
	20	1.9503	3.9176	6.0061
	30	1.9666	3.9423	5.9915
2.00	5	1.8611	3.8460	5.9829
	10	1.9630	3.9693	6.0571
	15	1.9997	4.0223	6.0965
	20	2.0186	4.0517	6.1210
	30	2.0379	4.0833	6.1499

TABLE 9-21

Sphere

Roots of characteristic equation (9-1-32), for various values of  $Bi_m$ ,  $Lu_p$ ,  $\epsilon$ ,  $Ko$ , and  $Pn$ \*

Dimensionless number		$\mu_1$	$\mu_2$	$\mu_3$	Dimensionless number		$\mu_1$	$\mu_2$	$\mu_3$
$Bi_m$	5	1.7034	3.3876	4.3156	$\epsilon$	0.25	1.9020	5.7124	9.5451
	10	1.7801	3.6820	4.7356		0.50	1.8017	3.6309	7.2194
	15	1.789	3.6328	4.9722		0.75	1.7518	3.5115	5.3046
	20	1.8017	3.6309	7.2194	$Pn$	0.05	2.3538	3.7291	4.7049
	30	1.8294	3.6746	7.3190		0.25	1.8017	3.6309	7.2194
$Lu_p$						0.45	1.5698	3.1494	4.7738
	100	1.8022	3.6320	7.2173	$Ko$	0.65	1.4089	2.8240	4.2562
	250	1.8018	3.6312	7.2190		3	2.2100	4.3900	6.6531
	500	1.8017	3.6309	7.2194		6	1.9747	3.9927	4.4904
	750	1.8016	3.6308	7.2196		9	1.8017	3.6309	7.2194
	1000	1.8016	3.6308	7.2197		12	1.6984	3.4108	6.7488

\* The table gives virtual variations of the dimensionless parameters, for which the following initial combination of values were assumed:  $Lu = 0.5$ ;  $Lu_p = 500$ ;  $\epsilon = 0.5$ ;  $Ko = 9$ ;  $Pn = 0.25$ ;  $Bi_g = 20$ ;  $Bi_m = 20$ .

TABLE 9-2.2

Values of constant coefficients  $C_{nq}$  for a sphere, for various values of  $Lu$  and  $Bl_q$   
 ( $\epsilon = 0.5$ ;  $Ko = 9$ ;  $Pn = 0.25$ ;  $Lu_p = 500$ ;  $Bl_m = 20$ )

Lu	Bl <sub>q</sub>	C <sub>n1</sub>			C <sub>n2</sub>			C <sub>n3</sub>		
		n=1	n=2	n=3	n=1	n=2	n=3	n=1	n=2	n=3
0.25	5	-1.1209	0.7241	-0.2129	0.6621·10 <sup>-2</sup>	0.7936·10 <sup>-2</sup>	-0.5789·10 <sup>-2</sup>	-0.8998	-2.3373	2.6150
	10	-1.1064	0.6134	-0.0899	0.1816·10 <sup>-2</sup>	0.1858·10 <sup>-2</sup>	-0.1389·10 <sup>-2</sup>	-0.3683	-0.7048	0.9099
	15	-1.1006	0.5881	-0.0661	0.2901·10 <sup>-2</sup>	0.5415·10 <sup>-2</sup>	-0.8254·10 <sup>-2</sup>	-0.1971	-0.3471	0.6231
	20	-1.02511	0.5362	-0.0533	-0.4525·10 <sup>-2</sup>	0.8451·10 <sup>-2</sup>	-0.5100·10 <sup>-2</sup>	-0.1134	-0.1989	0.3761
0.50	5	-1.0943	-0.05116	-0.3164	-0.1192·10 <sup>-2</sup>	-0.5153·10 <sup>-2</sup>	-0.3605·10 <sup>-2</sup>	-0.0295	0.4491	0.0381
	10	-1.5175	0.9075	0.1849	0.2650·10 <sup>-1</sup>	0.1507·10 <sup>-1</sup>	0.1557·10 <sup>-2</sup>	-1.1278	-1.2437	0.2904
	15	-1.4729	0.8413	0.2757	0.1038·10 <sup>-1</sup>	0.6568·10 <sup>-2</sup>	0.1506·10 <sup>-2</sup>	-0.4915	-0.6006	0.2538
	20	-1.4471	0.8155	0.2968	0.5093·10 <sup>-2</sup>	0.3444·10 <sup>-2</sup>	0.7651·10 <sup>-2</sup>	-0.2795	-0.3554	0.1714
0.75	5	-1.4379	0.8018	0.3072	0.2483·10 <sup>-2</sup>	0.1824·10 <sup>-2</sup>	0.3882·10 <sup>-2</sup>	-0.1746	-0.2263	0.1146
	10	-1.5338	0.8154	0.3222	-0.9954·10 <sup>-2</sup>	0.1638·10 <sup>-2</sup>	-0.2328·10 <sup>-2</sup>	-0.0699	-0.0926	0.7600
	15	-1.6285	0.8723	-0.0253	0.2733·10 <sup>-1</sup>	0.1254·10 <sup>-1</sup>	-0.1535·10 <sup>-2</sup>	-1.1036	-0.8452	-0.0525
	20	-1.5764	0.8455	0.2306	0.1198·10 <sup>-1</sup>	0.6301·10 <sup>-2</sup>	0.1114·10 <sup>-2</sup>	-0.5051	-0.4541	0.2852
1.00	5	-1.5560	0.8297	0.2860	0.6995·10 <sup>-2</sup>	0.3875·10 <sup>-2</sup>	0.9268·10 <sup>-2</sup>	-0.3077	-0.2931	0.2249
	10	-1.5451	0.8241	0.3079	0.4549·10 <sup>-2</sup>	0.2591·10 <sup>-2</sup>	0.6701·10 <sup>-2</sup>	-0.2099	-0.2059	0.1649
	15	-1.5338	0.8154	0.3222	0.2137·10 <sup>-2</sup>	0.1268·10 <sup>-2</sup>	0.4045·10 <sup>-2</sup>	-0.1131	-0.1143	0.1016
	20	-1.6986	0.8960	-1.1203	0.3735·10 <sup>-1</sup>	0.1576·10 <sup>-1</sup>	0.1140·10 <sup>-1</sup>	-1.1665	-0.7385	-2.0658
2.00	5	-1.6359	0.8687	-1.0331	0.1688·10 <sup>-1</sup>	0.8321·10 <sup>-2</sup>	0.8247·10 <sup>-2</sup>	-0.5422	-0.4093	-1.4526
	10	-1.6114	0.8540	-0.9627	0.1024·10 <sup>-1</sup>	0.5329·10 <sup>-2</sup>	0.6059·10 <sup>-2</sup>	-0.3365	-0.2706	-1.0562
	15	-1.5984	0.8454	-0.9127	0.6973·10 <sup>-2</sup>	0.3735·10 <sup>-2</sup>	0.4581·10 <sup>-2</sup>	-0.2348	-0.1950	-0.7980
	20	-1.5847	0.8357	-0.8501	-0.3757·10 <sup>-2</sup>	0.2076·10 <sup>-2</sup>	0.2781·10 <sup>-2</sup>	-0.1341	-0.1151	-0.4926
2.00	5	-1.8002	0.9316	-0.7244	0.7780·10 <sup>-1</sup>	0.2971·10 <sup>-1</sup>	0.1558·10 <sup>-1</sup>	-1.4143	-0.6696	-0.5605
	10	-1.7180	0.9002	-0.6829	0.3654·10 <sup>-1</sup>	0.1664·10 <sup>-1</sup>	0.1005·10 <sup>-1</sup>	-0.6715	-0.3838	-0.3711
	15	-1.6859	0.8821	-0.6606	0.2314·10 <sup>-1</sup>	0.1119·10 <sup>-1</sup>	0.7202·10 <sup>-2</sup>	-0.4282	-0.2612	-0.2727
	20	-1.6686	0.8709	-0.6468	0.1657·10 <sup>-1</sup>	0.8247·10 <sup>-2</sup>	0.5603·10 <sup>-2</sup>	-0.3084	-0.1943	-0.2126
3.00	5	-1.6505	0.8579	-0.6305	0.1011·10 <sup>-1</sup>	0.5176·10 <sup>-2</sup>	0.3699·10 <sup>-2</sup>	-0.1903	-0.1238	-0.1428



TABLE 9-23

Values of constant coefficients  $C_{nj}$  for a sphere, for various values of  $Lu_p$   
 ( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $Pn = 0.25$ ;  $Ko = 9$ ;  $Bl_q = 20$ ;  $Bl_m = 20$ )

$Lu_p$	$C_{n1}$			$C_{n2}$			$C_{n3}$		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
100	-1.4493	0.8028	0.2987	0.006509	0.004829	0.001101	-0.1768	-0.2252	0.1122
250	-1.4477	0.8020	0.3053	0.003649	0.002687	0.0005820	-0.1751	-0.2260	0.1140
500	-1.4471	0.8018	0.3072	0.002483	0.001824	0.0003882	-0.1746	-0.2263	0.1146
750	-1.4469	0.8017	0.3078	0.002002	0.001469	0.0003109	-0.1745	-0.2264	0.1148
1000	-1.4468	0.8016	0.3081	0.001722	0.001264	0.0002668	-0.1744	-0.2265	0.1148

TABLE 9-24

Values of constant coefficients  $C_{nj}$  for a sphere, for various values of  $Bl_m$   
 ( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $Pn = 0.25$ ;  $Ko = 9$ ;  $Lu_p = 500$ ;  $Bl_q = 20$ )

$Bl_m$	$C_{n1}$			$C_{n2}$			$C_{n3}$		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
5	-1.5162	0.8308	-0.2994	-0.006860	-0.004351	-0.003486	0.1478	0.2090	0.5591
10	-1.4654	0.8352	-0.3158	-0.000624	-0.000034	-0.003069	-0.0776	-0.1126	0.8521
15	-1.4426	0.7959	-0.2540	0.001208	0.000994	-0.001614	-0.1445	-0.1950	0.7582
20	-1.4471	0.8018	0.3072	0.002483	0.001824	0.000388	-0.1746	-0.2263	0.1146
30	-1.4170	0.7466	0.3320	0.002891	0.001701	0.000608	-0.2063	-0.2526	0.1643

TABLE 9-25

Values of constant coefficients  $C_{n,j}$  for a sphere, for various values of  $Ko$   
 $(Lu = 0.5; \epsilon = 0.5; Pn = 0.25; Lu_p = 500; Bl_g = 20; Bl_m = 20)$

Ko	$C_{n1}$			$C_{n2}$			$C_{n3}$		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
3	-1.0298	0.4465	-0.3279	-0.000242	-0.000289	-0.0000591	-0.0575	0.1509	0.0550
6	-1.2706	0.8023	-0.2000	0.000577	0.000906	-0.0015190	-0.1158	-0.3824	0.9403
9	-1.4471	0.8018	0.3072	0.002483	0.001824	0.0003882	-0.1746	-0.2263	0.1146
12	-1.5485	0.8115	0.3107	0.004125	0.002354	0.0005286	-0.2390	-0.2232	0.2010

TABLE 9-26

Values of constant coefficients  $C_{n,j}$  for a sphere, for various values of  $Pn$   
 $(Lu = 0.5; \epsilon = 0.5; Ko = 3; Lu_p = 500; Bl_g = 20; Bl_m = 20)$

Pn	$C_{n1}$			$C_{n2}$			$C_{n3}$		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
0.05	-3.4902	0.3098	1.4443	-0.001322	-0.002581	-0.000648	-0.1319	2.6614	0.1281
0.25	-1.4471	0.8018	0.3072	0.002483	0.001824	0.000388	-0.1746	-0.2263	0.1146
0.45	-0.9872	0.5007	-0.3771	0.004617	0.002403	0.002043	-0.1953	-0.1440	-0.2717
0.65	-0.7680	0.3841	-0.2606	0.006785	0.003374	0.002286	-0.2095	-0.1295	-0.1344

TABLE 9-27

Values of constant coefficients  $D_{nj}$  for a sphere, for various values of  $Lu$  and  $Bl_q$   
 ( $\epsilon = 0.5$ ;  $Ko = 9$ ;  $Pn = 9$ ;  $Pu = 500$ ;  $Bl_m = 20$ )

Lu	$Bl_q$	$D_{n1}$			$D_{n2}$			$D_{n3}$		
		$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
0.25	5	-1.2207	-0.3645	0.07178	0.000455	0.001127	-0.000937	-0.9048	-2.9863	3.0710
	10	-1.1684	-0.3084	0.01635	0.000128	0.000264	-0.000239	-0.3761	-0.8996	-0.9998
	15	-1.1499	-0.2956	0.00857	0.000020	0.000077	-0.000144	-0.2022	-0.4430	0.6681
	20	-1.0151	-0.2615	-0.00388	0.001040	-0.000009	0.000383	-0.1159	-0.2574	0.3795
0.50	5	-1.1310	0.0042	-0.10139	-0.000085	-0.000092	-0.000077	-0.0305	0.4703	0.0251
	10	-1.7105	-0.4575	-0.02402	0.002860	0.003384	0.006654	-1.0347	-1.6261	0.0742
	15	-1.5785	-0.4236	-0.05336	0.001161	0.001493	0.000574	-0.4648	-0.7845	0.0420
	20	-1.5301	-0.4099	-0.06380	0.000577	0.000786	0.000341	-0.2674	-0.4640	0.0218
0.75	5	-1.5050	-0.4026	-0.08822	0.000283	0.000418	0.000174	-0.1677	-0.2954	0.0123
	10	-1.4795	-0.3949	-0.04874	-0.000011	0.000038	-0.000727	-0.0675	-0.1208	0.7927
	15	-1.8273	-0.4381	0.00121	0.002181	0.002061	-0.000047	-0.9268	-1.0946	-0.0349
	20	-1.6677	-0.4256	-0.04070	0.000999	0.001062	0.000362	-0.4398	-0.5908	0.1500
1.00	5	-1.6083	-0.4168	-0.06137	0.000714	0.000659	0.000308	-0.2712	-0.3818	0.1047
	10	-1.5775	-0.4113	-0.06967	0.000477	0.000443	0.000225	-0.1862	-0.2684	0.0716
	15	-1.5458	-0.4051	-0.07638	0.000246	0.000318	0.000138	-0.1009	-0.1491	0.0402
	20	-1.9160	-0.4489	-0.3554	0.003035	0.002639	0.003080	-0.8968	-0.9281	-2.3917
2.00	5	-1.7329	-0.4375	0.00316	0.000439	0.001432	0.002215	-0.4341	-0.5194	-1.6911
	10	-1.6642	-0.4280	-0.3101	0.000887	0.000927	0.001622	-0.2732	-0.3446	-1.2335
	15	-1.6284	-0.4217	-0.2948	0.000609	0.000654	0.001224	-0.1919	-0.2488	-0.9338
	20	-1.5916	-0.4144	-0.2753	0.000331	0.000366	0.000741	-0.1104	-0.1472	-0.5777
2.00	5	-2.0475	-0.4645	-0.2312	0.006499	0.005117	0.004158	-0.8382	-0.7218	-0.7314
	10	-1.8243	-0.4537	-0.2216	0.003219	0.002957	0.002715	-0.4180	-0.4223	-0.4848
	15	-1.7395	-0.4421	-0.2151	0.002077	0.002014	0.001976	-0.2711	-0.2898	-0.3564
	20	-1.6953	-0.4341	-0.2107	0.001501	0.001496	0.001529	-0.1969	-0.2166	-0.2779
2.00	5	-1.6495	-0.4243	-0.2052	0.000925	0.000946	0.001014	-0.1226	-0.1386	-0.1868

TABLE 9-28

Values of constant coefficients  $D_{nj}$  for a sphere, for various values of  $Lu_p$   
 $(Lu = 0.5; \epsilon = 0.5; Pn = 0.25; Ko = 9; Bl_q = 20; Bl_m = 20)$

$Lu_p$	$D_{n1}$			$D_{n2}$			$D_{n3}$		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
100	-1.5076	-0.4031	-0.06610	0.001667	0.002441	0.001020	-0.1687	-0.2941	0.01510
250	-1.5057	-0.4027	-0.06674	0.000589	0.000867	0.000362	-0.1679	-0.2950	0.01296
500	-1.5050	-0.4026	-0.06822	0.000283	0.000418	0.000174	-0.1677	-0.2954	0.01227
750	-1.5048	-0.4025	-0.06838	0.000186	0.000275	0.000114	-0.1676	-0.2955	0.01205
1 000	-1.5048	-0.4025	-0.06846	0.000139	0.000205	0.000085	-0.1676	-0.2956	0.0119

TABLE 9-29

Values of constant coefficients  $D_{nj}$  for a sphere, for various values of  $Bl_m$   
 $(Lu = 0.5; \epsilon = 0.5; Pn = 0.25; Ko = 9; Lu_p = 500; Bl_q = 20)$

$Bl_m$	$D_{n1}$			$D_{n2}$			$D_{n3}$		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
5	-1.7030	-0.3995	0.06524	-0.000523	-0.000658	-0.000671	0.1360	0.2734	0.6819
10	-1.5523	-0.4211	-0.01738	-0.000050	-0.000005	-0.000648	-0.0738	-0.1472	0.9511
15	-1.4958	-0.3995	-0.04704	0.000098	0.000161	-0.000358	-0.1391	-0.2545	0.7912
20	-1.5050	-0.4026	-0.06822	0.000283	0.000418	0.000174	-0.1677	-0.2954	0.0123
30	-1.4372	-0.3716	-0.07802	0.000237	0.000279	0.000197	-0.2006	-0.3291	-0.0104

TABLE 9-30

Values of constant coefficients  $D_{n,j}$  for a sphere, for various values of  $K_0$   
( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $Pn = 0.25$ ;  $Lu_p = 500$ ;  $Bl_p = 20$ ;  $Bl_m = 20$ )

$K_0$	$D_{n,1}$			$D_{n,2}$			$D_{n,3}$		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
3	-1.0749	-0.2251	-0.10447	-0.000024	-0.000056	-0.000017	-0.0712	0.1343	-0.0135
6	-1.3129	-0.4009	0.06788	0.000051	0.000161	-0.000304	-0.1264	-0.4629	1.0022
9	-1.5050	-0.4026	-0.06822	0.000283	0.000417	0.000174	-0.1677	-0.2954	0.0123
12	-1.5787	-0.4050	-0.07033	0.000314	0.000359	0.000158	-0.2072	-0.2897	0.1035

TABLE 9-31

Values of constant coefficients  $D_{n,j}$  for a sphere, for various values of  $Pn$   
( $Lu = 0.5$ ;  $\epsilon = 0.5$ ;  $K_0 = 9$ ;  $Lu_p = 500$ ;  $Bl_p = 20$ ;  $Bl_m = 20$ )

$Pn$	$D_{n,1}$			$D_{n,2}$			$D_{n,3}$		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
0.05	-3.6632	-0.0426	-0.7282	-0.000139	-0.000429	-0.000136	-0.1704	2.8363	0.0715
0.25	-1.5050	-0.4026	-0.0682	0.000283	0.000418	0.000174	-0.1677	-0.2954	0.0123
0.45	-0.9997	-0.2493	-0.1227	0.000324	0.000338	0.000435	-0.1478	-0.1778	-0.3422
0.65	-0.7711	-0.1904	-0.0848	0.000428	0.000426	0.000434	-0.1301	-0.1414	-0.1751

On the basis of the solutions obtained in this section it is easy to solve various particular problems. For example, problems in which the value of the Biot heat-exchange number is very large ( $Bi_q = \infty$ ) are of great practical interest. Such conditions exist, for example, at the boundary between a porous wall and its surroundings during the dynamic cooling of various equipment and devices (such as powerful turbogenerators or rockets). When  $Bi_q = \infty$  heat-exchange boundary condition of the third kind (9-1-4) is transformed into the following boundary condition of the first kind:

$$T(1, Fo) = 1.$$

The general solution, in spite of the fact that the previous notation is retained, is simplified considerably due to the simplification of the quantities  $A_{nj}$  and  $Q_{nj}$  appearing in it, as well as due to the fact that the compound dimensionless number  $K_1$  is zero; for a plate, cylinder, and sphere  $A_{nj}$  and  $Q_{nj}$  are respectively:

$$\begin{aligned} \text{for } \Gamma = 0, A_{nj} &= \sin v_j \mu_n, & Q_{nj} &= \cos v_j \mu_n; \\ \text{for } \Gamma = 1, A_{nj} &= J_1(v_j \mu_n), & Q_{nj} &= J_0(v_j \mu_n); \\ \text{for } \Gamma = 2, A_{nj} &= \cos v_j \mu_n, & Q_{nj} &= \sin v_j \mu_n. \end{aligned}$$

If the particular solutions are found by means of taking limits (for example, as  $Pn \rightarrow 0$ ), then the remarks made in § 6-5 should be kept in mind.

## 9-2. Unsteady Fields of Heat-Transfer and Mass-Transfer Potential, with Boundary Conditions of the Third Kind. Mass Exchange at the Body Surface a Function of Time

During some high-intensity processes the specific flux of the mass being removed from the material varies continually with time. In such cases limiting conditions (9-1-4) through (9-1-6) for a symmetrical one-dimensional body have the general form:

$$\frac{\partial T(1, Fo)}{\partial X} - Bi_q [1 - T(1, Fo)] + (1 - \epsilon) Ko Lu Ki_m(Fo) = 0; \quad (9-2-1)$$

$$-\frac{\partial \theta(1, Fo)}{\partial X} + Pn \frac{\partial T(1, Fo)}{\partial X} + \frac{Bu}{Ko} \frac{Lu_p}{Lu} \frac{\partial P(1, Fo)}{\partial X} + Ki_m(Fo) = 0; \quad (9-2-2)$$

$$P(1, Fo) = \varphi_s(Fo); \quad (9-2-3)$$

$$\frac{\partial T(0, Fo)}{\partial X} = \frac{\partial \theta(0, Fo)}{\partial X} = \frac{\partial P(0, Fo)}{\partial X} = 0; \quad (9-2-4)$$

$$T(X, 0) = F_1(X); \theta(X, 0) = F_2(X); P(X, 0) = F_3(X). \quad (9-2-5)$$

Here the dimensionless transfer potentials are defined as

$$T = \frac{t - t_*}{t_*}; \quad \theta = \frac{\theta_* - \theta}{\theta_*}; \quad P = \frac{p - p_*}{p_*}.$$

System of equations (9-1-1) through (9-1-3), subject to limiting conditions (9-2-1) through (9-2-5), can be solved by a combined application of Fourier and Laplace integral transformations, just as described in detail in § 6-4. Let us once more go through the basic steps of the solution method, using the determination of the potential fields for macroscopic-molecular transfer in an infinite plate as an example. For convenience, in the following calculations we denote the dimensionless transfer potentials as  $\theta_l$  ( $l=1, 2, 3$ ), where  $T = \theta_1$ ,  $\theta = \theta_2$ , and  $P = \theta_3$ .

First let us consider an auxiliary problem. We will determine the functions  $\theta_k(X, Fo)$  which satisfy system of equations (9-1-1) through (9-1-3), conditions (9-2-3) through (9-2-5), and also the conditions

$$\theta_k(1, Fo) = \varphi_k(Fo) \quad (k=1, 2), \quad (9-2-6)$$

the latter being taken, for the time being, instead of boundary conditions (9-2-1) and (9-2-2). Next, after applying successively finite Fourier cosine transformation (6-4-6) with respect to the variable  $X$  and Laplace integral transformation (6-4-9) with respect to  $F_0$ , we obtain, taking into account formulas (6-4-8) and (6-4-10), the following system of algebraic equations in terms of the transforms:

$$\begin{aligned} (s + \Lambda_n^2) \{\Theta_i\}_{cL} + \epsilon \text{Ko } s \{\Theta_s\}_{cL} &= \{F_i\}_c + \epsilon \text{Ko} \{F_s\}_c + (-1)^n \Lambda_n \{\varphi_i\}_L; \\ -\Lambda_n^2 \text{Lu Pn} \{\Theta_i\}_{cL} + (s + \Lambda_n^2 \text{Lu}) \{\Theta_s\}_{cL} - \Lambda_n^2 \text{Lu}_p \frac{\text{Bu}}{\text{Ko}} \{\Theta_s\}_{cL} &= \\ = \{F_s\}_c + (-1)^n \Lambda_n \text{Lu} \left[ \{\varphi_s\}_L - \text{Pn} \{\varphi_i\}_L - \frac{\text{Lu}_p}{\text{Lu}} \cdot \frac{\text{Bu}}{\text{Ko}} \{\varphi_s\}_L \right]; \\ -\epsilon \frac{\text{Ko}}{\text{Bu}} s \{\Theta_s\}_{cL} + (s + \Lambda_n^2 \text{Lu}_p) \{\Theta_s\}_{cL} &= -\epsilon \frac{\text{Ko}}{\text{Bu}} \{F_s\}_c + \{F_s\}_c + (-1)^n \Lambda_n \text{Lu}_p \{\varphi_s\}_L, \end{aligned}$$

where

$$\begin{aligned} \{\varphi_i\}_L &= \int_0^{\infty} \varphi_i(F_0) \exp(-s F_0) d F_0; \\ \{F_i\}_c &= \int_0^1 F_i(X) \cos \Lambda_n X d X; \\ \Lambda_n &= \left(n + \frac{1}{2}\right) \pi. \end{aligned}$$

When this system of equations is solved for  $\{\Theta_i\}_{cL}$  and the transform solution inverted, we obtain

$$\Theta_i(X, F_0) = \sum_{s=1}^3 \sum_{j=1}^3 [A_{sj}^i P_{sj} + B_{sj}^i Q_{sj}] \quad (i = 1, 2, 3). \quad (9-2-7)$$

Here  $A_{sj}^i$  and  $B_{sj}^i$  are certain constants which depend on the dimensionless parameters involved in system of equations (9-1-1) through (9-1-3). The definitions of these resemble those of the corresponding coefficients introduced in § 6-4:

$$P_{sj} = 2 \sum_{n=0}^{\infty} \exp(-\Lambda_n^2 \nu_j F_0) \int_0^1 F_s(\xi) \cos \Lambda_n \xi d \xi \cos \Lambda_n X; \quad (9-2-8)$$

$$Q_{sj} = 2 \sum_{n=0}^{\infty} \exp(-\Lambda_n^2 \nu_j F_0) F_0 (-1)^n \Lambda_n \varphi_s(F_0) \cos \Lambda_n X; \quad (9-2-9)$$

where  $\Lambda_n^2 \nu_j$  are the roots of the cubic equation

$$(\beta_s) s^3 + (\Lambda_n^2 \beta_s) s^2 + (\Lambda_n^4 \beta_1) s + \Lambda_n^6 = 0,$$

in which

$$\begin{aligned} \beta_1 &= 1 + (1 - \epsilon) \frac{1}{\text{Lu}} + \frac{1}{\text{Lu}_p} + \epsilon \text{Ko Pn}; \\ \beta_2 &= (1 - \epsilon) \frac{1}{\text{Lu}} + \left(1 + \frac{1}{\text{Lu}} + \epsilon \text{Ko Pn}\right) \frac{1}{\text{Lu}_p}; \\ \beta_3 &= \frac{1}{\text{Lu Lu}_p}. \end{aligned}$$

In this case it is assumed that the determinant constructed from the coefficients (the dimensionless parameters) in the system of differential equations for the transforms is not zero:

$$\begin{vmatrix} s + \Lambda_n^2 & \epsilon \text{Ko } s & 0 \\ -\Lambda_n^2 \text{Lu Pn} & s + \Lambda_n^2 \text{Lu} & -\Lambda_n^2 \text{Lu}_p \frac{\text{Bu}}{\text{Ko}} \\ 0 & -\epsilon \frac{\text{Ko}}{\text{Bu}} s & s + \Lambda_n^2 \text{Lu}_p \end{vmatrix} \neq 0.$$

The symbol  $\overset{\text{Fo}}{\star}$  [in equation (9-2-9)] once again denotes the convolution operation. The auxiliary problem is now solved. The second stage, the determination of the functions  $\varphi_k(\text{Fo})$  ( $k=1, 2$ ), in no way differs from the analogous stage of the solution considered in § 6-4, a.

Let us write (9-2-8) and (9-2-9) in terms of the theta function:

$$P_{s,j} = \frac{1}{2} \int_0^1 \left\{ \vartheta_s \left[ \frac{X-\xi}{2}, \nu_j \text{Fo} \right] + \vartheta_s \left[ \frac{X+\xi}{2}, \nu_j \text{Fo} \right] \right\} F_s(\xi) d\xi;$$

$$Q_{s,j} = \int_0^{\text{Fo}} \frac{\partial}{\partial X} \vartheta_s \left[ \frac{1-X}{2}, \nu_j (\text{Fo} - \text{Fo}^*) \right] \varphi_s(\text{Fo}^*) d\text{Fo}^*;$$

$$\vartheta_s(X, \text{Fo}) = \frac{1}{\sqrt{\pi \text{Fo}}} \sum_{n=-\infty}^{\infty} (-1)^n \exp \left[ -\frac{(X+n)^2}{\text{Fo}} \right].$$

If we differentiate (9-2-7) with respect to  $X$  and replace the second derivative of  $\vartheta_s$  with respect to  $X$  by its derivative with respect to  $\text{Fo}$ , then we obtain

$$\frac{\partial \Theta_l(X, \text{Fo})}{\partial X} = \sum_{s=1}^3 \sum_{j=1}^3 (A_{s,j}^l M_{s,j} + B_{s,j}^l N_{s,j}) \quad (l=1, 2, 3), \quad (9-2-10)$$

where

$$M_{s,j}(X, \text{Fo}) = \frac{1}{2} \int_0^1 \frac{\partial}{\partial X} \left\{ \vartheta_s \left[ \frac{X-\xi}{2}, \nu_j \text{Fo} \right] + \vartheta_s \left[ \frac{X+\xi}{2}, \nu_j \text{Fo} \right] \right\} F_s(\xi) d\xi;$$

$$N_{s,j}(X, \text{Fo}) = \frac{1}{\nu_j} \int_0^{\text{Fo}} \frac{\partial}{\partial (\text{Fo} - \text{Fo}^*)} \vartheta_s \left[ \frac{1-X}{2}, \nu_j (\text{Fo} - \text{Fo}^*) \right] \varphi_s(\text{Fo}^*) d\text{Fo}^*. \quad (9-2-11)$$

Since as  $\text{Fo}$  approaches zero the derivative  $\frac{\partial}{\partial (\text{Fo} - \text{Fo}^*)} \vartheta_s$  goes to infinity as  $\text{Fo}^{-3/2}$ , when  $X=1$  and  $\text{Fo} \rightarrow \text{Fo}^*$ , boundary conditions (9-2-1) and (9-2-2) may be replaced by others which are equivalent to them. In order to obtain these, we integrate (9-2-1) and (9-2-2) with respect to  $\text{Fo}$  from 0 to 1:

$$\int_0^{\text{Fo}} \frac{\partial \Theta_1(l, \text{Fo}^*)}{\partial X} d\text{Fo}^* - \text{Bi}_s \int_0^{\text{Fo}} [1 - \Theta_1(l, \text{Fo}^*)] d\text{Fo}^* + (1 - \epsilon) \text{Ko Lu} \int_0^{\text{Fo}} \text{Ki}_m(\text{Fo}^*) d\text{Fo}^* = 0; \quad (9-2-12)$$

$$-\int_0^{\text{Fo}} \frac{\partial \Theta_2(l, \text{Fo}^*)}{\partial X} d\text{Fo}^* + \text{Pe}_n \int_0^{\text{Fo}} \frac{\partial \Theta_1(l, \text{Fo}^*)}{\partial X} d\text{Fo}^* + \frac{\text{Bu Lu}_2}{\text{Ko Lu}} \int_0^{\text{Fo}} \frac{\partial \Theta_2(l, \text{Fo}^*)}{\partial X} d\text{Fo}^* +$$

$$+\int_0^{\text{Fo}} \text{Ki}_m(\text{Fo}^*) d\text{Fo}^* = 0. \quad (9-2-13)$$

Now, if we integrate (9-2-10) with respect to  $\text{Fo}$ , we obtain

$$\int_0^{\text{Fo}} \frac{\partial \Theta_l(X, \text{Fo}^*)}{\partial X} d\text{Fo}^* = \sum_{s=1}^3 \sum_{j=1}^3 (A_{s,j}^l M_{s,j}^* + B_{s,j}^l N_{s,j}^*) \quad (l=1, 2, 3), \quad (9-2-14)$$

where

$$M_{s,j}^*(X, \text{Fo}) = \int_0^{\text{Fo}} M_{s,j}(X, \text{Fo}^*) d\text{Fo}^*, \quad N_{s,j}^*(X, \text{Fo}) = \int_0^{\text{Fo}} N_{s,j}(X, \text{Fo}^*) d\text{Fo}^*.$$

Then, by integrating (9-2-11) with respect to  $\text{Fo}$  and changing the order of integration by means of



Dirichlet's formula, we obtain a relation which indicates that for all  $X < 1$  it is true that

$$N^{\circ}_{\alpha j}(X, \omega) = \frac{1}{v_j} \int_0^{\omega} \varphi_{\alpha}(Fo^{\circ}) \vartheta_{\alpha} \left[ \frac{1-X}{2}, v_j(\omega - Fo^{\circ}) \right] dFo^{\circ}. \quad (9-2-15)$$

Direct calculations on the basis of (9-2-15) show that this relation is also true for  $X = 1$ , that is,

$$N^{\circ}_{\alpha j}(1, \omega) = \frac{1}{v_j} \int_0^{\omega} \varphi_{\alpha}(Fo^{\circ}) \vartheta_{\alpha} [0, v_j(\omega - Fo^{\circ})] dFo^{\circ}. \quad (9-2-16)$$

The latter integral converges, since as  $Fo^{\circ} \rightarrow \omega$  the function  $\vartheta_{\alpha}$  goes to infinity as  $\omega^{-1/2}$ , for  $\omega \rightarrow 0$ .

If we now use relations (9-2-14) through (9-2-16) to form the corresponding terms in equations (9-2-12) and (9-2-13), taking into account (9-2-6), then we obtain the following system of Volterra integral equations of the first kind:

$$\sum_{\beta=1}^2 \int_0^{Fo} K_{\alpha\beta}(Fo, Fo^{\circ}) \varphi_{\beta}(Fo^{\circ}) dFo^{\circ} = g_{\alpha}(Fo) \quad (\alpha = 1, 2), \quad (9-2-17)$$

where the functions  $K_{\alpha\beta}(Fo, Fo^{\circ})$  and  $g_{\alpha}(Fo)$  have a form which makes it possible to reduce system of equations (9-2-17) to a system of generalized Abel integral equations, and then to convert the latter, in turn, to a system of Volterra integral equations of the second kind:

$$\varphi_{\alpha}(Fo) = \sum_{\beta=1}^2 \chi_{\alpha\beta}(Fo, Fo^{\circ}) \varphi_{\beta}(Fo^{\circ}) dFo^{\circ} = \omega_{\alpha}(Fo) \quad (\alpha = 1, 2). \quad (9-2-18)$$

The solution of the problem at hand is obtained by finding  $\varphi_1(Fo)$  and  $\varphi_2(Fo)$  from (9-2-18) and then substituting them into equations (9-2-7) and (9-2-9).

### 9-3. The Reduction of the System of Equations Describing Macroscopic-Molecular Transfer to a System of "Uncoupled" Parabolic-Type Equations

We showed in Chapter V that it is possible to reduce system of differential heat-transfer and mass-transfer equations (4-1-2) and (4-1-3) to a system of two "uncoupled" equations similar to the heat-conduction equation. The latter equations enable us to extend the experience accrued in solving differential equations for uncoupled transfer to a new group of problems. Smirnov /26/ indicated the possibility of applying Henry's method of solving systems of equations describing macroscopic-molecular heat and mass transfer. If this is done, the system of differential equations reduces to a system of three ordinary heat-conduction equations for the combined function  $Z_j = p_j T + q_j \theta + r_j P$ , in which  $p_j$ ,  $q_j$ , and  $r_j$  are given numbers.

Let us now apply these methods to the reduction of the three-dimensional system of differential equations describing macroscopic-molecular transfer. We have the system of equations

$$\frac{\partial T}{\partial Fo} = \nabla^2 T - \varepsilon Ko \frac{\partial \theta}{\partial Fo}; \quad (9-3-1)$$

$$\frac{\partial \theta}{\partial Fo} = Lu \nabla^2 \theta - Lu P n \nabla^2 T - Lu_p \frac{Bu}{Ko} \nabla^2 P; \quad (9-3-2)$$

$$\frac{\partial P}{\partial Fo} = Lu_p \nabla^2 P + \varepsilon \frac{Ko}{Bu} \cdot \frac{\partial \theta}{\partial Fo}. \quad (9-3-3)$$

By determining  $\nabla^2 T$  from equation (9-3-1) and  $\nabla^2 P$  from equation (9-3-3) and substituting these into equation (9-3-2), we obtain

$$\nabla^2 T = \frac{\partial T}{\partial F_0} + \varepsilon K_0 \frac{\partial \theta}{\partial F_0}; \quad (9-3-4)$$

$$\nabla^2 \theta = \left[ \frac{1}{Lu} + \varepsilon K_0 Pn - \frac{\varepsilon}{Lu} \right] \frac{\partial \theta}{\partial F_0} + Pn \frac{\partial T}{\partial F_0} + \frac{Bu}{Lu K_0} \frac{\partial P}{\partial F_0}; \quad (9-3-5)$$

$$\nabla^2 P = \frac{1}{Lu_p} \frac{\partial P}{\partial F_0} - \frac{\varepsilon}{Lu_p} \frac{K_0}{Bu} \frac{\partial \theta}{\partial F_0}. \quad (9-3-6)$$

If we now multiply equations (9-3-4), (9-3-5), and (9-3-6) respectively by the numbers  $p$ ,  $q$ , and  $r$  and add them, we have

$$\begin{aligned} \nabla^2 \{pT + q\theta + rP\} = & \frac{\partial}{\partial F_0} \left\{ (p + q Pn) T + \left[ q \left( \frac{1}{Lu} + \varepsilon K_0 Pn - \frac{\varepsilon}{Lu} \right) + \right. \right. \\ & \left. \left. + p \varepsilon K_0 - r \frac{\varepsilon}{Lu_p} \frac{K_0}{Bu} \right] \theta + \left( q \frac{Bu}{Lu K_0} + r \frac{1}{Lu_p} \right) P \right\}. \end{aligned} \quad (9-3-7)$$

Equations (9-3-7) will be converted into an equation of the heat-conduction type for the function  $Z = pT + q\theta + rP$ , provided the following conditions are satisfied:

$$\begin{aligned} \frac{p + q Pn}{p} = & \frac{q \left( \frac{1}{Lu} + \varepsilon K_0 Pn - \frac{\varepsilon}{Lu} \right) + p \varepsilon K_0 - r \frac{\varepsilon}{Lu_p} \frac{K_0}{Bu}}{q} = \\ = & \frac{q \frac{Bu}{Lu K_0} + r \frac{1}{Lu_p}}{q} = \nabla^2. \end{aligned} \quad (9-3-8)$$

Let us now determine the numbers  $\nabla^2$ ,  $p$ ,  $q$ , and  $r$  and express them in terms of the dimensionless parameters entering into system of equations (9-3-1) through (9-3-3).

When we eliminate  $\frac{q}{p}$ ,  $\frac{r}{p}$ , and their reciprocals from equations (9-3-8), we obtain a cubic equation in  $\nabla^2$ :

$$\nabla^6 - \beta_1 \nabla^4 + \beta_2 \nabla^2 - \beta_3 = 0, \quad (9-3-9)$$

the solution of which gives three values for  $\nabla^2$ :

$$\nabla_j^2 = \Lambda_j + \frac{\beta_1}{3} \quad (j=1, 2, 3),$$

where  $\Lambda_j$ ,  $\beta_1$ ,  $\beta_2$ , and  $\beta_3$  are defined by the respective expressions in § 9-1. Let us consider here only the case when the roots of the cubic equation in  $\nabla_j^2$  are real and unequal, that is, when

$$\frac{1}{4} \Pi_2^2 + \frac{1}{27} \Pi_1^3 < 0.$$

By inserting successively the numbers  $\nabla_1^2$ ,  $\nabla_2^2$ , and  $\nabla_3^2$  into equations (9-3-8), we obtain six independent equations, enabling the calculation of six ratios or nine independent numbers. If in these equations we set  $p_1$ ,  $q_1$ , and  $r_1$  equal to unity, then we obtain

$$\begin{aligned} p_1 = 1; \quad q_1 = \frac{\nabla_1^2 - 1}{Pn}; \quad r_1 = \frac{Lu_p Bu Pn}{(\nabla_1^2 Lu_p - 1) Lu K_0 (\nabla_1^2 - 1)}; \\ p_2 = \frac{Pn}{\nabla_2^2 - 1}; \quad q_2 = 1; \quad r_2 = \frac{Lu_p Bu}{(\nabla_2^2 Lu_p - 1) Lu K_0}; \end{aligned}$$

$$p_s = \frac{P_n \text{Lu}_p \text{Bu}}{(v_3^2 - 1)(v_3^2 \text{Lu}_p - 1) \text{Lu Ko}}; \quad q_s = \frac{(v_3^2 \text{Lu}_p - 1) \text{Lu Ko}}{\text{Lu}_p \text{Bu}}; \quad r_s = 1.$$

Consequently, system of equations (9-3-1) through (9-3-3) can be rewritten as the equivalent system of "uncoupled" equations

$$\nabla^2 Z_j = v_j^2 \frac{\partial}{\partial F_0} Z_j \quad (j=1, 2, 3). \quad (9-3-10)$$

Here, if we define  $\frac{1}{v_1^2} = K_{11}$ ;  $\frac{1}{v_2^2} = K_{22}$ , and  $\frac{1}{v_3^2} = K_{33}$ , we finally obtain

$$\frac{\partial}{\partial F_0} Z_1 = K_{11} \nabla^2 Z_1; \quad (9-3-11)$$

$$\frac{\partial}{\partial F_0} Z_2 = K_{22} \nabla^2 Z_2; \quad (9-3-12)$$

$$\frac{\partial}{\partial F_0} Z_3 = K_{33} \nabla^2 Z_3. \quad (9-3-13)$$

The solution of equations (9-3-11) through (9-3-13), subject to the corresponding limiting conditions, leads to solutions for the combined function  $Z_j$ . If we denote these solutions as  $f_j$  ( $j=1, 2, 3$ ), then each of the functions  $T$ ,  $\theta$ , and  $P$  can be expressed directly in terms of the functions  $f_1$ ,  $f_2$  and  $f_3$ :

$$T = \frac{\Delta_T}{\Delta}; \quad \theta = \frac{\Delta_\theta}{\Delta}; \quad P = \frac{\Delta_P}{\Delta}, \quad (9-3-14)$$

where

$$\Delta = \begin{vmatrix} 1 & q_1 & r_1 \\ p_2 & 1 & r_2 \\ p_3 & q_3 & 1 \end{vmatrix} \neq 0;$$

$$\Delta_T = \begin{vmatrix} f_1 & q_1 & r_1 \\ f_2 & 1 & r_2 \\ f_3 & q_3 & 1 \end{vmatrix}; \quad \Delta_\theta = \begin{vmatrix} 1 & f_1 & r_1 \\ p_2 & f_2 & r_2 \\ p_3 & f_3 & 1 \end{vmatrix}; \quad \Delta_P = \begin{vmatrix} 1 & q_1 & f_1 \\ p_2 & 1 & f_2 \\ p_3 & q_3 & f_3 \end{vmatrix}.$$

The solution of the problem in the form (9-3-14) is of practical interest when calculating the rates of variation of the corresponding transfer potentials. It makes it possible to determine quite simply the extent of the effects of individual factors on the process development and to derive the simplified analytical expressions which are so necessary in engineering practice.

#### 9-4. Effects of the Dimensionless Parameters on Macroscopic-Molecular Heat and Mass Transfer

In our study of molecular heat and mass transfer (§ 6-6) we analyzed the effects of a large number of similarity criteria [dimensionless parameters] on the distribution of the dimensionless potentials. The modification of the transfer mechanism which accompanies the formation of a macroscopic flux of matter and heat changes, in turn, the nature of the effects which some of the above-considered dimensionless parameters have on the transfer process. For example, the Posnov and Kossovich numbers are no longer self-similar with respect to the fields of certain potentials. For high-intensity

processes some new dimensionless parameters, which are characteristic only of macroscopic transfer, begin to be important. A comparison of the effects of the individual parameters when molecular and macroscopic processes exist together gives us a better understanding of the nature of the parameter itself, which is very important for a correct interpretation of these new and highly complex phenomena.

The parameters  $Bi_q$  and  $Bi_m$  for surface heat and mass exchange have the same general effect on the heat and mass transfer as that observed during the molecular process alone. Small values of the Biot number correspond to undeveloped potential fields, and the transfer of mass and energy is gradual. For Biot numbers from ten to, say, fifty or sixty, marked potential gradients appear in the body, due to which the redistribution of heat and mass is intensified. In the ordered-regime stage ( $Fo > 0.7$ ) the Biot heat-exchange number affects only the heat transfer (Figure 9-16, a), whereas the Biot mass-exchange number affects only the mass transfer (Figure 9-16, b).

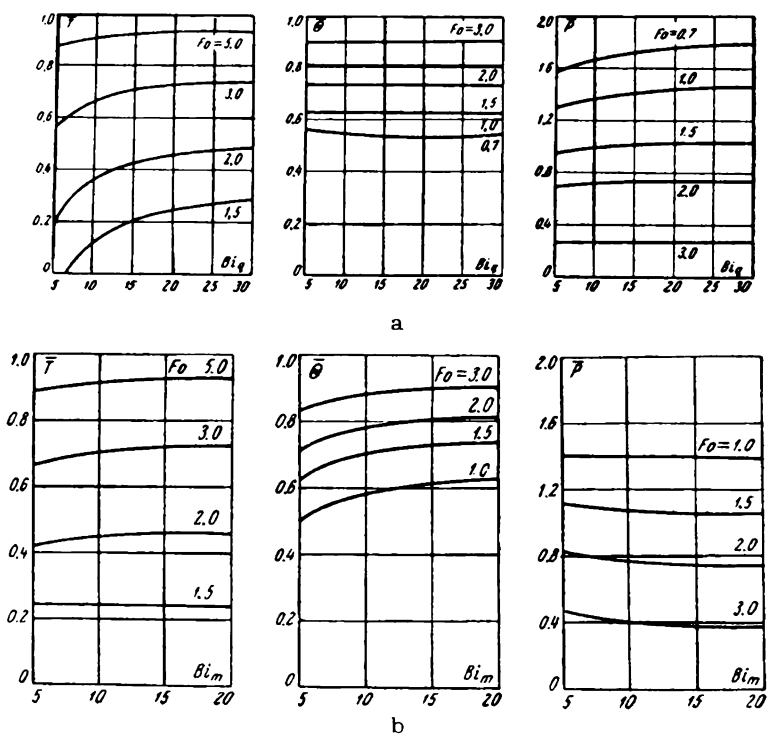


FIGURE 9-16. Transfer potentials  
a) as functions of  $Bi_q$ ; b) as functions of  $Bi_m$

The field of the mass-transfer seepage potential becomes self-similar with respect to both parameters. The fact that the Biot number is independent of the seepage potential, and consequently of the macroscopic transfer as well, compels us to assume that the surface heat-exchange and mass-exchange numbers are coupled mainly to the molecular transfer mechanism. Figure 9-16 shows

that the field of the potential  $\bar{\theta}$  is affected less by  $Bi_m$  than the field of the potential  $\bar{T}$  is affected by  $Bi_q$ . This is due to the fact that the propagation velocity of the field of  $\bar{\theta}$  is lower than that of the field of  $\bar{T}$ , for  $Lu < 1$ .

An analysis of the solutions obtained shows that in the ordered-regime stage the rates of variation of the mass-transfer potentials ( $d\theta/dFo$  and  $dP/dFo$ ) decrease linearly with increasing  $\theta$  and are independent of  $Bi_q$ .

The roles of the individual dimensionless parameters can be better understood if we consider the process in the three-dimensional space  $(\bar{T}, \bar{\theta}, \bar{P})$ . As we know, with the passing of time the temperature and the mass-transfer potentials  $\theta$  and  $P$  vary simultaneously; here, the variations in these potentials are coupled. This set of three potentials gives a more correct picture of the real transfer conditions and thus enables a better determination of the optimum technological conditions for the process. In a three-dimensional system  $(\bar{T}, \bar{\theta}, \bar{P})$ , the heat and mass transfer is characterized by some curve, each point of which corresponds to a generalized

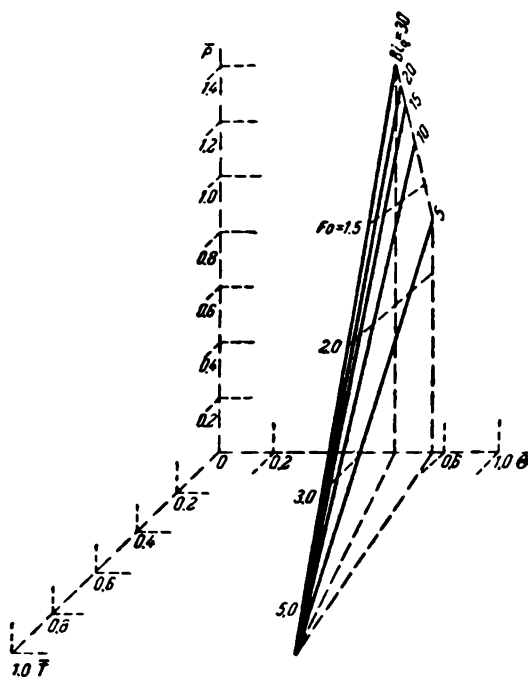


FIGURE 9-17. Heat and mass transfer curves for various values of  $Bi_q$ , plotted in  $(\bar{T}, \bar{\theta}, \bar{P})$  space

dimensionless time ( $Fo$ ). If the set of dimensionless parameters does not vary with time, then the curve describing the process represents a single-valued function of  $\bar{T}$ ,  $\bar{\theta}$ , and  $\bar{P}$ . By varying one of the parameters, we shift the process curve, thus sweeping out the process surface describing the dimensionless parameter under consideration. Figure 9-17 shows the process surface for  $Bi_q$ . It is evident from the figure that when the process depends on the mass-transfer surface potential, the transfer development follows

a straight line (for  $Lu=0.3$ ;  $Lu_p=500$ ;  $Bi_m=20$ ;  $Pn=0.25$ ;  $Ko=9$ ;  $\epsilon=0.7$ ;  $Bu=1 \cdot 10^{-3}$ ), whereas the process surface for  $Bi_q$  corresponds to a surface which approximates that of a plane triangle. The higher  $Bi_q$  is, the higher will be the average temperature of the material and the mass-transfer seepage potential.

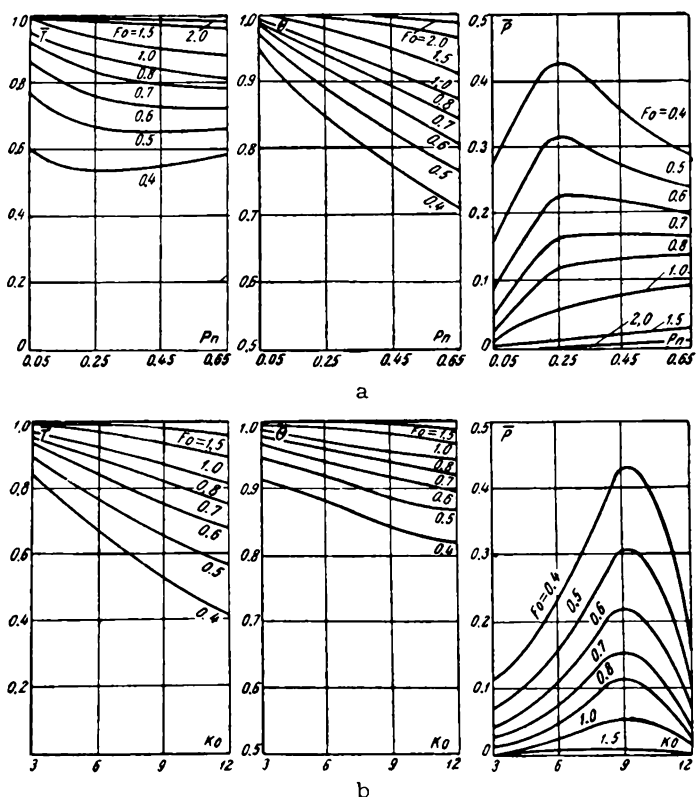


FIGURE 9-18. Transfer potentials for a sphere  
a) as functions of  $Pn$ ; b) as functions of  $Ko$

Consequently the process will be intensified and its duration shortened. The lines of intersection of the process surface with the coordinate surfaces each represent the dependence of an individual potential on a given dimensionless parameter, when the two other potentials are held constant. The fact that the function is single-valued makes it possible, in turn, to observe the effect of the individual potential on the dimensionless parameter, and on this basis to compare theoretical conclusions with experiment. For instance, the variation in the surface heat-exchange and mass-exchange numbers as  $\bar{\theta}$  increases shows a good qualitative agreement with the experimental results of Lebedev /14/.

The presence in the material of macroscopic along with molecular mass transfer modifies the previously observed dependence of the distributions of the dimensionless potentials on  $Pn$  and  $Ko$ , as well as their dependence on the phase-transition ratio  $\epsilon$  (compare Figure 6-61 with Figure 9-18).

Calculations carried out for a sphere (Figure 9-18, a) show that, beginning with  $Pn = 0.25$  for low values of  $Fo$ , the average temperature of the body rises with an increase in the Posnov number. However, as time goes by, this effect weakens and for  $Fo > 0.6$  just the opposite effect is observed. For an infinite plate (Figure 9-19) this dependence shows up much more strongly,

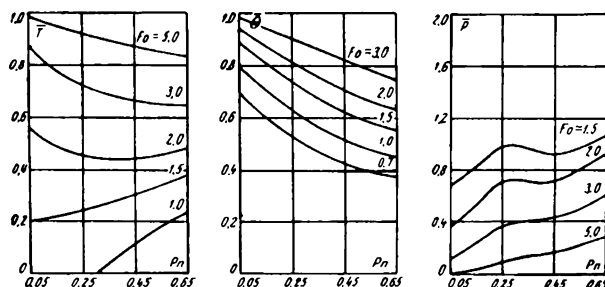


FIGURE 9-19.  $\bar{T}$ ,  $\bar{\theta}$ , and  $\bar{P}$  as functions of the Posnov number, for an infinite plate

and for low  $Fo$  extends over the entire range of variation of  $Pn$ ; the drop in average temperature with increasing  $Pn$  here occurs much later than for a sphere.

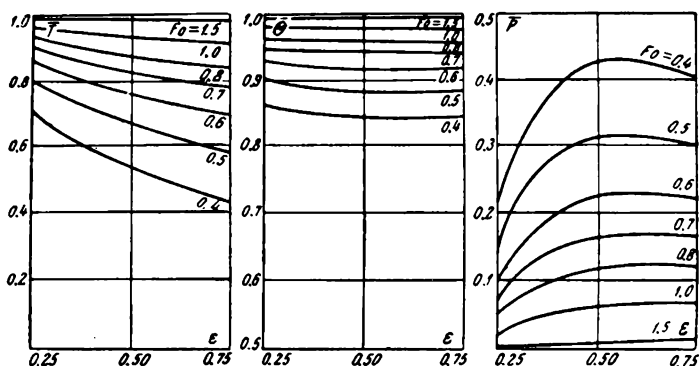


FIGURE 9-20. Transfer potentials for a sphere, as functions of the phase-transition ratio

Certain special features may be observed in the distribution of the mass-transfer seepage potential with respect to  $Pn$ . For both an infinite plate and a sphere, at  $Pn = 0.25$  and low  $Fo$  a maximum is observed, the magnitude of which decreases with time. For a sphere the seepage potential at first increases, until  $Pn = 0.25$  and  $Fo = 0.7$ , and then decreases. Above  $Fo = 0.7$ , the average mass-transfer seepage potential increases. If we recall the analogous distribution of  $\bar{P}$  for an infinite plate, we may conclude that the observed drop in  $\bar{P}$  for low  $Fo$  is local, that is, that it only continues up to some certain value of the Posnov number. The dependence of the mass-transfer seepage potential on  $Pn$ , together with the new aspect of the effect

which  $P_n$  has on the temperature distribution, indicates that the Posnov number is related to both the molecular and the macroscopic transfer mechanisms. Similar conclusions can be drawn with regard to the Kossovich number (Figure 9-18, b) and the phase-transition ratio (Figure 9-20). In relation to this, it is important to note that variations of all these dimensionless parameters ( $P_n$ ,  $Ko$ , and  $\epsilon$ ) lead to the appearance of an extremum in the distribution of the seepage potential  $P$ . The extremum is more marked for  $P_n$  and  $Ko$  and less marked (except for  $Fo < 0.5$ ) for  $\epsilon$ . The physical significance of these extrema will be clear only after a further investigation of the macroscopic-molecular heat and mass transfer.

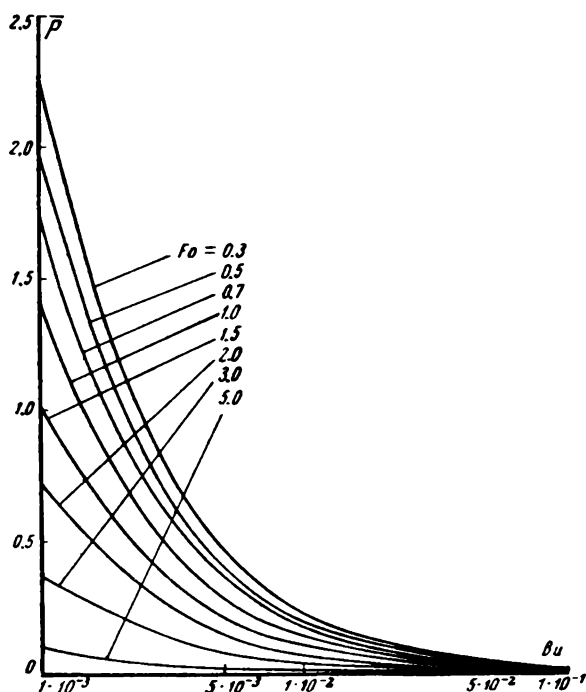


FIGURE 9-21. Mass-transfer seepage potential as a function of  $Bu$

The establishment of an intensive macroscopic transfer in the material primarily affects the redistribution of mass, since the redistribution of heat is actually a result of the mass transfer. The Posnov number, which for molecular transfer describes the internal mass-exchange processes, must indicate the appearance of the new mechanism more sharply than the Kossovich number, which describes the internal heat-exchange processes. An analysis of the solutions of system of equations (9-1-1) through (9-1-3) confirms this: the variation which took place for  $P_n$  as a function of  $\bar{\theta}$  was greater than that for  $Ko$  as a function of  $\bar{\theta}$ . As  $P_n$  increases, the dimensionless mass-transfer potential decreases. This indicates that, during equal time intervals, at lower Posnov numbers a greater amount of mass is removed from the material.  $Ko$  and  $\epsilon$  primarily affect the temperature



field. In this respect the previously mentioned resemblance of the Kossovich number to the Biot heat-exchange number is preserved.

The Bulygin number  $Bu = \frac{P_0}{c_q} \cdot \frac{P_0}{t_0 - t_0}$  is a new dimensionless parameter which is specific to macroscopic-molecular heat and mass transfer. This number is related to the seepage type of vapor transfer. It characterizes the relative heat loss during the evaporation of the part of the liquid from which the vapor participating in the macroscopic motion of the vapor-gas mixture is formed. Thus, just as does the Kossovich number, Bu is a measure of the accumulation capacity of the body and is one of the group of heat-transfer and mass-transfer numbers. The Bulygin number refers only to macroscopic transfer, and therefore it influences only the distribution of the pressure potential  $P$  (has no effect on the distributions of  $T$  and  $\Theta$ ). The mass-transfer seepage potential is shown in Figure 9-21 as a function of Bu. The Bulygin number varies over a wide range, sometimes by 3 or 4 orders of magnitude. The potential  $P$  is inversely proportional to Bu.

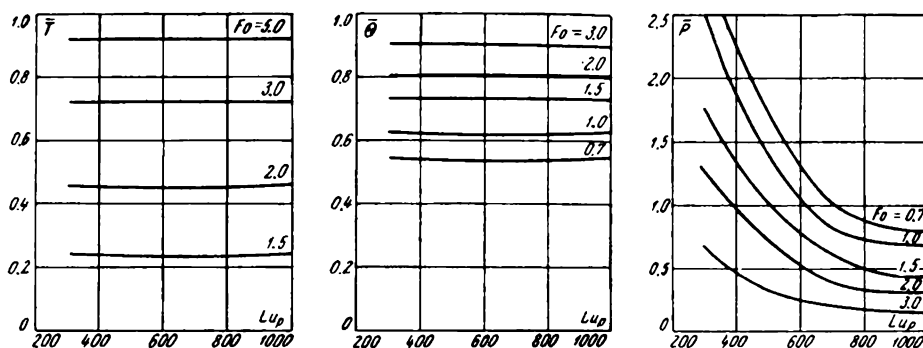


FIGURE 9-22. The transfer potentials as functions of  $Lu_p$ .

Another dimensionless parameter which is macroscopic in nature is  $Lu_p$ , a quantity characterizing the propagation velocity of the field of  $P$  in comparison with the development of the temperature field.  $Lu_p$  is thus a measure of the coupling between these fields, or of the inertia of the mass-transfer seepage potential fields compared to that of the temperature field. Figure 9-22 shows that  $Lu_p$  is self-similar with respect to  $\bar{T}$  and  $\bar{\Theta}$ . This dimensionless parameter also varies over a wide range. For example, for a moist dispersed medium, it varies from several hundred for convective heating to several tens of thousands for high-frequency heating.

The nature of the effect which the parameter  $Lu_p$ , describing the coupling between heat and mass transfer, has on the potentials  $T$  and  $\Theta$  is in part the same as before. For  $Lu < 1.0$  the development of the fields of mass-transfer potential lags behind that of the temperature fields, whereas for  $Lu > 1.0$  just the opposite is true. A linear relation between  $T$  and  $\Theta$  is observed at individual points of the body, and increases in temperature and in the dimensionless mass-transfer potential take place smoothly. At the same time, the macroscopic transfer introduces many new factors. First of all, the

symmetry boundary, with respect to the rates of development of the fields of  $T$  and  $\Theta$ , is shifted still further toward values lower than  $Lu = 1.0$ .

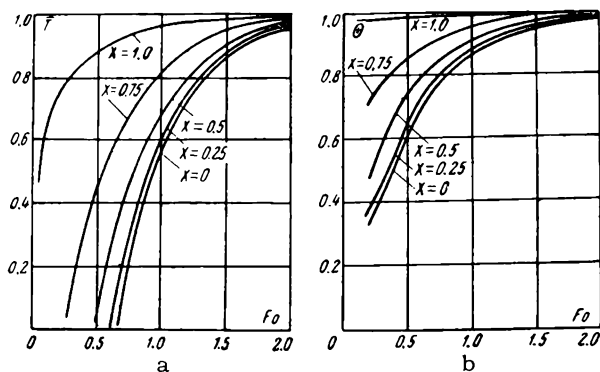


FIGURE 9-23. Variations of the fields of potentials  $T$  and  $\Theta$  in a sphere, for  $Lu = 1.0$

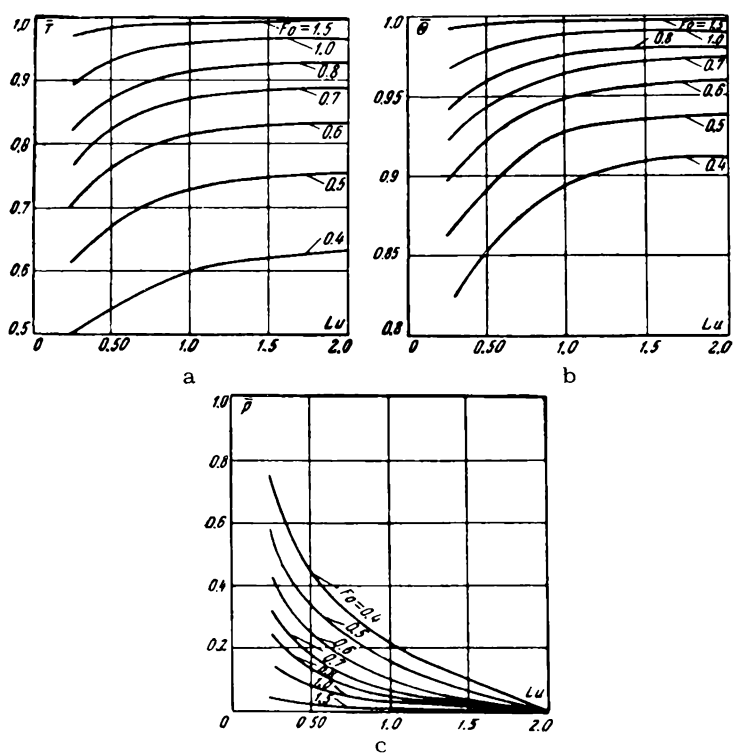


FIGURE 9-24. Dimensionless transfer potentials as functions of  $Lu$

In other words, for  $Lu = 1.0$  the development of the field of mass-transfer potential (Figure 9-23, b) already leads considerably the development of

the temperature field (Figure 9-23, a). This situation is partly due to the way in which the dimensionless temperature and mass-transfer potential are functions of  $Lu$ . A very rapid rise in average temperature is observed up to  $Lu$  from 0.5 to 0.7 (Figure 9-24, a), while the dimensionless mass-transfer potential rises rapidly up to  $Lu$  from 0.7 to 1.0 (Figure 9-24, b). The rise in the potential  $\theta$  is more significant than the rise in  $T$ .

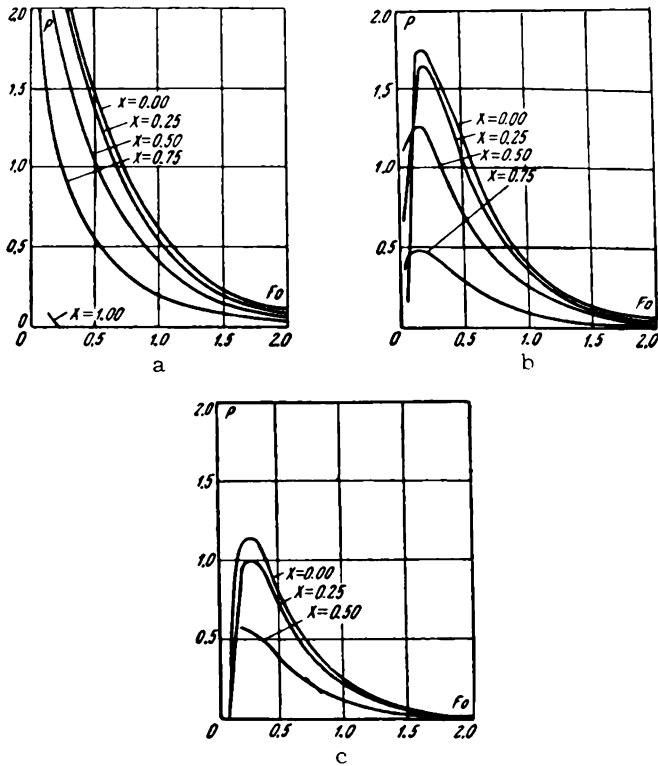


FIGURE 9-25. Evolution of the fields of mass-transfer seepage potential  
a)  $Lu = 0.5$ ; b)  $Lu = 1.0$ ; c)  $Lu = 2.0$

With an increase in  $Lu$ , the seepage potential  $\bar{P}$  decreases. For  $Lu > 1.7$  its effect becomes weaker, while for  $Lu > 2.0$  it is already negligible (Figure 9-24, c). The parameter  $Lu$  has a substantial effect on the evolution of the fields of mass-transfer seepage potential. Figures 9-25 and 9-26 show that at the beginning of the process the mass-transfer seepage potential increases rapidly, attains a maximum, and then begins to decrease. The lower the value of  $Lu$ , the higher will be the rate of increase of  $P$ . This potential goes through a maximum for lower  $Fo$  and the absolute value of the maximum is greater for lower  $Lu$ . Later in the process (for large  $Fo$ ) the potential relaxes completely. For small  $Lu$  the rise in excess pressure occurs so rapidly that the rising part of the  $P$  curve (Figure 9-25, a) practically follows the ordinate axis.

Intensive heating of the material from surface to center causes a gradient of an unrelaxed [persistent] mass-transfer seepage potential to appear,

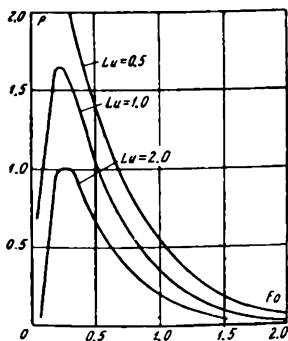


FIGURE 9-26. Variation in the potential  $P$  for a sphere ( $X=0.25$ ), for various values of  $Lu$

primarily in the layers near the surface. However, the very low resistance of the skeleton to seepage transfer makes the forming extremum of  $P$  a highly unstable one. Therefore, although in these layers the maximum of  $P$  is formed earlier, it is quite small and begins to relax sooner. Further heating of the body causes an excess pressure to appear in all the layers which are more remote from the body surface, and the increase in the resistance to seepage transfer sets up conditions under which its absolute value increases further. This is evident from Figure 9-25, b. The reason for the early formation of sharp, high maxima for low  $Lu$  is indicated by a study of the dynamics of the temperature-field development. It is just at low  $Lu$  that the most intensive evolution of the temperature field takes place, leading to rapid heating of the body, to an intensification of the phase transitions, and consequently to a greater increase in the seepage potential and in its stability.

This pattern of variation of the fields of mass-transfer seepage potential has been confirmed by the experiments of Lebedev and Zuev [10, 15], Zhmakin [9], Shubin [25], and others. Figure 9-27 gives the results of measurements of the excess pressure at the center and near the surface of a wood sample, for various temperatures of the liquid medium (after Zuev and Lebedev).

In conclusion, let us note that analytical studies show very clearly how, depending on  $Lu$ , the zone in which the pressure corresponding to that of the surroundings penetrates more deeply into the material. Whereas for low  $Lu$  ( $Lu < 0.6$ ) zero excess pressure is easily maintained throughout the whole process, already at  $Lu = 1.0$  the "zero zone" moves rapidly into the interior of the material at the beginning of the process and then returns slowly to the surface as  $\bar{P}$  decreases. The latter fact still requires experimental verification.

We have considered the effects of the basic dimensionless parameters on macroscopic-molecular heat and mass transfer. The foregoing analysis makes it possible to estimate the portion of the effect which each of the given parameters has on the variation of the potentials  $T$ ,  $\theta$ , and  $P$ . Such an overall evaluation is represented graphically in Figure 9-28. On the basis of these curves, it is possible to propose simplified equations, in terms of the dimensionless parameters, for the description and calculation of high-intensity heat and mass transfer. The general form of these equations, for the average potentials, is

$$\left. \begin{aligned} \bar{T} &= \bar{T}(Lu, Bi_g, Ko, Pn, \epsilon, Fo); \\ \bar{\theta} &= \bar{\theta}(Lu, Bi_m, Ko, Pn, \epsilon, Fo); \\ \bar{P} &= \bar{P}(Lu, Lu_p, Bu, Ko, Pn, \epsilon, Fo). \end{aligned} \right\} \quad (9-4-1)$$

Thus, for example, experimental data on the kinetics of the high-intensity

heating of various moist dispersed media should be presented in the form of equations (9-4-1). Sometimes certain of the dimensionless parameters

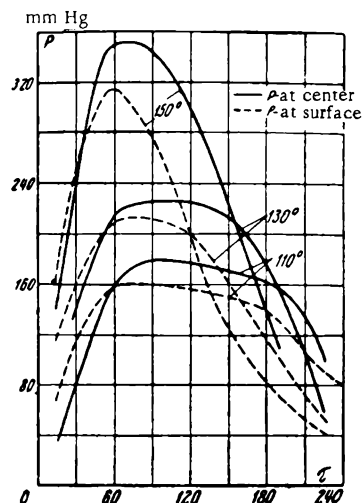


FIGURE 9-27. Variation of excess pressure in wood, for various temperatures of the liquid medium (paraffin), according to the experiments of Lebedev and Zuev

in equations (9-4-1) form compound parameters, just as for molecular heat and mass transfer, in which case (9-4-1) may be simplified even more.

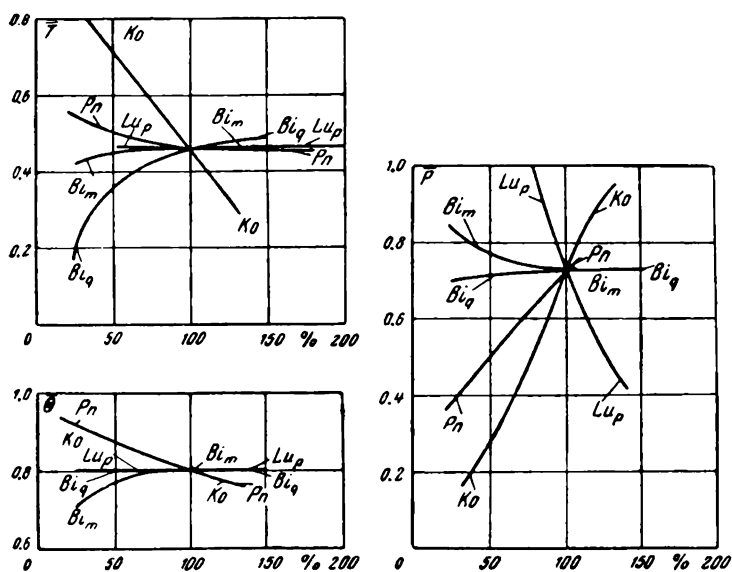


FIGURE 9-28. Overall evaluation of the effects of individual dimensionless parameters on the heat-transfer and mass-transfer potentials

## 9-5. Heat and Mass Transfer in the Case of a Pressure Drop

Let us consider an application of macroscopic-molecular heat and mass transfer. In various branches of industry pressure drops are utilized to intensify technological processes (such as drying, crushing and grinding, and processes designed to improve the physicomachanical properties of a material; see /16/ and /27/ through /30/). For example, when drying is accompanied by a pressure drop, the drying rate is some tens, and sometimes even some hundreds, of times greater than the rate of high-temperature convective drying; at the same time, the strength properties of the material are improved considerably. The physical basis for the various applications of pressure drops is the fact that such a drop enables full utilization of the effects of an intensive macroscopic vapor transfer and of its motive force, namely the persistent total-pressure gradient. At the beginning of the process the material is heated under pressure, after which the pressure is rapidly reduced. Due to the heat accumulated throughout the whole mass of the body, a violent vaporization process takes place, which leads in turn to the formation of a powerful macroscopic vapor flux. Depending on the particular requirements, it is possible to regulate the boiling and the macroscopic transfer in such a way that either up to 40% of the bound matter in the liquid phase is removed from the material together with the vapor (an effect similar to mechanical dehydration) or else the process effectively destroys or modifies the structure of the material.

The mathematical theory describing the transfer in the case of a pressure drop is especially interesting, since experimental studies of such problems are quite difficult. The analytical relationships which follow from theory facilitate the study of the process in general and the determination of the transfer coefficients in particular. On the basis of experimental results, together with the general assumptions of the above phenomenological theory describing heat and mass transfer, the system of nonlinear differential equations (see § 2-3), as applied to transfer phenomena in the case of a pressure drop, can be generalized as follows:

$$c_m \gamma \frac{\partial t}{\partial \tau} = \operatorname{div}(\lambda_q \nabla t) + \epsilon' \rho c_m \gamma \frac{\partial \theta}{\partial \tau} + \sum_i c_{pi} \lambda_{pi} \nabla p \nabla t; \quad (9-5-1)$$

$$c_m \gamma \frac{\partial \theta}{\partial \tau} = \operatorname{div} \left[ \sum_{i=1}^2 (\lambda_{mi} \nabla \theta + \lambda_{mi} \delta_i \nabla t + \lambda_{pi} \nabla p) \right]; \quad (9-5-2)$$

$$c_{ps} \gamma \frac{\partial p}{\partial \tau} = \operatorname{div}(\lambda_{ps} \nabla p) - m b_T \frac{\partial t}{\partial \tau} - \epsilon' c_m \gamma \frac{\partial \theta}{\partial \tau}, \quad (9-5-3)$$

where  $m$  is the porosity of the material;  $b_T$  is a coefficient characterizing the rate of expansion of vapor in the capillaries due to heating (for an ideal gas  $m b_T = -\gamma c_{ps} \frac{273+t}{p}$ );  $\epsilon'$  is the phase-transition ratio in the case of a pressure drop; the subscript  $i=1$  corresponds to the liquid phase, while  $i=2$  corresponds to the vapor phase of the bound matter;  $\lambda_{pi} = k$  is the coefficient of air conduction; and  $c_{pa} = c_a$  [where the subscript  $a$  stands for air].

The phase-transition ratio  $\epsilon'$  in the case of a pressure drop is related to the ordinary phase-transition ratio  $\epsilon$  as follows:  $\epsilon' = (1-\kappa)\epsilon$ . The parameter  $\kappa$  characterizes the thermomechanical entrainment of liquid bound matter

from the material; for  $\kappa=0$  there is no entrainment of liquid by the vapor, while for  $\kappa=1$  the dehydration of the material is, at least theoretically, completely determined by thermomechanical entrainment effects. In practice, as shown by experiment,  $\kappa$  lies between 0.05 and 0.4.

Since the dominant factor in the process when there is a pressure drop is the macroscopic mass transfer, therefore

$$\sum_{i=1}^2 \lambda_{pi} \nabla p \gg \sum_{i=1}^2 (\lambda_{mi} \nabla \theta + \lambda_{mi} \delta_i \nabla t).$$

In addition, an evaluation of the convective-heat-transfer term  $\sum_i c_{pi} \lambda_{pi} \nabla p \nabla t$ ,

in comparison with the other terms of heat-transfer equation (9-5-1), shows that the convection term is less than 2.5% of the phase-transition term. In order to derive a complete solution of the system of equations, it is also necessary to assume that the thermodynamic characteristics and the transfer coefficients are constant. Taking into consideration the above remarks, we obtain system of equations (9-5-1) through (9-5-3) for a one-dimensional body in the following dimensionless form:

$$\frac{\partial T(X, F_0)}{\partial F_0} = \left[ \frac{\partial^2 T(X, F_0)}{\partial X^2} + \frac{\Gamma}{X} \cdot \frac{\partial T(X, F_0)}{\partial X} \right] + s' K_0 \frac{\partial \theta(X, F_0)}{\partial F_0}; \quad (9-5-4)$$

$$\frac{\partial \theta(X, F_0)}{\partial F_0} = Ra Lu_p (1 + \lambda^*_{p1}) \left[ \frac{\partial^2 P(X, F_0)}{\partial X^2} + \frac{\Gamma}{X} \cdot \frac{\partial P(X, F_0)}{\partial X} \right]; \quad (9-5-5)$$

$$\begin{aligned} \frac{\partial P(X, F_0)}{\partial F_0} = & Lu_p \left[ \frac{\partial^2 P(X, F_0)}{\partial X^2} + \frac{\Gamma}{X} \cdot \frac{\partial P(X, F_0)}{\partial X} \right] - \\ & - Le \frac{\partial T(X, F_0)}{\partial F_0} - \frac{s'}{Ra} \cdot \frac{\partial \theta(X, F_0)}{\partial F_0}, \end{aligned} \quad (9-5-6)$$

where for constant initial conditions

$$T = \frac{t_s - t}{t_{c0} - t_s}; \quad \theta = \frac{\theta_s - \theta}{\theta_s}; \quad P = \frac{p_s - p}{p_s};$$

and where  $Lu_p = \frac{a_{p2}}{a_q}$  ( $a_{p2} = \lambda_{p2}/c_{p2}\gamma$  is the coefficient describing the seepage potential conductivity for the vapor); and  $Ra = \gamma c_{p2} p_0 / \gamma c_m \theta_0$  is the Ramzin number, characterizing the vapor content for macroscopic transfer, expressed as a fraction of the mass content of the matter participating in the molecular transfer. The Ramzin number replaces here the previously introduced Bulygin number Bu and is related to the latter by the equation  $Ra = Bu/K_0$ . The parameter  $Le = mb_T (t_{c0} - t_s) / \gamma c_{p2} p_0$  is the Lebedev number, representing the ratio between the flux due to molecular expansion and the total flux for the macroscopic transfer. The dimensionless parameter  $\lambda^*_{p1} = \lambda_{p1}/\lambda_{p2}$  is the ratio between the macroscopic mass conductivity for the liquid phase and that for the vapor phase, while the subscript 0 denotes the initial state of the body and the subscript c refers as usual to conditions in the surroundings.

Various kinds of pressure drops can be produced. For example, if it is desired to harden the material as well as to dry it, then the duration of the pressure drop should be long, while, conversely, if the main goal is to obtain the maximum effect of the pressure drop in order to dehydrate or to destroy the material, then the drop should be rapid, with a duration of some seconds or tens of seconds. The nature of the pressure drop and the corresponding temperature drop in the medium can be approximated by a linear or

exponential function:

$$p_c(\tau) = p_0 - e_p \tau \quad \text{and} \quad t_c(\tau) = t_{c0} - e_T \tau$$

or

$$p_c(\tau) = p_0 \exp(-e_p \tau) \quad \text{and} \quad t_c(\tau) = t_{c0} \exp(-e_T \tau).$$

The boundary conditions for the mathematical model of heat and mass transfer represented by (9-5-4) through (9-5-6) can be obtained from the laws of conservation of matter and energy. If the foregoing remarks are taken into account, then for symmetrical distributions of the dimensionless potentials these conditions have the form:

for a linear pressure drop,

$$\begin{aligned} \frac{\partial T(1, Fo)}{\partial X} + Bi_q [1 + T(1, Fo) - Pd_q Fo] - (1 - \kappa - \\ - \epsilon') Ko Ra Lu_p (1 + \lambda_p^*) \frac{\partial P(1, Fo)}{\partial X} = 0; \end{aligned} \quad (9-5-7)$$

$$P(1, Fo) = Lu_p Pd_p Fo; \quad (9-5-8)$$

for an exponential pressure drop,

$$\begin{aligned} \frac{\partial T(1, Fo)}{\partial X} + Bi_q [1 + T(1, Fo) + \Phi (1 - \exp(-Pd'_q Fo))] - \\ - (1 - \kappa - \epsilon') Ko Ra Lu_p (1 + \lambda_p^*) \frac{\partial P(1, Fo)}{\partial X} = 0; \end{aligned} \quad (9-5-9)$$

$$P(1, Fo) = 1 - \exp(-Lu_p Pd'_p Fo). \quad (9-5-10)$$

Here  $Pd_q = \frac{e_T R^2}{a_q (t_{c0} - t_0)}$  and  $Pd_p = \frac{e_p R^2}{a_{p1} p_0}$  are the Predvoditelev heat-exchange and mass-exchange numbers, characterizing the variations in surrounding temperature and in pressure for a linear pressure drop ( $e_p$  and  $e_T$  are the rates of variation of pressure and temperature);  $Pd'_q = \frac{e_T R^2}{a_q}$  and  $Pd'_p = \frac{e_p R^2}{a_{p1}}$  are the Predvoditelev heat-exchange and mass-exchange numbers for an exponential pressure drop; and  $\Phi = \frac{t_{c0}}{t_{c0} - t_0}$  is the dimensionless temperature parameter.

System of differential equations (9-5-4) through (9-5-6), subject to boundary conditions (9-5-7) and (9-5-8) or (9-5-9) and (9-5-10), and with a symmetrical distribution of the transfer potentials:

$$\frac{\partial T(0, Fo)}{\partial X} = \frac{\partial \theta(0, Fo)}{\partial X} = \frac{\partial P(0, Fo)}{\partial X} = 0,$$

can be solved using a Laplace integral transformation. The procedure for solving this system of equations is similar to the procedure for solving the differential equations for macroscopic-molecular heat and mass transfer described in § 9-1. Consequently only the final results will be given here.

For a linear pressure drop the solution of the system of differential equations, for constant initial conditions, has the form:

for an infinite plate,

$$T(X, Fo) = K_T + \sum_{n=1}^{\infty} \sum_{l=1}^2 C_{nl} M_l \cos v_l \mu_n X \exp(-\mu_n^2 Fo); \quad (9-5-11)$$

$$\theta(X, Fo) = K_\theta - Ra Lu_p (1 + \lambda_p^*) \sum_{n=1}^{\infty} \sum_{l=1}^2 C_{nl} v_l^2 \cos v_l \mu_n X \exp(-\mu_n^2 Fo); \quad (9-5-12)$$



$$P(X, Fo) = K_p - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \cos v_i \mu_n X \exp(-\mu_n^2 Fo); \quad (9-5-13)$$

for an infinite cylinder,

$$T(X, Fo) = K_r + \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} M_i J_0(v_i \mu_n X) \exp(-\mu_n^2 Fo); \quad (9-5-14)$$

$$\theta(X, Fo) = K_\theta - Ra Lu_p (1 + \lambda_p^*) \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} v_i^2 J_0(v_i \mu_n X) \exp(-\mu_n^2 Fo); \quad (9-5-15)$$

$$P(X, Fo) = K_p - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} J_0(v_i \mu_n X) \exp(-\mu_n^2 Fo); \quad (9-5-16)$$

for a sphere,

$$T(X, Fo) = K_r - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} M_i \frac{\sin v_i \mu_n X}{X} \exp(-\mu_n^2 Fo); \quad (9-5-17)$$

$$\theta(X, Fo) = K_\theta + Ra Lu_p (1 + \lambda_p^*) \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} v_i^2 \frac{\sin v_i \mu_n X}{X} \exp(-\mu_n^2 Fo); \quad (9-5-18)$$

$$P(X, Fo) = K_p + \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} \frac{\sin v_i \mu_n X}{X} \exp(-\mu_n^2 Fo). \quad (9-5-19)$$

In these solutions the following notation was used:

$$\begin{aligned} K_r &= -\left\{1 - Pd_q Fo + \frac{1}{N} Pd_q \left(1 - X^2 + \frac{2}{Bi_q}\right) - \right. \\ &\quad \left. - (Lu_p Pd_p + Le Pd_q) \frac{\epsilon' Ko Ra (1 + \lambda_p^*)}{N [1 - \epsilon' (1 + \lambda_p^*)]} \left(1 - X^2 + \frac{1 - \epsilon}{\epsilon'} \frac{2}{Bi_q}\right)\right\}; \\ K_\theta &= \frac{Ra Le (1 + \lambda_p^*)}{1 - \epsilon' (1 + \lambda_p^*)} \left[-1 - \frac{1}{N} Pd_q \left(1 - X^2 + \frac{2}{Bi_q}\right) + \right. \\ &\quad \left. + \frac{Lu_p Pd_p + Le Pd_q}{Le} \left\{Fo - \frac{1}{N Lu_p [1 - \epsilon' (1 + \lambda_p^*)]} \times \right. \right. \\ &\quad \left. \left. \times \left[1 - X^2 - \epsilon' Ko Ra Le Lu_p (1 + \lambda_p^*) \left(1 - X^2 + \frac{1 - \epsilon}{\epsilon'} \frac{2}{Bi_q}\right)\right]\right\}\right]; \\ K_p &= Lu_p Pd_p Fo - \frac{Lu_p Pd_p + Le Pd_q}{N Lu_p [1 - \epsilon' (1 + \lambda_p^*)]} (1 - X^2); \\ C_{n1} &= 2 \frac{\left(1 + \frac{Pd_q}{\mu_n^2}\right) P_{n1} + \frac{Lu_p Pd_p}{\mu_n^2} Q_{n1}}{\mu_n \psi_n}; \\ C_{n2} &= -2 \frac{\left(1 + \frac{Pd_q}{\mu_n^2}\right) P_{n1} + \frac{Lu_p Pd_p}{\mu_n^2} Q_{n1}}{\mu_n \psi_n}; \\ M_i &= \frac{1 - Lu_p [1 - \epsilon' (1 + \lambda_p^*)] v_i^2}{Le}; \\ \psi_n &= v_1 A_{n1} P_{n2} + v_2 B_{n2} Q_{n1} - v_2 A_{n2} P_{n1} - v_1 B_{n1} Q_{n2}; \end{aligned} \quad (9-5-20)$$

$$v_i^2 = \frac{-V_1 + (-1)^i \sqrt{V_1^2 - 4V_2}}{2V_3} \quad (i = 1, 2); \quad (9-5-21)$$

$$V_1 = Lu_p \left[ \varepsilon' (1 + \lambda^*_p) (1 + Ko Ra Le) - \frac{1}{Lu_p} - 1 \right];$$

$$V_2 = Lu_p [1 - \varepsilon' (1 + \lambda^*_p)].$$

The characteristic-equation roots  $\mu_n$  satisfy the equation

$$Q_{ni} P_{n2} - Q_{n2} P_{ni} = 0. \quad (9-5-22)$$

For an infinite plate,

$$P_{ni} = \cos v_i \mu_n; \quad (9-5-23)$$

$$Q_{ni} = M_i \left( \cos v_i \mu_n - \frac{1}{Bi_q} v_i \mu_n \sin v_i \mu_n \right) - \\ - (1 - \kappa - \varepsilon') \frac{Ko Ra Lu_p (1 + \lambda^*_p)}{Bi_q} v_i \mu_n \sin v_i \mu_n; \quad (9-5-24)$$

$$A_{ni} = M_i \sin v_i \mu_n + [M_i + (1 - \kappa - \varepsilon') Ko Ra Lu (1 + \\ + \lambda^*_p)] \frac{\sin v_i \mu_n + v_i \mu_n \cos v_i \mu_n}{Bi_q}; \quad (9-5-25)$$

$$B_{ni} = \sin v_i \mu_n. \quad (9-5-26)$$

For an infinite cylinder,

$$P_{ni} = J_0(v_i \mu_n); \quad (9-5-27)$$

$$Q_{ni} = M_i \left[ J_0(v_i \mu_n) - \frac{1}{Bi_q} v_i \mu_n J_1(v_i \mu_n) \right] - \\ - (1 - \kappa - \varepsilon') \frac{Ko Ra Lu_p (1 + \lambda^*_p)}{Bi_q} v_i \mu_n J_1(v_i \mu_n); \quad (9-5-28)$$

$$A_{ni} = M_i J_1(v_i \mu_n) + [M_i + (1 - \kappa - \varepsilon') Ko Ra Lu_p (1 + \lambda^*_p)] \frac{v_i \mu_n J_0(v_i \mu_n)}{Bi_q}; \quad (9-5-29)$$

$$B_{ni} = J_1(v_i \mu_n). \quad (9-5-30)$$

For a sphere,

$$P_{ni} = \sin v_i \mu_n; \quad (9-5-31)$$

$$Q_{ni} = M_i \sin v_i \mu_n - [M_i + (1 - \kappa - \varepsilon') Ko Ra Lu_p (1 + \\ + \lambda^*_p)] \frac{\sin v_i \mu_n - v_i \mu_n \cos v_i \mu_n}{Bi_q}; \quad (9-5-32)$$

$$A_{ni} = M_i \cos v_i \mu_n - [M_i + (1 - \kappa - \varepsilon') Ko Ra Lu_p (1 + \lambda^*_p)] \frac{v_i \mu_n \sin v_i \mu_n}{Bi_q}; \quad (9-5-33)$$

$$B_{ni} = \cos v_i \mu_n. \quad (9-5-34)$$

The number  $N$  is 2 for an infinite plate, 4 for a cylinder, and 6 for a sphere.

The solution of system of differential equations (9-5-4) through (9-5-6) in the case of an exponential pressure drop has the same form for the infinite plate (9-5-11) through (9-5-13), for the infinite cylinder (9-5-14) through (9-5-16), and for the sphere (9-5-17) through (9-5-19). However, in the exponential case,

$$K_r = - \left\{ 1 + \vartheta + \sum_{i=1}^2 M_i [A_i^p F_i^p \exp(-Lu_p Pd'_p Fo) - \right. \\ \left. - \vartheta A_i^q F_i^q \exp(-Pd'_q Fo)] \right\};$$

$$\begin{aligned}
K_\theta &= \frac{Ra(1+\lambda^*_p)[1-Le(1+\theta)]}{1-\varepsilon'(1+\lambda^*_p)} + Lu_p Ra(1+\lambda^*_p) \sum_{i=1}^2 v_i^2 [A_i^p F_i^p \times \\
&\quad \times \exp(-Lu_p Pd'_p Fo) - \theta A_i^q F_i^q \exp(-Pd'_q Fo)]; \\
K_p &= 1 + \sum_{i=1}^2 [A_i^p F_i^p \exp(-Lu_p Pd'_p Fo) - \theta A_i^q F_i^q \exp(-Pd'_q Fo)]; \\
C_{n1} &= 2 \frac{\frac{\mu_n^2 - (1+\theta)Pd'_q}{\mu_n^2 - Pd'_q} P_{n1} + \frac{Lu_p Pd'_p}{\mu_n^2 - Lu_p Pd'_p} Q_{n1}}{\mu_n \psi_n}; \\
C_{n3} &= -2 \frac{\frac{\mu_n^2 - (1+\theta)Pd'_q}{\mu_n^2 - Pd'_q} P_{n1} + \frac{Lu_p Pd'_p}{\mu_n^2 - Lu_p Pd'_p} Q_{n1}}{\mu_n \psi_n},
\end{aligned}$$

where  $\psi_n$  and  $v_i$  are defined by equations (9-5-20) and (9-5-21);  $P_{ni}$ ,  $Q_{ni}$ ,  $A_{ni}$ , and  $B_{ni}$  are defined for a plate by (9-5-23) through (9-5-26), for a cylinder by (9-5-27) through (9-5-30), and for a sphere by (9-5-31) through (9-5-34). The characteristic-equation roots  $\mu_n$  satisfy equations (9-5-22). In addition,

$$\begin{aligned}
A_1^p &= \frac{Q_2^p}{Q_1^p P_2^p - Q_2^p P_1^p}; \quad A_2^p = -\frac{Q_1^p}{Q_1^p P_2^p - Q_2^p P_1^p}; \\
A_1^q &= \frac{P_2^q}{Q_1^q P_2^q - Q_2^q P_1^q}; \quad A_2^q = -\frac{P_1^q}{Q_1^q P_2^q - Q_2^q P_1^q}.
\end{aligned}$$

For an infinite plate, we have

$$\begin{aligned}
F_i^p &= \cos(v_i \sqrt{Lu_p Pd'_p} X); \quad F_i^q = \cos(v_i \sqrt{Pd'_q} X); \\
P_i^p &= \cos(v_i \sqrt{Lu_p Pd'_p}); \quad P_i^q = \cos(v_i \sqrt{Pd'_q}); \\
Q_i^p &= M_i [\cos(v_i \sqrt{Lu_p Pd'_p}) - \frac{1}{Bi_q} v_i \sqrt{Lu_p Pd'_p} \sin(v_i \sqrt{Lu_p Pd'_p})] - \\
&\quad - (1 - \varkappa - \varepsilon') \frac{Ko Ra Lu_p (1 + \lambda^*_p)}{Bi_q} v_i \sqrt{Lu_p Pd'_p} \sin(v_i \sqrt{Lu_p Pd'_p}); \\
Q_i^q &= M_i [\cos(v_i \sqrt{Pd'_q}) - \frac{1}{Bi_q} v_i \sqrt{Pd'_q} \sin(v_i \sqrt{Pd'_q})] - \\
&\quad - (1 - \varkappa - \varepsilon') \frac{Ko Ra Lu_p (1 + \lambda^*_p)}{Bi_q} v_i \sqrt{Pd'_q} \sin(v_i \sqrt{Pd'_q});
\end{aligned}$$

for an infinite cylinder,

$$\begin{aligned}
F_i^p &= J_0(v_i \sqrt{Lu_p Pd'_p} X); \quad F_i^q = J_0(v_i \sqrt{Pd'_q} X); \\
P_i^p &= J_0(v_i \sqrt{Lu_p Pd'_p}); \quad P_i^q = J_0(v_i \sqrt{Pd'_q}); \\
Q_i^p &= M_i J_0(v_i \sqrt{Lu_p Pd'_p}) - [M_i + (1 - \varkappa - \varepsilon') Ko Ra Lu_p (1 + \\
&\quad + \lambda^*_p)] \frac{v_i \sqrt{Lu_p Pd'_p} J_1(v_i \sqrt{Lu_p Pd'_p})}{Bi_q}; \\
Q_i^q &= M_i J_0(v_i \sqrt{Pd'_q}) - [M_i + (1 - \varkappa - \varepsilon') Ko Ra Lu_p (1 + \\
&\quad + \lambda^*_p)] \frac{v_i \sqrt{Pd'_q} J_1(v_i \sqrt{Pd'_q})}{Bi_q};
\end{aligned}$$

for a sphere,

$$\begin{aligned}
 F_i^p &= \frac{\sin(v_i \sqrt{\text{Lu}_p \text{Pd}'_p} X)}{X}; \quad F_i^q = \frac{\sin(v_i \sqrt{\text{Pd}'_q} X)}{X}; \\
 P_i^p &= \sin(v_i \sqrt{\text{Lu}_p \text{Pd}'_p}); \quad P_i^q = \sin(v_i \sqrt{\text{Pd}'_q}); \\
 Q_i^p &= M_i \sin(v_i \sqrt{\text{Lu}_p \text{Pd}'_p}) + [M_i + (1 - \kappa - \varepsilon') \text{Ko Ra Lu}_p (1 + \\
 &\quad + \lambda^*_p)] \frac{v_i \sqrt{\text{Lu}_p \text{Pd}'_p} \cos(v_i \sqrt{\text{Lu}_p \text{Pd}'_p}) - \sin(v_i \sqrt{\text{Lu}_p \text{Pd}'_p})}{\text{Bi}_q}; \\
 Q_i^q &= M_i \sin(v_i \sqrt{\text{Pd}'_q}) + [M_i + (1 - \kappa - \varepsilon') \text{Ko Ra Lu}_p (1 + \\
 &\quad + \lambda^*_p)] \frac{v_i \sqrt{\text{Pd}'_q} \cos(v_i \sqrt{\text{Pd}'_q}) - \sin(v_i \sqrt{\text{Pd}'_q})}{\text{Bi}_q}.
 \end{aligned}$$

The expressions obtained for the unsteady distributions of the dimensionless potentials in the material refer to the general case of transfer with a pressure drop. The use of these equations for calculations is associated with a large amount of computational work. Thus, it is of interest to derive approximate solutions describing the heat and mass transfer, since these will be more convenient for practical use.

Let us note first that the molecular-heat-transfer term  $\frac{\lambda_q}{c_q \gamma} \nabla^2 t$  in equations (9-5-1) is small in comparison with the phase-transition term  $\frac{\varepsilon' \rho c_m}{c_q} \cdot \frac{\partial \theta}{\partial \tau}$ . According to our experimental data, this term represents less than 1% of the heat consumed during phase transition. In addition, in the case of a pressure drop vapor formation occurs almost entirely inside the material, so that in boundary conditions (9-5-7) through (9-5-9) we have  $1 - \kappa - \varepsilon' = 0$ . If we also take into consideration that, on account of the convective heat supply in the case of a pressure drop, it is possible to extract from the material 0.5 to 0.9% of the bound matter of the entire removed mass, then we are justified in replacing the heat-exchange boundary condition of the third kind by a condition of the first kind. However, in the solutions it is enough just to use one boundary condition, since the variations in the temperature and pressure of the surroundings are coupled. For example, in the case of a linear time-variation of the pressure ( $\text{Pd}_p$ ) and the temperature ( $\text{Pd}_q$ ) we have

$$\text{Pd}_q = \frac{(1 - \kappa) \text{Ko Ra Lu}_p (1 + \lambda^*_p)}{1 - (1 - \kappa)(1 + \lambda^*_p)(1 + \text{Ko Ra Le})} \text{Pd}_p + \frac{1}{\text{Fo}}.$$

The solution of the simplified system of differential equations gives the following relations:

for a linear pressure variation,

$$T(X, \text{Fo}) = \frac{(1 - \kappa) \text{Ko Ra Lu}_p^2 (1 + \lambda^*_p) \text{Pd}_p}{V^2} F(X, \text{Fo}); \quad (9-5-35)$$

$$\Theta(X, \text{Fo}) = \frac{\text{Ra Lu}_p^2 (1 + \lambda^*_p) \text{Pd}_p}{V^2} F(X, \text{Fo}); \quad (9-5-36)$$

$$P(X, \text{Fo}) = \frac{\text{Lu}_p \text{Pd}_p}{V} F(X, \text{Fo}); \quad (9-5-37)$$

for an exponential pressure variation,

$$T(X, \text{Fo}) = \frac{(1 - \kappa) \text{Ko Ra Lu}_p (1 + \lambda^*_p)}{V} \Phi(X, \text{Fo}); \quad (9-5-38)$$

$$\theta(X, Fo) = \frac{Ra Lu_p (1 + \lambda_p^*)}{V} \Phi(X, Fo); \quad (9-5-39)$$

$$P(X, Fo) = \Phi(X, Fo), \quad (9-5-40)$$

where

$$F(X, Fo) = V Fo - \frac{1}{N} (1 - X^*) + \sum_{n=1}^{\infty} \frac{2}{\mu_n^3} f(\mu_n X) \exp(-\mu_n^2 V Fo);$$

$$\begin{aligned} \Phi(X, Fo) = & 1 - \varphi\left(\sqrt{\frac{Lu_p Pd'_p}{V}} X\right) \exp(-Lu_p Pd'_p Fo) + \\ & + \sum_{n=1}^{\infty} \frac{2}{\mu_n} \cdot \frac{Lu_p Pd'_p}{\mu_n^2 V - Lu_p Pd'_p} f(\mu_n X) \exp(-\mu_n^2 V Fo); \\ V = & Lu_p [1 - (1 - \kappa)(1 + \lambda_p^*)(1 + Ko Ra Le)]; \end{aligned}$$

for an infinite plate,

$$\begin{aligned} f(\mu_n X) = & (-1)^{n+1} \cos \mu_n X; \quad \varphi\left(\sqrt{\frac{Lu_p Pd'_p}{V}} X\right) = \frac{\cos\left(\sqrt{\frac{Lu_p Pd'_p}{V}} X\right)}{\cos\left(\sqrt{\frac{Lu_p Pd'_p}{V}}\right)}; \\ \mu_n = & (2n - 1) \frac{\pi}{2}; \end{aligned}$$

for a cylinder,

$$f(\mu_n X) = \frac{J_0(\mu_n X)}{J_1(\mu_n)}; \quad \varphi\left(\sqrt{\frac{Lu_p Pd'_p}{V}} X\right) = \frac{J_0\left(\sqrt{\frac{Lu_p Pd'_p}{V}} X\right)}{J_0\left(\sqrt{\frac{Lu_p Pd'_p}{V}}\right)};$$

and the characteristic roots  $\mu_n$  satisfy the equation

$$J_0(\mu_n) = 0;$$

for a sphere,

$$\begin{aligned} f(\mu_n X) = & (-1)^{n+1} \frac{\sin \mu_n X}{X}; \quad \varphi\left(\sqrt{\frac{Lu_p Pd'_p}{V}} X\right) = \frac{\sin \sqrt{\frac{Lu_p Pd'_p}{V}} X}{X \sin \sqrt{\frac{Lu_p Pd'_p}{V}}}; \\ \mu_n = & n\pi. \end{aligned}$$

Solutions (9-5-35) through (9-5-37) and (9-5-38) through (9-5-40) indicate that there is a definite relationship between the dimensionless potentials  $T$ ,  $\theta$ , and  $P$ . The following relation between the dimensionless temperature and mass-transfer potential is especially interesting:

$$\theta = \frac{1}{(1 - \kappa) Ko} T. \quad (9-5-41)$$

A direct determination, and still more a continuous recording, of the mass-transfer potential (or the mass content) during the pressure drop is extremely difficult. However, if  $\kappa$  as a function of  $T$  and  $\theta$  is known, then it is easy to find the approximate potential using formula (9-5-41). Conversely, if the distributions of  $\theta$  and  $T$  throughout the material are known, it is possible to use this formula to find the amount of matter in the liquid phase removed from the material during the pressure drop. In conclusion,

it should be noted that calculations using formula (9-5-41) will be more accurate, the less the final specific heat of the material differs from its initial value.

By assigning various values to the dimensionless numbers and parameters entering into the solutions, it is possible to calculate the unsteady fields of transfer potential in the case of a pressure drop. This enables us, on one hand, to find out which combinations of dimensionless parameters have the main effects on heat and mass transfer, and, on the other hand, to find ways of determining effectively the values of these compound parameters (or the transfer coefficients involved in them) from a minimum number of experiments.

#### 9-6. Heat and Mass Transfer under the Effect of Numerous Thermodynamic Forces

On the basis of our analysis of unsteady potential fields, let us go on to consider heat and mass transfer when one, two, or three thermodynamic forces are acting. In general, the transfer of energy and matter may be determined by the action of a much larger number of forces and fluxes (as in the transfer phenomena accompanying phase and chemical transformations in multicomponent systems, heat and mass transfer in anisotropic media, and transfer under the effects of electromagnetic and other forces). Therefore, in the general case Onsager's system of linear equations has the form

$$j_l = \sum_{k=1}^n L_{lk} X_k \quad (l=1, 2, \dots, n).$$

Onsager's theorem states that if the fluxes  $j_l$  and the forces  $X_k$  are properly chosen, then the matrix composed of the kinetic coefficients must be symmetric, that is,

$$L_{lk} = L_{kl} \quad (k, l=1, 2, \dots, n).$$

If we also assume that the thermodynamic forces are gradient functions of the corresponding potentials, namely  $X_k = -n_k \nabla \theta_k$ , where  $n_k$  is a proportionality factor depending on the choice of fluxes and forces (as a particular case it may be unity), then it can be shown that the system of equations describing transfer phenomena in a system at rest has the form

$$c_l \gamma \frac{\partial \theta_l}{\partial \tau} = \operatorname{div} \sum_{k=1}^n L_{lk} n_k \nabla \theta_k + w_l \quad (l=1, 2, \dots, n);$$

or, for constant transfer coefficients and characteristics,

$$\frac{\partial \theta_l}{\partial \tau} = \sum_{k=1}^n K_{lk} \nabla^2 \theta_k + \frac{w_l}{c_l \gamma} \quad (l=1, 2, \dots, n). \quad (9-6-1)$$

Here  $K_{lk}$  are the transfer coefficients for the corresponding potentials,  $w_l/c_l \gamma$  are the strengths of the additional sources, and  $c_l$  represents the capacities for the given potentials.

General system of equations (9-6-1) has been solved by a number of authors. The method for solving the one-dimensional system of four equations (three-phase state of the matter and generalized boundary conditions of the third kind) has been considered by Prudnikov /7/. The three-dimensional problem for a semi-infinite medium, with boundary conditions of the first and second kinds, was investigated by Tsoi /8/, while the two-dimensional problem with boundary conditions of the first and third kinds was investigated by Kim and Ivanova /31/. A quite general method for solving system (9-6-1) in a plane for boundary conditions of the second kind has been given by Mikhailov /32/. Finally, some other problems were considered by Zagorskii /33/ and others. The methods for solving some of the above problems will now be presented.

### 1. Semi-infinite three-dimensional medium. Boundary conditions of the first kind

Let us consider system of equations (9-6-1) for a space  $\Omega$  ( $0 \leq x < \infty$ ,  $-\infty < y < \infty$ ,  $-\infty < z < \infty$ ), with arbitrary limiting conditions of the first kind:

$$\text{for } \tau = 0 \quad \theta_i(x, y, z, \tau) = f_i(x, y, z); \quad (9-6-2)$$

$$\text{for } x = 0 \quad \theta_i(x, y, z, \tau) = \varphi_i(y, z, \tau). \quad (9-6-3)$$

In order for the solution of system (9-6-1) subject to conditions (9-6-2) and (9-6-3) to exist and to be single-valued, it is necessary to impose certain restrictions on the functions  $w_i = \frac{w}{c_1 \gamma}$ ,  $f_i$ , and  $\varphi_i$ . A class of the unknown functions  $\theta_i(x, y, z, \tau)$  will be sought for which a Fourier transformation with respect to the spatial coordinates and a Laplace transformation with respect to time will be applicable in the domain  $\Omega$ .

We assume that the functions  $w_i(x, y, z, \tau)$ ,  $f_i(x, y, z)$ , and  $\varphi_i(y, z, \tau)$  belong to the class  $L\{\Omega\}$ . In view of the above-mentioned restrictions, all the functions must be regular at infinity. In particular,  $\theta_i$  will approach zero as  $r^2 = x^2 + y^2 + z^2$  goes to infinity. If we define

$$u_i(x, \eta, \zeta, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_i(x, y, z, \tau) \exp[-i(y\eta + z\zeta)] dy dz,$$

then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\partial^2 \theta_i}{\partial y^2} + \frac{\partial^2 \theta_i}{\partial z^2} \right) \exp[-i(y\eta + z\zeta)] dy dz = -(\eta^2 + \zeta^2) u_i(x, \eta, \zeta, \tau).$$

If the Fourier sine transformation

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} u_i(x, \eta, \zeta, \tau) \sin x\xi dx = \{u_i(\xi, \eta, \zeta, \tau)\}_s = \{u_i\}_s$$

is now applied, then taking into account boundary conditions (9-6-3), obtained after the transformation, we will find, after integrating by parts twice, that

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u_i}{\partial x^2} \sin x\xi dx = -\xi^2 \{u_i\}_s + \sqrt{\frac{2}{\pi}} \Phi_i(\eta, \zeta, \tau),$$

where

$$\Phi_i(\eta, \zeta, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_i(y, z, \tau) \exp[-i(y\eta + z\zeta)] dy dz.$$

After the application of a two-coordinate Fourier integral transformation with respect to variables  $y$  and  $z$  and a single-coordinate Fourier sine transformation with respect to variable  $x$ , system of differential

equations (9-6-1) becomes, taking into account previous calculations,

$$\frac{\partial}{\partial \tau} \{u_i\}_s = - \sum_{k=1}^n K_{ik} p^s \{u_k\}_s + \sqrt{\frac{2}{\pi}} \xi \sum_{k=1}^n K_{ik} \Phi_k + \{w_i\}_s \quad (9-6-4)$$

$$(p^s = \xi^s + \eta^s + \zeta^s) \quad (i = 1, 2, \dots, n)$$

For system (9-6-4) the initial conditions will be

$$\{F_i(\xi, \eta, \zeta)\}_s = \sqrt{\frac{2}{\pi}} \xi \int_0^\infty F_i(x, \eta, \zeta) \sin x\xi dx. \quad (9-6-5)$$

Next we apply the Laplace transformation, denoting the transform as

$$L[\{u_i(\xi, \eta, \zeta, \tau)\}_s] = \int_0^\infty \{u_i\}_s \exp(-s\tau) d\tau = \{u_i\}_{sL},$$

and so, taking into account initial conditions (9-6-5), system (9-6-4) becomes

$$\sum_{k=1}^n (x_{ik}s + K_{ik}p^s) \{u_k\}_{sL} = \sqrt{\frac{2}{\pi}} \xi \sum_{k=1}^n K_{ik} \{\Phi_k\}_L + \{w_i\}_{sL} + \{F_i\}_s \quad (9-6-6)$$

$$(i = 1, 2, \dots, n),$$

where

$$x_{ik} = \begin{cases} 1 & \text{for } k = i; \\ 0 & \text{for } k \neq i. \end{cases}$$

By means of Cramer's rule we can derive from (9-6-6) the following transform solution of system of equations (9-6-1), subject to limiting conditions (9-6-2) and (9-6-3):

$$\{u_i\}_{sL} = \frac{D_i(p, s)}{D(p, s)},$$

where  $D(p, s)$  is the basic determinant of system (9-6-6):

$$D(p, s) = \begin{vmatrix} s + K_{11}p^s & K_{12}p^s & \dots & K_{1n}p^s \\ K_{21}p^s & s + K_{22}p^s & \dots & K_{2n}p^s \\ \dots & \dots & \dots & \dots \\ K_{n1}p^s & K_{n2}p^s & \dots & s + K_{nn}p^s \end{vmatrix} = ||(x_{ik}s + K_{ik}p^s)||.$$

Let us consider the numerator of the transform  $\{u_i\}_{sL}$ :

$$D_i(p, s) = \begin{vmatrix} \sqrt{\frac{2}{\pi}} \xi \sum_{k=1}^n K_{1k} \{\Phi_k\}_L + \{w_1\}_{sL} + \{F_1\}_s & K_{12}p^s & \dots & K_{1n}p^s \\ \sqrt{\frac{2}{\pi}} \xi \sum_{k=1}^n K_{2k} \{\Phi_k\}_L + \{w_2\}_{sL} + \{F_2\}_s & (s + K_{22}p^s) & \dots & K_{2n}p^s \\ \dots & \dots & \dots & \dots \\ \sqrt{\frac{2}{\pi}} \xi \sum_{k=1}^n K_{nk} \{\Phi_k\}_L + \{w_n\}_{sL} + \{F_n\}_s & K_{n2}p^s & \dots & (s + K_{nn}p^s) \end{vmatrix} =$$



$$= \sqrt{\frac{2}{\pi}} \xi \begin{vmatrix} \sum_{k=1}^n K_{1k} \{\Phi_k\}_L & K_{12}\rho^2 & \dots & K_{1n}\rho^2 \\ \sum_{k=1}^n K_{2k} \{\Phi_k\}_L & (s + K_{22}\rho^2) & \dots & K_{2n}\rho^2 \\ \dots & \dots & \dots & \dots \\ \sum_{k=1}^n K_{nk} \{\Phi_k\}_L & K_{n2}\rho^2 & \dots & (s + K_{nn}\rho^2) \end{vmatrix} +$$

$$+ \begin{vmatrix} \{w_1\}_{sL} & K_{12}\rho^2 & \dots & K_{1n}\rho^2 \\ \{w_2\}_{sL} & (s + K_{22}\rho^2) & \dots & K_{2n}\rho^2 \\ \dots & \dots & \dots & \dots \\ \{w_n\}_{sL} & K_{n2}\rho^2 & \dots & (s + K_{nn}\rho^2) \end{vmatrix} + \begin{vmatrix} \{F_1\}_s & K_{12}\rho^2 & \dots & K_{1n}\rho^2 \\ \{F_2\}_s & (s + K_{22}\rho^2) & \dots & K_{2n}\rho^2 \\ \dots & \dots & \dots & \dots \\ \{F_n\}_s & K_{n2}\rho^2 & \dots & (s + K_{nn}\rho^2) \end{vmatrix}$$

or

$$D_1(\rho, s) = \sqrt{\frac{2}{\pi}} \xi \sum_{i=1}^n \left[ A_{i1}(\rho, s) \sum_{k=1}^n K_{ik} \{\Phi_k\}_L \right] +$$

$$+ \sum_{i=1}^n \{w_i\}_{sL} A_{i1}(\rho, s) + \sum_{i=1}^n \{F_i\}_s A_{i1}(\rho, s),$$

where  $A_{i1}(\rho, s)$  denotes the cofactor of the matrix element formed by the  $i$ th row and the first column of the basic determinant.

Similarly, the determinant  $D_l(\rho, s)$  may be written as

$$D_l(\rho, s) = \sqrt{\frac{2}{\pi}} \xi \sum_{i=1}^n \left[ A_{il}(\rho, s) \sum_{k=1}^n K_{ik} \{\Phi_k\}_L \right] + \sum_{i=1}^n \{w_i\}_{sL} A_{il}(\rho, s) +$$

$$+ \sum_{i=1}^n \{F_i\}_s A_{il}(\rho, s)$$

We may now expand the transform  $\{u_i\}_{sL} = \{u_i(\xi, \eta, \zeta, s)\}_{sL}$  to

$$\{u_i\}_{sL} = \frac{\sqrt{\frac{2}{\pi}} \xi \sum_{i=1}^n [A_{il}(\rho, s) \sum_{k=1}^n K_{ik} \{\Phi_k\}_L]}{D(\rho, s)} + \frac{\sum_{i=1}^n A_{il}(\rho, s) \{w_i\}_{sL}}{D(\rho, s)} + \frac{\sum_{i=1}^n \{F_i\}_s A_{il}(\rho, s)}{D(\rho, s)}. \quad (9-6-7)$$

Let us assume that the basic determinant  $D(\rho, s)$  has real negative roots with respect to the parameter  $s$  (namely  $s_n = -c_n^2 \rho^2$ ). If even one of the roots of determinant  $D(\rho, s)$  is positive, then after performing the inversion operation with respect to the parameter  $s$  it is evident that  $\{u_i(\xi, \eta, \zeta, \tau)\} \rightarrow \infty$  as  $\tau \rightarrow \infty$  (and consequently that the function  $\{u_i\}_s$  is unstable).

It is known from the thermodynamics of irreversible processes that  $K_{li} \gg K_{il}$  for  $k \neq l$ . Obviously, for  $K_{li} \gg K_{il}$  the roots of the determinant  $D(\rho, s)$  will be negative, and for  $K_{il} \rightarrow 0$  ( $k \neq l$ ) we find that  $c_l^2 \rightarrow K_{li}$ .

In order to perform the inversion of the transform  $\{u_i\}_{sL}$  with respect to the parameter  $s$ , we will use the formula for the case when the transform is a regular fractional-rational function with first-order poles:

$$L^{-1} \left[ \frac{Q_m(s)}{R_p(s)} \right] = \sum_{n=1}^m \frac{Q_m(s_n)}{R'_p(s_n)} \exp(s_n \tau).$$

Here  $s_n$  are the simple roots of the entire function  $R_p(s)$ , and  $p > m$ . When we apply the last formula and the corresponding convolution theorem,

$$\int_0^\tau f(\tau^*) \varphi(\tau - \tau^*) d\tau^* = F(s) \Phi(s),$$

to the individual terms of (9-6-7), we obtain

$$\begin{aligned} \{u_i(\xi, \eta, \zeta, \tau)\}_s &= \sum_{m=1}^n \frac{1}{D'_s(1; -c_m^2)} \times \\ &\times \left\{ \sqrt{\frac{2}{\pi}} \int_0^{\tau} \xi \sum_{i=1}^n \left[ A_{ii}(1; -c_m^2) \sum_{k=1}^n K_{ik} \Phi_k(\eta, \zeta, \tau^*) \right] \times \right. \\ &\times \exp[-c_m^2 \rho^2 (\tau - \tau^*)] d\tau^* + \int_0^{\tau} \sum_{i=1}^n [A_{ii}(1; -c_m^2) \times \\ &\times \{w_i\}_s] \exp[-c_m^2 \rho^2 (\tau - \tau^*)] d\tau^* + \\ &\left. + \sum_{i=1}^n A_{ii}(1; -c_m^2) F_i(\xi, \eta, \zeta) \exp(-c_m^2 \rho^2 \tau) \right\}. \end{aligned}$$

Here the obvious relations

$$D'_s(\rho, s_m) = \rho^{2(n-1)} D'_s(1; -c_m^2) \quad \text{and} \quad A_{ii}(\rho, s_m) = \rho^{2(n-1)} A_{ii}(1; -c_m^2)$$

were used, and also the quantity  $\rho^{2(n-1)}$  was canceled.

When an inverse Fourier sine transformation with respect to  $\xi$  is applied to the function  $\{u_i(\xi, \eta, \zeta, \tau)\}_s$ , then, considering that a change in integration order is permissible, we obtain

$$\begin{aligned} u_i(x, \eta, \zeta, \tau) &= \sum_{m=1}^n \frac{2/\pi}{D'(1; -c_m^2)} \left\{ \int_0^{\tau} \left[ \sum_{i=1}^n A_{ii}(1; -c_m^2) \times \right. \right. \\ &\times \sum_{k=1}^n K_{ik} \Phi_k(\eta, \zeta, \tau^*) \left. \right] \exp[-c_m^2 (\eta^2 + \zeta^2)(\tau - \tau^*)] d\tau^* \times \\ &\times \int_0^{\infty} \xi \exp[-c_m^2 \xi^2 (\tau - \tau^*)] \sin x \xi d\xi + \int_0^{\tau} \int_0^{\infty} \left[ \sum_{i=1}^n A_{ii}(1; -c_m^2) \times \right. \\ &\times w_i(a, \eta, \zeta, \tau^*) \exp[-c_m^2 (\eta^2 + \zeta^2)(\tau - \tau^*)] d\tau^* d\alpha \times \\ &\times \int_0^{\infty} \exp[-c_m^2 (\tau - \tau^*) \xi^2] \sin x \xi \sin a \xi d\xi + \\ &+ \int_0^{\infty} \left[ \sum_{i=1}^n A_{ii}(1; -c_m^2) F_i(a, \eta, \zeta) \right] \exp[-c_m^2 (\eta^2 + \\ &\left. + \zeta^2) \tau] d\alpha \int_0^{\infty} \exp(-c_m^2 \xi^2 \tau) \sin \xi x \sin a \xi d\xi \right\}. \end{aligned}$$

Since the improper integrals may be evaluated using the formulas

$$\begin{aligned} \int_0^{\infty} \xi \exp(-a^2 \xi^2 \tau) \sin \xi x d\xi &= \frac{x \sqrt{\pi}}{4a^3 \sqrt{\tau^3}} \exp\left(-\frac{x^2}{4a^2 \tau}\right); \\ \int_0^{\infty} \exp(-a^2 \xi^2 \tau) \cos x \xi d\xi &= \frac{\sqrt{\pi}}{2a \sqrt{\tau}} \exp\left(-\frac{x^2}{4a^2 \tau}\right), \end{aligned}$$

and since  $2 \sin a \xi \sin x \xi = \cos(x - a) \xi - \cos(x + a) \xi$ , the inner integrals may be calculated, and so

we have

$$\begin{aligned}
 u_i(x, \eta, \zeta, \tau) = & \sum_{m=1}^n \frac{1}{2c_m \sqrt{\pi} D'(1; -c_m^2)} \left\{ \frac{x}{c_m^2} \times \right. \\
 & \times \int_0^{\tau} \sum_{i=1}^n A_{ii}(1; -c_m^2) \sum_{k=1}^n K_{ik} \Phi_k(\eta, \zeta, \tau^*) \times \\
 & \times \frac{\exp \left[ -\frac{x^2}{4c_m^2(\tau - \tau^*)} \right]}{(\tau - \tau^*)^{3/2}} d\tau^* + \int_0^{\tau} \int_0^{\infty} \left[ \sum_{i=1}^n A_{ii}(1; -c_m^2) \times \right. \\
 & \times w_i(\alpha, \eta, \zeta, \tau^*) \left. \right] \frac{\exp[-c_m^2(\eta^2 + \zeta^2)(\tau - \tau^*)]}{\sqrt{\tau - \tau^*}} \times \\
 & \times \left[ \exp \left( -\frac{(x - \alpha)^2}{4c_m^2(\tau - \tau^*)} \right) - \exp \left( -\frac{(x + \alpha)^2}{4c_m^2(\tau - \tau^*)} \right) \right] d\tau^* d\alpha + \\
 & + \frac{1}{\sqrt{\tau}} \int_0^{\infty} \left[ \sum_{i=1}^n A_{ii}(1; -c_m^2) F_i(\alpha, \eta, \zeta) \right] \exp[-c_m^2(\eta^2 + \zeta^2)\tau] \times \\
 & \times \left[ \exp \left( -\frac{(x - \alpha)^2}{4c_m^2\tau} \right) - \exp \left( -\frac{(x + \alpha)^2}{4c_m^2\tau} \right) \right] d\alpha \left. \right\}. \quad (9-6-8)
 \end{aligned}$$

To carry out the inversion with respect to the other coordinates, let us apply an inverse two-coordinate Fourier transformation with respect to  $\eta$  and  $\zeta$  to the function  $u_k(x, \eta, \zeta, \tau)$ . By direct integration it is easy to show that the function

$$G(\eta, \zeta) = \exp[-(\eta^2 + \zeta^2)a^2\tau]$$

is the transform of a two-coordinate Fourier transformation of the function

$$g(y, z) = \frac{1}{2a^2\tau} \exp \left( -\frac{y^2 + z^2}{4a^2\tau} \right).$$

When this result is substituted into the following convolution formula for the Fourier transformation:

$$\begin{aligned}
 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\eta, \zeta, \tau) G(\eta, \zeta) \exp[-i(y\eta + z\zeta)] d\eta d\zeta = \\
 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\beta, \gamma, \tau) g(y - \beta, z - \gamma) d\beta d\gamma,
 \end{aligned}$$

then we find that the first integral under the general summation sign on the right side of (9-6-8) is, after the Fourier transformation, equal to

$$\begin{aligned}
 \frac{x}{8c_m^4 \sqrt{\pi}} \int_0^{\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sum_{i=1}^n A_{ii}(1; -c_m^2) \sum_{k=1}^n K_{ik} \varphi_k(\beta, \gamma, \tau^*)}{\sqrt{(\tau - \tau^*)^3}} \times \\
 \times \exp \left[ -\frac{x^2 + (y - \beta)^2 + (z - \gamma)^2}{4c_m^2(\tau - \tau^*)} \right] d\tau^* d\beta d\gamma.
 \end{aligned}$$

The inverse transforms for the second and third integrals are obtained in exactly the same way.

Thus, as a result of our preliminary calculations, after an inverse integral transformation with respect to  $\eta$

and  $\zeta$  the functions  $u_i(x, y, z, \tau)$  have the following form:

$$\begin{aligned} \theta_i(x, y, z, \tau) = & \sum_{m=1}^n \frac{1}{D_s(1; -c_m^2)(2c_m \sqrt{\pi})^3} \times \\ & \times \left\{ \frac{x}{c_m^2} \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\varphi_{mi}^*(\beta, \gamma, \tau^*)}{(\tau - \tau^*)^{5/2}} \exp \left[ -\frac{x^2 + (y - \beta)^2 + (z - \gamma)^2}{4c_m^2(\tau - \tau^*)} \right] \times \right. \\ & \times d\tau^* d\beta d\gamma + \int_0^\tau \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{w_{mi}^*(\alpha, \beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} \times \\ & \times E_m(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) d\alpha d\beta d\gamma d\tau^* + \\ & \left. + \frac{1}{\tau^{3/2}} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f_{mi}^*(\alpha, \beta, \gamma) E_m(x, y, z, \tau; \alpha, \beta, \gamma, 0) d\alpha d\beta d\gamma \right\}, \end{aligned} \quad (9-6-9)$$

where

$$\begin{aligned} E_m(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) = & \exp \left[ -\frac{(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2}{4c_m^2(\tau - \tau^*)} \right] \left[ 1 - \exp \left( -\frac{\alpha x}{c_m^2(\tau - \tau^*)} \right) \right]; \\ \varphi_{mi}^*(y, z, \tau) = & \sum_{i=1}^n A_{ii}(1; -c_m^2) \sum_{k=1}^n K_{ik} \varphi_k(y, z, \tau); \\ w_{mi}^*(x, y, z, \tau) = & \sum_{i=1}^n A_{ii}(1; -c_m^2) w_i(x, y, z, \tau); \\ f_{mi}^*(x, y, z) = & \sum_{i=1}^n A_{ii}(1; -c_m^2) f_i(x, y, z). \end{aligned} \quad (9-6-10)$$

The functions  $\theta_i(x, y, z, \tau)$  given by (9-6-9) represent a solution of system of equations (9-6-1), with very general limiting conditions of the first kind.

Let us now consider a particular case of the problem, namely when  $K_{ik} = 0$  for  $i < k$  and  $K_{ik} \neq 0$  for  $i \geq k$ . For these restrictions the first equation in system (9-6-1) is converted into an ordinary heat-conduction

equation with an internal heat-source term  $w_1(x, y, z, \tau)$ . We know that  $D(\rho, s) = \prod_{m=1}^n (s + K_{mm}\rho^2)$ ; this

means that  $s_m = -K_{mm}\rho^2$  or  $c_m^2 = K_{mm}$ . Obviously  $A_{ii}(\rho, s) = 0$  for  $i \neq 1$  and  $A_{11}(\rho, s) = \prod_{m=2}^n (s + K_{mm}\rho^2)$ ,

from which we have  $A_{11}(1; -K_{kk}) = 0$  for  $k \neq 1$  and  $D_s(1; -K_{11}) = A_{11}(1; -K_{11}) \neq 0$ . When these expressions are substituted into formulas (9-6-9) and (9-6-10), we obtain, after certain simplifications,

$$\begin{aligned} \theta_1^0(x, y, z, \tau) = & \frac{1}{(2\sqrt{K_{11}\pi})^3} \left\{ \frac{x}{K_{11}} \times \right. \\ & \times \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\varphi_1(\beta, \gamma, \tau^*)}{(\tau - \tau^*)^{5/2}} \exp \left[ -\frac{x^2 + (y - \beta)^2 + (z - \gamma)^2}{4K_{11}(\tau - \tau^*)} \right] d\tau^* d\beta d\gamma + \\ & + \int_0^\tau \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{w_1(\alpha, \beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} E_1(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) d\tau^* d\alpha d\beta d\gamma + \\ & \left. + \frac{1}{\tau^{3/2}} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f_1(\alpha, \beta, \gamma) E_1(x, y, z, \tau; \alpha, \beta, \gamma, 0) d\alpha d\beta d\gamma \right\}, \end{aligned} \quad (9-6-11)$$

where

$$E_1(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) = \exp \left[ \frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4K_{11}(\tau-\tau^*)} \right] \left[ 1 - \exp \left( -\frac{\alpha x}{K_{11}(\tau-\tau^*)} \right) \right].$$

The theory of parabolic-type differential equations states that the solution of the heat-conduction equation is correct (that is, minor variations in the limiting conditions and in the thermal diffusivity correspond to minor variations in the solutions). It may be assumed that the solution of system (9-6-1) is also correct. This means that the function  $\theta_1(x, y, z, \tau)$  defined by (9-6-9) for  $l=1$  approaches the function  $\theta_1^0(x, y, z, \tau)$  as  $K_{11} \rightarrow 0$  for all  $l > 1$ . Consequently, formula (9-6-11) represents a solution of the nonhomogeneous heat-conduction equation for a semi-infinite three-dimensional medium, for limiting conditions of the first kind. The presence of kernels of the form  $\exp[-\alpha^2 r^2]$  in formulas (9-6-10) and (9-6-11), where  $r^2 = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$ , ensures convergence of the improper integrals for a wide range of integrand functions  $f_i(x, y, z)$ ,  $w_i(x, y, z, \tau)$  and  $\varphi_i(y, z, \tau)$ .

It should be noted that some functions which are encountered in the solution of physical problems do not belong to the class  $L\{\Omega\}$ . In particular, for  $\varphi_i(y, z, \tau) = \varphi_i^0$  constant,  $w_i(x, y, z, \tau) = w_i^0$  constant, and  $f_i(x, y, z) = f_i^0$  constant, the functions do not belong to  $L\{\Omega\}$  and integral transformation is inapplicable.

For those cases when  $\varphi_i(y, z, \tau)$ ,  $f_i(x, y, z)$ , and  $w_i(x, y, z, \tau)$  are not included in the class of functions to which integral transformations are applicable, we will follow certain authors in calling formulas (9-6-10) and (9-6-11) "generalized solutions." The limits of applicability of formulas (9-6-10) and (9-6-11), if they are considered as "generalized solutions," can thus be extended considerably to include a wide range of integrand functions.

## 2. Unsteady fields of transfer potential for a semi-infinite three-dimensional medium. Boundary conditions of the second kind

It is required to solve system (9-6-1) for the space  $\Omega$ , with the limiting conditions

$$\theta_i(x, y, z, 0) = f_i(x, y, z); \quad (9-6-12)$$

$$\frac{\partial \theta_i(0, y, z, \tau)}{\partial x} = \varphi_i(y, z, \tau). \quad (9-6-13)$$

As in the previous problem, we apply a two-coordinate Fourier integral transformation with respect to  $y$  and  $z$ , a cosine transformation with respect to  $x$ , and a Laplace transformation with respect to the time  $\tau$ . The original system of differential conditions (9-6-12) and (9-6-13) is then reduced to a system of linear algebraic equations in the required function, for the transform domain:

$$\sum_{k=1}^n (x_{ik}s + K_{ik}p^2) \{u_k\}_{cL} = -\sqrt{\frac{2}{\pi}} \sum_{k=1}^n K_{ik} \{\Phi_k\}_L + \{w_i\}_{cL} + \{F_i\}_c. \quad (9-6-14)$$

We may use Cramer's rule to determine from (9-6-14) the transform solutions of the system:

$$\{u_i(\xi, \eta, \zeta, s)\}_{cL} = \frac{D_i(p, s)}{D(p, s)},$$

where

$$D_i(p, s) = -\sqrt{\frac{2}{\pi}} \sum_{i=1}^n [A_{ii}(p, s) \cdot \sum_{k=1}^n K_{ik} \{\Phi_k\}_L] + \sum_{i=1}^n \{w_i\}_{cL} A_{ii}(p, s) + \sum_{i=1}^n \{F_i\}_c A_{ii}(p, s).$$

Here  $A_{ii}(p, s)$  is the cofactor of the matrix element formed by the  $i$ th row and the  $i$ th column of the basic determinant  $D(p, s)$ .

Let us now write the transform  $\{u_i\}_{cL}$  in the expanded form

$$\begin{aligned} \{u_i(\xi, \eta, \zeta, s)\}_{cL} = & -\frac{\sqrt{\frac{2}{\pi}} \sum_{l=1}^n \left[ A_{il}(\rho, s) \sum_{k=1}^n K_{ik} \{\Phi_k\}_L \right]}{D(\rho, s)} + \\ & + \frac{\sum_{l=1}^n A_{il}(\rho, s) \{w_l\}_{cL}}{D(\rho, s)} + \frac{\sum_{l=1}^n A_{il}(\rho, s) \{F_l\}_0}{D(\rho, s)}. \end{aligned} \quad (9-6-15)$$

If we apply inverse integral transformations to the transform  $\{u_i(\xi, \eta, \zeta, s)\}_{cL}$ , using the same sequence as in the previous problem, then the desired functions  $\theta_i(x, y, z, \tau)$  may be written as

$$\begin{aligned} \theta_i(x, y, z, \tau) = & \sum_{m=1}^n \frac{1}{(2c_m \sqrt{\pi})^3 D'_s(1; -c_m^2)} \times \\ & \times \left\{ - \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\Psi_{mi}^*(\beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} E_m(x, y, z, \tau; 0, \beta, \gamma, \tau^*) d\tau^* d\beta d\gamma + \right. \\ & + \int_0^\tau \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\Psi_{mi}^*(\alpha, \beta, \gamma, \tau^*)}{(\tau - \tau^*)^{3/2}} E_m(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) d\tau^* d\alpha d\beta d\gamma + \\ & \left. + \frac{1}{\tau^{3/2}} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f_{mi}^*(\alpha, \beta, \gamma) E_m(x, y, z, \tau; \alpha, \beta, \gamma, 0) d\alpha d\beta d\gamma \right\}; \end{aligned} \quad (9-6-16)$$

where

$$\begin{aligned} E_m(x, y, z, \tau; \alpha, \beta, \gamma, \tau^*) = & \exp \left[ -\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4c_m^2(\tau - \tau^*)} \right] \times \\ & \times \left[ 1 + \exp \left( -\frac{\alpha x}{c_m^2(\tau - \tau^*)} \right) \right]; \\ \Psi_{mi}^*(y, z, \tau) = & \sum_{l=1}^n A_{il}(1; -c_m^2) \sum_{k=1}^n K_{ik} \Psi_k(y, z, \tau); \\ f_{mi}^*(x, y, z) = & \sum_{l=1}^n A_{il}(1; -c_m^2) f_l(x, y, z); \\ w_{mi}^*(x, y, z, \tau) = & \sum_{l=1}^n A_{il}(1; -c_m^2) w_l(x, y, z, \tau). \end{aligned}$$

In the derivation of inverse transform (9-6-16) (inversion with respect to the first parameter), the following evaluation of the improper integral was used:

$$\int_0^\infty \exp(-a^2 \xi^2 \tau) \cos(x \xi) d\xi = \frac{\sqrt{\pi}}{2a \sqrt{\tau}} \exp \left( -\frac{x^2}{4a^2 \tau} \right);$$

and in addition it was assumed that  $s_k = -c_k^2 \rho^2$  are the simple roots of the determinant  $D(\rho, s) = 0$ . The inversions with respect to the remaining coordinates are simply a repetition of the second half of the solution of the previous problem.

3. Unsteady fields of transfer potential in a two-dimensional plane domain.  
Boundary conditions of the first and third kinds

Let us consider system of equations (9-6-1) when it contains neither source terms nor sink terms ( $w_i=0$ );

$$\frac{\partial \theta_l}{\partial \tau} = \sum_{k=1}^n K_{lk} \nabla^2 \theta_k \quad (l=1, 2, \dots, n), \quad (9-6-17)$$

where  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ , and where the constant coefficients  $K_{lk}$  are such that all the roots of the equation

$$|M - \mu E| = 0 \quad (9-6-18)$$

are different and  $\text{Re } \mu > 0$ . Here  $M = \|K_{lm}\|$  is the matrix composed of the transfer coefficients  $K_{lk}$  and  $E$  is the unit matrix.

The solution of system of equations (9-6-17), which pertains to a domain  $D$  bounded by a piecewise-smooth closed curve  $c$ , must satisfy the initial condition

$$\theta_l(x, y, 0) = f_l(x, y) \quad (9-6-19)$$

and the boundary condition along the curve  $c$

$$\theta_l = \varphi_l(s, \tau) \quad (l=1, 2, \dots, n) \quad (9-6-20)$$

or

$$\frac{\partial \theta_l}{\partial n} + \sum_{k=1}^n \alpha_{lk}(s, \tau) \theta_k = \varphi_l(s, \tau) \quad (l=1, 2, \dots, n), \quad (9-6-21)$$

where  $f_l(x, y)$  are continuous bounded functions in the domain for boundary conditions (9-6-20) and functions with bounded first-order derivatives in domain  $D$  for conditions (9-6-21). An additional requirement is imposed on the functions  $f_l$  and  $\varphi_l$  and on the curve  $c$ , namely that the curve  $c$  can be broken up into a finite number of arcs, within which  $\varphi_l(s, \tau)$  and  $\alpha_{lk}(s, \tau)$  are continuous, according to Hölder's definition /31/.

The solution of system (9-6-17) will have the following form:

$$G_l^{(k)}(x-a, y-\beta, \tau-\tau^*) = \sum_{m=1}^n A_m^k B_{lm} \frac{\exp \left[ -\frac{r^2}{4\mu_m(\tau-\tau^*)} \right]}{4\pi\mu_m(\tau-\tau^*)} \quad (k, l=1, 2, \dots, n), \quad (9-6-22)$$

where  $r = \sqrt{(x-a)^2 + (y-\beta)^2}$  is the distance between the two points with coordinates  $r(x, y)$  and  $r_1(a, \beta)$ . Since the functions

$$g_m(x-a, y-\beta, \tau-\tau^*) = \frac{1}{4\pi\mu_m(\tau-\tau^*)} \exp \left[ -\frac{r^2}{4\mu_m(\tau-\tau^*)} \right]$$

satisfy the equation

$$\frac{\partial g_m}{\partial \tau} = \mu_m \nabla^2 g_m,$$

therefore it follows from (9-6-22) that

$$\begin{aligned} \sum_{i=1}^n K_{li} \nabla^2 G_i^k &= \sum_{m=1}^n A_m^k \left( \sum_{i=1}^n K_{li} B_{im} \right) \nabla^2 g_m = \\ &= \sum_{m=1}^n A_m^k \left( \sum_{i=1}^n K_{li} B_{im} \right) \frac{1}{\mu_m} \cdot \frac{\partial}{\partial \tau} g_m = \frac{\partial}{\partial \tau} \sum_{m=1}^n A_m^k \left( \sum_{i=1}^n K_{li} B_{im} \right) \frac{1}{\mu_m} g_m. \end{aligned} \quad (9-6-23)$$

Here  $\mu_m$  and  $B_{lm}$  are defined in such a way that functions (9-6-22) satisfy system of equations (9-6-17). If they are so chosen that

$$\sum_{i=1}^n K_{li} B_{im} = \mu_m B_{lm} \quad (l=1, 2, \dots, n), \quad (9-6-24)$$

then it follows from (9-6-23) that

$$\sum_{l=1}^n K_{il} \nabla G_l^{(k)} = -\frac{\partial G_i^{(k)}}{\partial \tau} \quad (k, l = 1, 2, \dots, n).$$

System (9-6-24) has a nontrivial solution  $B_{lm}$  provided that  $\mu_m$  satisfies characteristic equation (9-6-18). By definition, the roots of this equation are different; therefore the matrix  $\|B_{lm}\|$  is singular.

We determine  $A_m^k$  from the following system of linear algebraic equations:

$$\sum_{m=1}^n A_m^k B_{lm} = \begin{cases} 0, & l \neq k; \\ 1, & l = k, \end{cases} \quad (9-6-25)$$

which have single-valued solutions due to the singularity of the matrix  $\|B_{lm}\|$ .

Direct differentiation indicates that the functions

$$\begin{aligned} v_l^0(x, y, \tau) &= \sum_{k=1}^n \sum_{m=1}^n A_m^k B_{lm} \int_D g_m(x-\alpha, y-\beta, \tau-\tau^*) f_k(\alpha, \beta) d\alpha d\beta; \\ v_l(x, y, \tau) &= \sum_{k=1}^n \sum_{m=1}^n A_m^k B_{lm} \int_0^\tau d\tau^* \int_c \frac{\psi_k(\sigma, \tau^*)}{2\pi(\tau-\tau^*)} \exp\left[-\frac{\rho^2}{4\mu_m(\tau-\tau^*)}\right] d\sigma; \\ w_l(x, y, \tau) &= \sum_{k=1}^n \sum_{m=1}^n A_m^k B_{lm} \int_0^\tau d\tau^* \int_c \frac{\phi_k(\sigma, \tau^*)}{4\pi\mu_m(\tau-\tau^*)} \times \\ &\times \exp\left[-\frac{\rho^2}{4\mu_m(\tau-\tau^*)}\right] \rho \cos(n_{r_1}, \rho) d\sigma \quad (l = 1, 2, \dots, n) \end{aligned}$$

satisfy system of equations (9-6-17) in the domain  $D$  for any functions  $f_k$  and  $\phi_k$ . In the given integrals  $\sigma$  is the coordinate of the point  $r_1(\alpha, \beta)$  in curvilinear coordinates, and  $n_{r_0}$  is the internal normal. It is also easy to show [31] that, provided the functions  $f_k$  are continuous and bounded in the domain  $D$ , we may write

$$\lim_{\tau \rightarrow 0} v_l^0(x, y, \tau) = f_l(x, y).$$

If in the vicinity of the point  $r_0(x_0, y_0)$  the arc of curve  $c$  is smooth, while the functions  $\phi_k(\sigma, \tau)$  are continuous on this arc, then on the basis of the properties of the thermal potentials [34] and taking into account formula (9-6-25), we have

$$\begin{aligned} w_l^{(1)}(x_0, y_0, \tau) &= \phi_l(s, \tau) + \sum_{k=1}^n \sum_{m=1}^n A_m^k B_{lm} \int_0^\tau d\tau^* \int_c \frac{\phi_k(\sigma, \tau^*)}{4\pi\mu_m(\tau-\tau^*)} \times \\ &\times \exp\left[-\frac{r^2}{4\mu_m(\tau-\tau^*)}\right] r \cos(n_{r_1}, r) d\sigma; \end{aligned} \quad (9-6-26)$$

$$\begin{aligned} w_l^k(x_0, y_0, \tau) &= -\phi_l(s, \tau) + \sum_{k=1}^n \sum_{m=1}^n A_m^k B_{lm} \int_0^\tau d\tau^* \int_c \frac{\phi_k(\sigma, \tau^*)}{4\pi\mu_m(\tau-\tau^*)} \times \\ &\times \exp\left[-\frac{r^2}{4\mu_m(\tau-\tau^*)}\right] r \cos(n_{r_1}, r) d\sigma; \end{aligned} \quad (9-6-27)$$

$$\begin{aligned} \left(\frac{\partial v_l}{\partial n}\right)_l &= -\phi_l(s, \tau) + \sum_{k=1}^n \sum_{m=1}^n A_m^k B_{lm} \int_0^\tau d\tau^* \int_c \frac{\phi_k(\sigma, \tau^*)}{4\pi\mu_m(\tau-\tau^*)} \times \\ &\times \exp\left[-\frac{r^2}{4\mu_m(\tau-\tau^*)}\right] r \cos(n_{r_0}, r) d\sigma; \end{aligned} \quad (9-6-28)$$

$$\begin{aligned} \left(\frac{\partial v_l}{\partial n}\right)_k &= \phi_l(s, \tau) + \sum_{k=1}^n \sum_{m=1}^n A_m^k B_{lm} \int_0^\tau d\tau^* \int_c \frac{\phi_k(\sigma, \tau^*)}{4\pi\mu_m(\tau-\tau^*)} \times \\ &\times \exp\left[-\frac{r^2}{4\mu_m(\tau-\tau^*)}\right] r \cos(n_{r_0}, r) d\sigma. \end{aligned} \quad (9-6-29)$$



By means of these functions, it is possible to reduce the problem at hand to the solution of a system of integral equations.

The solution for boundary conditions of the first kind must have the form

$$\theta_l(x, y, \tau) = v_l^{(0)}(x, y, \tau) + w_l(x, y, \tau) \quad (l = 1, 2, \dots, n), \quad (9-6-30)$$

while for boundary conditions of the third kind it is

$$\theta_l(x, y, \tau) = v_l^{(0)}(x, y, \tau) + v_l(x, y, \tau) \quad (l = 1, 2, \dots, n). \quad (9-6-31)$$

In relation to the foregoing, it is obvious that these functions satisfy system (9-6-17) and initial condition (9-6-19). Let us now choose  $\psi_k$  in such a way that systems of functions (9-6-30) and (9-6-31) satisfy boundary conditions (9-6-20) and (9-6-21) respectively. To do this, the functions  $\theta_l$  defined by (9-6-30) and (9-6-31) are substituted into (9-6-20) and (9-6-21). We then obtain the following system of integral equations:

$$\psi_l(s, \tau) = \sum_{m=1}^n \int_0^{\tau} d\tau^* \int_c L_{lm}(r_0, r_1, \tau, \tau^*) \psi_m(s, \tau^*) ds + F_l(s, \tau) \quad (l = 1, 2, \dots, n), \quad (9-6-32)$$

where for boundary conditions (9-6-20)

$$L_{lm}(r_0, r_1, \tau, \tau^*) = - \sum_{k=1}^n A_k^m B_{lk} \frac{\exp \left[ -\frac{r^2}{4\mu_k(\tau - \tau^*)} \right]}{4\pi\mu_k(\tau - \tau^*)^2} r \cos(n_{r_1}, r_k) \\ F_l(s, \tau) = \varphi_l(s, \tau) - v_l^{(0)}|_c$$

and for conditions (9-6-21)

$$L_{lm}(r_0, r_1, \tau, \tau^*) = \sum_{k=1}^n A_k^m B_{lk} \frac{\exp \left[ -\frac{r^2}{4\mu_k(\tau - \tau^*)} \right]}{4\pi\mu_k(\tau - \tau^*)^2} r \cos(n_{r_0}, r) + \\ + \sum_{j=1}^n \sum_{k=1}^n a_{lj} A_k^m B_{jk} \frac{\exp \left[ -\frac{r^2}{4\mu_k(\tau - \tau^*)} \right]}{2\pi(\tau - \tau^*)}; \\ F_l(s, \tau) = \varphi_l(s, \tau) - \frac{\partial v_l^0}{\partial n} \Big|_c - \sum_{k=1}^n a_{lk} v_k^0 \Big|_c.$$

When the restrictions mentioned at the very beginning of this subsection are imposed on the functions  $f_l(x, y)$ , the system of integral equations can be solved by successive approximations.

## Chapter X

### HEAT AND MASS TRANSFER WITH VARIABLE TRANSFER COEFFICIENTS

#### 10-1. General Observations

The transfer coefficients and the thermodynamic properties of a material or medium may in general have different values at different points of a system. Sometimes these quantities also vary considerably with the transfer potentials. Thus, the solutions of a large number of research and engineering problems can be made much more exact by introducing corrections which compensate for the variations of the coefficients. This has become particularly important in connection with the widespread implementation of high-intensity processes in various branches of engineering. Moreover, if the proper substitutions are made, many problems in convective diffusion and heat conduction, in the hydrodynamics of a viscous fluid, and in other cases as well, can be solved using differential equations of the heat-conduction type with variable coefficients. Consequently, it is useful to collect and to generalize the solutions obtained for the nonhomogeneous and nonlinear equations describing heat and mass conduction, and also to develop further the methods for solving these equations.

#### 10-2. Heat and Mass Transfer with a Variable Parameter of Phase or Chemical Transformation

In the derivation of the system of differential heat-transfer and mass-transfer equations

$$\frac{\partial t}{\partial \tau} = a_q \nabla^2 t + \varepsilon \rho \frac{c_m}{c_q} \frac{\partial \theta}{\partial \tau}; \quad \frac{\partial \theta}{\partial \tau} = a_m \nabla^2 \theta + a_m \delta \nabla^2 t \quad (10-2-1)$$

no restrictions were imposed on the parameter  $\varepsilon$  for phase (or chemical) transformations. Therefore it may be assumed that in general this parameter is a function of the spatial coordinates and of time (as well as of the transfer potentials). That this is the case is indicated by direct measurements of  $\varepsilon$  during heat and mass transfer. By studying  $\varepsilon$  as a function of the coordinates, we can specify more exactly the influence which phase (chemical) transformations have on the transfer kinetics. However, the way in which  $\varepsilon$  depends on the coordinates and time is also of fundamental interest in a somewhat different respect. This will now be explained using as an example the transfer of heat and moisture in freezing or thawing soils.

The process of freezing (or thawing) is usually described by setting up a problem in which a moving freezing boundary sharply separates the soils in the frozen and thawed states. Above the freezing boundary moisture is

present only in the form of ice, while below it all the moisture is in liquid form. In such soil there is no moisture transfer.

The actual freezing mechanism, as experiments show, is considerably more complicated, since all the moisture is not transformed simultaneously into ice at some given temperature but rather a gradual freezing takes place as the soil temperature drops. Because of the different ways in which moisture can be bound to the skeleton of the soil system, a temperature spectrum ranging from 0°C (for moisture in microcapillaries) to -70°C (for moisture bound by the strongest adsorption forces) is observed.

As a first approximation, the freezing mechanism can be represented as follows [1, 2]. As the soil cools, the moisture is gradually transformed into ice. This phase-transition process takes place in the freezing zone, in which the temperature is below 0°C. The freezing zone is in contact with the melting soil, the boundary between them being the isothermal surface corresponding to the temperature 0°C. Since in the freezing zone part of the water has been transformed into ice, a mass-content gradient is established, causing the transfer of liquid from the still-thawed portion of the soil to the freezing zone. This liquid transfer is intensified by the thermal mass conduction, that is, the transfer is due not only to the mass-content gradient (more precisely, to the mass-transfer potential) but also to the temperature gradient. System of equations (10-2-1), as applied to this process mechanism, has the following form for a one-dimensional half-space:

$$\left. \begin{aligned} \frac{\partial t}{\partial \tau} &= a_q \frac{\partial^2 t}{\partial z^2} + \frac{\epsilon \rho c_m}{c_q} \frac{\partial \theta}{\partial \tau}, \\ \frac{\partial \theta}{\partial \tau} &= a_m \frac{\partial^2 \theta}{\partial z^2} + \epsilon \frac{\partial \theta}{\partial \tau}, \end{aligned} \right\} \quad (10-2-2)$$

where  $\epsilon$  is the phase-transition ratio for the conversion of water into ice.

System (10-2-2) describes completely the basic features of a real freezing process. The presence of coupling between the heat and mass transfer makes it possible to represent the transformation of liquid moisture into ice at various temperatures. To do this, it is necessary to assume that the parameter  $\epsilon$  is a function of temperature and of the spatial coordinate, or of the spatial coordinate alone, since the soil temperature in the freezing zone depends mainly on  $z$ . It varies from unity at the soil surface, where all the moisture is ice, to zero at levels far from the surface, where all the moisture is in liquid form.

Thus, an improvement of the physical model describing soil freezing enables us to replace the problem of heat conduction with a limiting condition at a moving boundary to the solution of a system of differential heat-transfer and mass-transfer equations with limiting conditions at immobile boundaries. This simplifies the solution of the problem considerably.

For specific calculations it is desirable to assume some approximate dependence of the phase-transition parameter on the spatial coordinates and time. For freezing soils this dependence can be established, for example, by comparing the function  $\epsilon = f(x, \tau)$  with the condition  $\xi = b\sqrt{\tau}$  representing the displacement of the freezing boundary into the interior (in Stefan's problem). A comparison [2] shows that in this case

$$\epsilon = f(z, \tau) = \exp\left(-\frac{z}{b\sqrt{\tau}}\right).$$

For low values of the time  $\tau$  we have  $\varepsilon=0$ . For high  $\tau$  we have  $\varepsilon \approx 1$ , which agrees with the general physical picture of the problem, as well as with experimental data. Figure 10-1 shows the variation in phase-transition ratio throughout the soil depth at various times, according to the experimental data of Yablonskaya /3/. The curves show that as the distance from

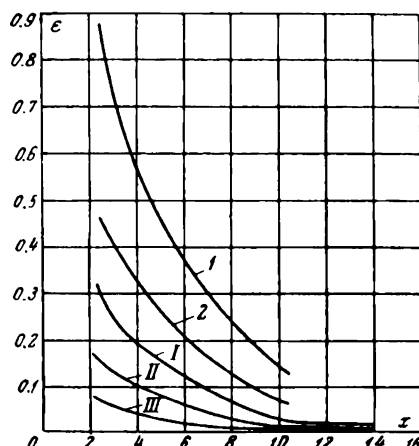


FIGURE 10-1. Variation of phase-transition parameter with soil depth  $x$  (cm): for Vorkuta loam, curve 1 at 7 p. m., curve 2 at 9 p. m.; for loam from the Moscow vicinity, curve I at 11:30 a. m., curve II at 6 p. m., curve III at 3 a. m.

the soil surface increases, or as time goes by,  $\varepsilon$  decreases. Within a certain time range ( $\tau_1 \leq \tau \leq \tau_2$ ) the parameter  $\varepsilon$  can be considered to be a function of position only. For this range  $\Delta\tau$  we may assume that

$$\varepsilon = f_1(z). \quad (10-2-3)$$

1. Let us solve system (10-2-2) for the case when (10-2-3) is true. When equation (10-2-3) is substituted into (10-2-2), the system of equations becomes

$$\frac{\partial t}{\partial \tau} = a_q \frac{\partial^2 t}{\partial z^2} + \rho \frac{c_m}{c_q} f_1(z) \frac{\partial \theta}{\partial \tau}; \quad (10-2-4)$$

$$\frac{\partial \theta}{\partial \tau} = a_m f_2(z) \frac{\partial^2 \theta}{\partial z^2}, \quad (10-2-5)$$

where

$$f_2(z) = \frac{1}{1-\varepsilon} = \frac{1}{1-f_1(z)}.$$

We assume the following limiting conditions for a finite medium:

$$t(z, 0) = f(z); \quad (10-2-6)$$

$$\theta(z, 0) = F(z); \quad (10-2-7)$$

$$t(0, \tau) = \varphi(\tau); \quad t(h, \tau) = t_h; \quad (10-2-8)$$

$$\theta(0, \tau) = \theta_0; \quad \theta(h, \tau) = \theta_h. \quad (10-2-9)$$

We will use the solution of this problem given in /2/. First, let us solve equation (10-2-5) with limiting conditions (10-2-7) and (10-2-9). To do this, let us represent the desired function  $\theta(z, \tau)$  as the sum

$$\theta(z, \tau) = \theta_0 + \frac{\theta_h - \theta_0}{h} z + \theta_1(z, \tau). \quad (10-2-10)$$

For the auxiliary function  $\theta_1(z, \tau)$  we obtain the differential equation

$$\frac{\partial \theta_1}{\partial \tau} = a_m f_2(z) \frac{\partial^2 \theta_1}{\partial z^2} \quad (10-2-11)$$

with the homogeneous boundary conditions

$$\theta_1(0, \tau) = 0 \quad \text{and} \quad \theta_1(h, \tau) = 0 \quad (10-2-12)$$

and the initial conditions

$$\theta_1(z, 0) = F(z) - \theta_0 - \frac{\theta_h - \theta_0}{h} z = \psi(z). \quad (10-2-13)$$

Differential equation (10-2-11) can be solved with conditions (10-2-12) and (10-2-13) by means of separation of variables, making use of the properties of the Sturm-Liouville problem. The solution is

$$\theta_1(z, \tau) = \sum_{n=1}^{\infty} C_n \varphi(z, \mu_n) \exp(-\mu_n^2 a_m \tau), \quad (10-2-14)$$

where

$$C_n = \frac{\int_0^h f_2^{-1}(\xi) \psi(\xi) \varphi(\xi, \mu_n) d\xi}{\int_0^h f_0^{-1}(\xi) \varphi^2(\xi, \mu_n) d\xi}; \quad (10-2-15)$$

and  $\mu_n$  are the roots of the transcendental equation

$$\begin{aligned} \varphi(h, \mu) &= 0; \\ \varphi(z, \mu) &= z - \mu \int_0^z dz_1 \int_0^{z_1} \frac{dz_2}{f_2(z_2)} + \mu^2 \int_0^z dz_1 \int_0^{z_1} \frac{dz_2}{f_2(z_2)} \int_0^{z_2} dz_3 \int_0^{z_3} \frac{dz_4}{f_4(z_4)} + \dots \end{aligned} \quad (10-2-16)$$

The substitution of (10-2-14) into (10-2-10) gives the final solution of equation (10-2-5):

$$\theta(z, \tau) = \theta_0 + \frac{\theta_h - \theta_0}{h} z + \sum_{n=1}^{\infty} C_n \varphi(z, \mu_n) \exp(-\mu_n^2 a_m \tau). \quad (10-2-17)$$

As before, coefficients  $C_n$  are defined by expressions (10-2-15), while the roots  $\mu_n$  obey equation (10-2-16).

The solution of heat-conduction equation (10-2-4) is now reduced to solving the solution of the differential equation

$$\frac{\partial t}{\partial \tau} = a_q \frac{\partial^2 t}{\partial z^2} + \Phi(z, \tau), \quad (10-2-18)$$

where

$$\Phi(z, \tau) = \rho \frac{c_m}{c_q} f_1(z) \frac{\partial \theta}{\partial \tau},$$

since the function  $f_1(z)$  is given and since  $\theta$  is determined by solution (10-2-17). The limiting conditions of (10-2-18) are (10-2-6) and (10-2-8).

Equation (10-2-10), subject to conditions (10-2-6) and (10-2-8), can be solved by the methods described in the previous chapters. The solution can be simplified still more if it is sought in the following form:

$$t(z, \tau) = \frac{t_h - \varphi(\tau)}{h} z + \varphi(\tau) + t_1(z, \tau) + t_2(z, \tau),$$

where the function  $t_1(z, \tau)$  satisfies the equation

$$\frac{\partial t_1}{\partial \tau} = a_q \frac{\partial^2 t_1}{\partial z^2}$$

with the conditions

$$t_1(z, 0) = \psi_1(z) = f(z) - \frac{t_h - \varphi(0)}{h} z + \varphi(0); \quad (10-2-19)$$

$$t_1(0, \tau) = 0; \quad t_1(h, \tau) = 0;$$

and where the function  $t_2(z, \tau)$  satisfies the equation

$$\frac{\partial t_2}{\partial \tau} = a_q \frac{\partial^2 t_2}{\partial z^2} + \Phi_1(z, \tau);$$

$$\Phi_1(z, \tau) = \Phi(z, \tau) - \varphi'(\tau) \left(1 - \frac{z}{h}\right)$$

with the conditions

$$t_2(z, 0) = 0;$$

$$t_2(0, \tau) = 0; \quad t_2(h, \tau) = 0.$$

We may now write

$$\begin{aligned} t(z, \tau) = & \frac{t_h - \varphi(\tau)}{h} z + \varphi(\tau) + \sum_{n=1}^{\infty} \left[ \frac{2}{h} \int_0^h \psi_1(\xi) \sin \frac{n\pi\xi}{h} d\xi \right] \sin \frac{n\pi z}{h} \times \\ & \times \exp \left( -a_q \frac{n^2\pi^2\tau}{h^2} \right) + \sum_{n=1}^{\infty} \left\{ \frac{2}{h} \int_0^{\tau} d\eta \int_0^h \Phi_1(\xi, \eta) \sin \frac{n\pi\xi}{h} \times \right. \\ & \left. \times \exp \left[ -a_q \frac{n^2\pi^2}{h^2} (\tau - \eta) \right] d\xi \right\} \sin \frac{n\pi z}{h}, \end{aligned} \quad (10-2-20)$$

where

$$\Phi_1(z, \tau) = \rho \frac{c_m}{c_q} f_1(z) \frac{\partial \theta}{\partial \tau} - \varphi'(\tau) \left(1 - \frac{z}{h}\right),$$

and  $\psi_1(z)$  is defined by (10-2-19).

Equation (10-2-17) gives the value of the mass-transfer potential at any point in the material and at any moment, while equation (10-2-20) does the same for the temperature.

2. If we consider system of differential heat-transfer and mass-transfer equations (10-2-1) in the absence of thermal mass conduction, then the relation for the phase-transition ratio as a function of time and space is much more simple. To derive the solutions in this case the very general solutions of the differential heat-conduction equations with a source term, considered in the previous chapter, may be used.

Let us first explain the general idea of the method and then illustrate it using a simple example. In the absence of thermal mass conduction, system (10-2-1) becomes

$$\frac{\partial t}{\partial \tau} = a_q \nabla^2 t + \epsilon \rho \frac{c_m}{c_q} \frac{\partial \theta}{\partial \tau}; \quad (10-2-21)$$

$$\frac{\partial \theta}{\partial \tau} = a_m \nabla^2 \theta, \quad (10-2-22)$$

where

$$\varepsilon = \varepsilon(x, y, z, \tau).$$

The solution of equation (10-2-22) does not involve any special difficulty. If this solution is written as

$$\theta = f(x, y, z, \tau),$$

then equation (10-2-21) becomes an equation in which the source term is explicit:

$$\frac{\partial \theta}{\partial \tau} = a_q \nabla^2 \theta + p \frac{c_m}{c_q} \varepsilon(x, y, z, \tau) \frac{\partial f(x, y, z, \tau)}{\partial \tau} = a_q \nabla^2 \theta + \omega(x, y, z, \tau),$$

where  $\omega(x, y, z, \tau)$  is a given function of space and time. If the general solution of the heat-conduction equation with a source term is known, for the body of interest and for the required limiting conditions, then we next substitute the expanded expression for  $\omega$  into this solution; after carrying out the necessary simplifications, we obtain the final solution of the problem.

Let us consider an example. In order to facilitate the use of the results obtained previously, let us derive the solution in dimensionless form. On one surface ( $X=0$ ) of a finite one-dimensional plate ( $0 \leq X \leq 1$ ), no heat or mass exchange with the surroundings takes place. On the other surface ( $X=1$ ) the heat exchange obeys boundary conditions of the second kind, while the mass exchange obeys boundary conditions of the third kind. The phase-transition parameter is a function of the spatial coordinate.

Thus, we must solve the system of equations

$$\frac{\partial T}{\partial Fo} = \frac{\partial^2 T}{\partial X^2} + \varepsilon K_0 \frac{\partial \theta}{\partial Fo}; \quad (10-2-23)$$

$$\frac{\partial \theta}{\partial Fo} = Lu \frac{\partial^2 \theta}{\partial X^2} \quad (10-2-24)$$

with the limiting conditions

$$T(X, 0) = F(X); \quad \theta(X, 0) = \Psi(X); \quad (10-2-25)$$

$$\frac{\partial T(0, Fo)}{\partial X} = \frac{\partial \theta(0, Fo)}{\partial X} = 0; \quad (10-2-26)$$

$$-\frac{\partial T(1, Fo)}{\partial X} + Ki_q(Fo) = 0; \quad (10-2-27)$$

$$\frac{\partial \theta(1, Fo)}{\partial X} - Bi_m[\theta_p - \theta(1, Fo)] = 0, \quad (10-2-28)$$

where

$$T = \frac{t - t_*}{t_*}; \quad \theta = \frac{\theta - \theta_*}{\theta_*}; \quad \theta_p = \frac{\theta_p - \theta_*}{\theta_*};$$

$$\varepsilon = \varepsilon(X);$$

and  $F(X)$  and  $\Psi(X)$  are arbitrary even functions. In addition, we require that the functions entering into the equation and into the uniqueness conditions satisfy Dirichlet's conditions.

The solution of equation (10-2-24) subject to conditions (10-2-25),

(10-2-26), and (10-2-28) is given by equation (6-5-17):

$$\theta(X, Fo) = \theta_p - \sum_{m=1}^{\infty} B_m \left[ \int_0^1 \Psi_1(X) \cos \mu_m X dX \right] \cos \mu_m X \exp(-\mu_m^2 Lu Fo), \quad (10-2-29)$$

where

$$B_m = \frac{2\mu_m}{\mu_m + \sin \mu_m \cdot \cos \mu_m}; \quad \Psi_1 = \theta_p - \Psi(X);$$

and  $\mu_m$  are the roots of the equation  $\cot \mu = \frac{1}{Bi_m} \mu$ .

To simplify the following calculations, let us determine the distribution of dimensionless temperature for the regular-regime stage of the mass transfer. In this regime ( $Fo > 0.1$ ) the rate of variation of dimensionless mass-transfer potential is

$$\frac{\partial \theta}{\partial Fo} = \frac{2 Lu \mu_1^3 \cos \mu_1 X}{\mu_1 + \sin \mu_1 \cos \mu_1} \exp(-\mu_1^2 Lu Fo) \int_0^1 \Psi_1(X) \cos \mu_1 X dX.$$

Consequently, equation (10-2-23) represents the familiar equation for heat conduction with a heat source:

$$Po(X, Fo) = \varepsilon(X) Ko Lu B_1 \mu_1^2 \cos \mu_1 X \exp(-\mu_1^2 Lu Fo) \int_0^1 \Psi_1(X) \cos \mu_1 X dX. \quad (10-2-30)$$

The general solution of this equation with conditions (10-2-25) through (10-2-27) was given in § 5-5. Taking into account (10-2-30), we obtain, after several transformations, the equation

$$\begin{aligned} T(X, Fo) = & \int_0^{Fo} Ki_q(Fo^*) dFo^* + \int_0^1 F(X) dX + Ko B_1 [1 - \exp(-\mu_1^2 Lu Fo)] \times \\ & \times \int_0^1 \varepsilon(X) \cos \mu_1 X dX \int_0^1 \Psi_1(X) \cos \mu_1 X dX + \\ & + \sum_{n=1}^{\infty} \cos n\pi X \exp(-n^2 \pi^2 Fo) \int_0^1 F(X) \cos n\pi X dX + \\ & + 2 \sum_{n=1}^{\infty} (-1)^n \cos n\pi X \int_0^{Fo} Ki_q(Fo^*) \exp[-n^2 \pi^2 (Fo - Fo^*)] dFo^* + \\ & + \sum_{n=1}^{\infty} \frac{2 B_1 Ko Lu \mu_1^2}{n^2 \pi^2 - \mu_1^2 Lu} [\exp(-\mu_1^2 Lu Fo) - \exp(-n^2 \pi^2 Fo)] \cos n\pi X \times \\ & \times \int_0^1 \varepsilon(X) \cos \mu_1 X \cos n\pi X dX \int_0^1 \Psi_1(X) \cos \mu_1 X dX. \end{aligned} \quad (10-2-31)$$

On the basis of solution (10-2-31) it is possible to obtain a series of particular solutions. One of these solutions, for  $Ki_q(Fo)$  as a periodic function of time ( $j_q = q_m \sin \omega \tau$ ) and for  $Bi_m = \infty$ , is given in /4/.

If  $Ki_q$  is constant and  $F(X) = \Psi(X) = 0$ , then, if we take into account the new expression for  $Po(X, Fo)$ , namely

$$Po(X, Fo) = \varepsilon(X) A Ko Lu \mu_1^2 \cos \mu_1 X \exp(-\mu_1^2 Lu Fo),$$

where

$$A = \frac{2 Bi_m \sqrt{Bi_m^2 + \mu_1^2}}{\mu_1 (Bi_m^2 + Bi_m + \mu_1^2)},$$



equation (10-2-31) simplifies to

$$T(X, Fo) = Kl_0 \left[ Fo - \frac{1}{6}(1 - 3X^2) - \sum_{n=1}^{\infty} (-1)^n \frac{2}{n^2 \pi^2} \cos n\pi X \exp(-n^2 \pi^2 Fo) \right] + \\ + A Ko [1 - \exp(-\mu_1^2 Lu Fo)] \int_0^1 \varepsilon(X) \cos \mu_1 X dX + \sum_{n=1}^{\infty} \frac{2A Ko Lu \mu_1^2}{n^2 \pi^2 - \mu_1^2 Lu} \times \\ \times [\exp(-\mu_1^2 Lu Fo) - \exp(-n^2 \pi^2 Fo)] \cos n\pi X \int_0^1 \varepsilon(X) \cos \mu_1 X \cos n\pi X dX.$$

In the particular case when

$$\varepsilon = \varepsilon_0 \exp(-kX)$$

the integrals involved in this solution may be evaluated as follows:

$$\int_0^1 \varepsilon \cos \mu_1 X dX = \varepsilon_0 / (\mu_1^2 + k^2) [k + \exp(-k)(\mu_1 \sin \mu_1 - k \cos \mu_1)]; \\ \int_0^1 \varepsilon \cos \mu_1 X \cos n\pi X dX = \frac{\varepsilon_0}{2} \left\{ \frac{k + \exp(-k)[(\mu_1 + n\pi) \sin(\mu_1 + n\pi) - k \cos(\mu_1 + n\pi)]}{k^2 + (\mu_1 + n\pi)^2} + \right. \\ \left. + \frac{k + \exp(-k)[(\mu_1 - n\pi) \sin(\mu_1 - n\pi) - k \cos(\mu_1 - n\pi)]}{k^2 + (\mu_1 - n\pi)^2} \right\}.$$

### 10-3. Heat or Mass Transfer in a Nonhomogeneous Medium

During a process the structural properties of a material change to some extent. However, when the properties of a body vary only slightly or in a very irregular way along the spatial coordinate, it is permissible, in the study of transfer phenomena, to consider the corresponding coefficients and thermodynamic properties as constant and equal to their average effective values. On the other hand, in some cases the nonuniformity of the physical properties is so large, and their variations along the coordinate are so regular, that this nonuniformity cannot be neglected. Thus, instead of solving the differential transfer equations with constant coefficients, we must solve equations in which all or some of the coefficients are functions of the spatial coordinates.

The solution of the differential transfer equations with variable coefficients is quite difficult. Therefore, up to the present time it has been possible to obtain exact analytical solutions for only a very limited range of problems. It is even more difficult to solve the systems of differential equations for which we still must be satisfied with various approximations or numerical methods of solution. In relation to this, the chief problem still encountered in the analytical theory of heat and mass transfer is the development of methods for solving systems of differential transfer equations with variable coefficients.

In this section we will consider mainly methods for solving the one-dimensional differential transfer equation

$$c\gamma \frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x} \left( \lambda \frac{\partial \theta}{\partial x} \right). \quad (10-3-1)$$

**1. Solution for a semi-infinite body, when the coefficients of heat or mass capacity and conductivity are power functions of the coordinates**

Let us consider some solutions of equation (10-3-1) with the following limiting conditions:

$$\theta(x, 0) = 0, \quad (0 < x < \infty); \quad (10-3-2)$$

$$\theta(\infty, \tau) = 0, \quad \theta(0, \tau) = \theta_0. \quad (10-3-3)$$

a) It is assumed that  $c\gamma$  is constant and that the conductivity  $\lambda$  depends on the spatial coordinate as follows:

$$\lambda = \lambda_0 x^n. \quad (10-3-4)$$

If a Laplace transformation is applied to equation (10-3-1), then, taking into account (10-3-2) and (10-3-4), we obtain

$$x^n \theta''_L(x, s) + nx^{n-1} \theta'_L(x, s) - \frac{s}{a_0} \theta_L(x, s) = 0, \quad (10-3-5)$$

where  $a_0 = \lambda_0 / c\gamma$ . If we set  $x = k\xi^m$ , then equation (10-3-5) becomes

$$m^2 k^2 x^{2m+n-2} \frac{d^2 \theta_L}{d\xi^2} + m(m+n-1) k x^{m+n-2} \frac{d\theta_L}{d\xi} - \frac{s}{a_0} \theta_L = 0. \quad (10-3-6)$$

In this substitution  $k$  and  $m$  were chosen arbitrarily; let us now require that they satisfy the conditions

$$2m+n-2=0 \quad \text{and} \quad k=1.$$

In this case (10-3-6) is transformed into the Bessel-type equation

$$\frac{d^2 \theta_L}{d\xi^2} + \frac{n}{2-n} \frac{1}{\xi} \frac{d\theta_L}{d\xi} - \frac{4}{(2-n)^2} \frac{s}{a_0} \theta_L = 0.$$

The particular solutions of this equation are

$$\xi^{\frac{1}{2}} I_{\nu} \left( \xi^{\frac{2}{2-n}} \sqrt{\frac{s}{a_0}} \right) \quad \text{and} \quad \xi^{\frac{1}{2}} K_{\nu} \left( \xi^{\frac{2}{2-n}} \sqrt{\frac{s}{a_0}} \right),$$

where  $I_{\nu}$  and  $K_{\nu}$  are Bessel functions of the first and second kinds, with imaginary arguments and with an order of  $\nu$ , where  $\nu = \frac{1-n}{2-n}$ ; and  $n$  is some arbitrary number other than 2. Now, returning to the initial variable  $x$  by means of the relation  $\xi = x^{\frac{2-n}{2}}$ , we obtain the solutions

$$x^{\frac{1}{2}(1-n)} I_{\nu} \left( x^{1-\frac{1}{2}n} \frac{2}{2-n} \sqrt{\frac{s}{a_0}} \right) \quad \text{and} \quad x^{\frac{1}{2}(1-n)} K_{\nu} \left( x^{1-\frac{1}{2}n} \frac{2}{2-n} \sqrt{\frac{s}{a_0}} \right). \quad (10-3-7)$$

In most cases  $n$  is less than unity ( $0 < n < 1$ ).

The first solution does not satisfy the first condition of (10-3-3); therefore we take the solution of equation (10-3-5) in the form

$$\theta_L(x, s) = A x^{\frac{1}{2}(1-n)} K_{\nu} \left( x^{1-\frac{1}{2}n} \frac{2}{2-n} \sqrt{\frac{s}{a_0}} \right),$$

where  $A$ , which is constant with respect to  $x$ , is defined by the second boundary condition of (10-3-3). For  $x \rightarrow 0$ , the potential  $\theta_L(x, s) \rightarrow \frac{\theta_0}{s}$ , whereas

for  $z \rightarrow 0$ , we have  $z^\nu K_\nu(z) \rightarrow 2^{\nu-1} \Gamma(\nu)$ , where  $\Gamma(\nu)$  is the gamma function. Thus, the constant  $A$  is

$$A = \frac{2\theta_c}{s\Gamma(\nu)(2-n)^\nu} \left(\frac{s}{a_0}\right)^{\nu/2}$$

Consequently, the solution for the transform has the form

$$\theta_L(x, s) = \theta_c \frac{2x^{\frac{1}{2}(1-n)}}{s(2-n)^\nu \Gamma(\nu)} \left(\frac{s}{a_0}\right)^{\nu/2} K_\nu \left( \frac{2x^{\frac{1}{2}-\frac{1}{2}n}}{2-n} \sqrt{\frac{s}{a_0}} \right). \quad (10-3-8)$$

By means of transform tables, we obtain from (10-3-8) the following solution for the inverse transform:

$$\frac{\theta(x, \tau)}{\theta_c} = \frac{1}{\Gamma(\nu)} \int_x^\infty e^{-u} u^{\nu-1} du = \frac{\Gamma(\nu, X)}{\Gamma(\nu)},$$

where

$$X = \frac{x^{2-n}}{(2-n)^2 a_0 \tau}.$$

Problems for zero initial conditions and a finite body ( $0 \leq x \leq R$ ) can be solved similarly. For  $n=2$  the solution of equation (10-3-5) will be

$$x^{[-1 \pm \sqrt{1+4s/a_0}]/2}.$$

b) If the capacity and conductivity coefficients depend on the  $n$ th power of the spatial coordinate:

$$c = c_0 x^n, \quad \lambda = \lambda_0 x^n, \quad \gamma = \text{const}, \quad (10-3-9)$$

then after application of the Laplace transformation, taking into account initial condition (10-3-2) and variation laws (10-3-9) for the coefficients, we obtain

$$\theta_L''(x, s) + \frac{n}{x} \theta_L'(x, s) - \frac{s}{a_0} \theta_L(x, s) = 0, \quad (10-3-10)$$

where

$$a_0 = \frac{\lambda_0}{c_0 \gamma}.$$

The solutions of this equation are

$$x^\nu I_\nu \left( \sqrt{\frac{s}{a_0}} x \right) \quad \text{and} \quad x^\nu K_\nu \left( \sqrt{\frac{s}{a_0}} x \right),$$

where

$$\nu = \frac{1}{2}(1-n).$$

If we use, as in the previous problem, the first boundary condition of (10-3-3), then we obtain

$$\theta_L(x, s) = A x^\nu K_\nu \left( \sqrt{\frac{s}{a_0}} x \right). \quad (10-3-11)$$

By substituting (10-3-11) into the second boundary condition of (10-3-3), we find that

$$A = \frac{\theta_0}{s \cdot 2^{\nu-1} \Gamma(\nu)} \left(\frac{s}{a_0}\right)^{\nu/2}.$$

Consequently, the solution of equation (10-3-10) is

$$\theta_L(x, s) = \theta_0 \frac{x^\nu}{s \cdot 2^{\nu-1} \Gamma(\nu)} \left(\frac{s}{a_0}\right)^{1/2} K_\nu \left(\sqrt{\frac{s}{a_0}} x\right).$$

The inverse transform of this solution is

$$\frac{\theta(x, \tau)}{\theta_0} = \frac{\sqrt{a_0}}{x} \frac{\Gamma(\nu, X)}{\Gamma(\nu)},$$

where

$$X = \frac{x^2}{4a_0\tau}.$$

c) If the capacity and conductivity coefficients obey the relations

$$c = c_0 x^n, \lambda = \lambda_0 x^m, \gamma = \text{const},$$

then after applying the Laplace transform we obtain the following equation for the transform:

$$\theta''_L(x, s) + \frac{n}{x} \theta'_L(x, s) - \frac{s}{a_0 x^{n-m}} \theta_L(x, s) = 0.$$

The solution of this equation is similar to that in subsection a, namely

$$x^{-\frac{1}{2}(n-1)} I_\nu \left[ \frac{2x^{\frac{1}{2}(m-n+2)}}{m-n+2} \sqrt{\frac{s}{a_0}} \right] \text{ and } x^{-\frac{1}{2}(n-1)} K_\nu \left[ \frac{2x^{\frac{1}{2}(m-n+2)}}{m-n+2} \sqrt{\frac{s}{a_0}} \right],$$

where

$$\nu = \frac{1-n}{m-n+2}.$$

Let us note that the solutions for cylindrical and spherical bodies, when the coefficients  $c$ ,  $\lambda$ , and  $\gamma$  are power functions of the spatial coordinates, represent particular cases of the last problem considered. As an example, for a sphere the equation for the transform is

$$\theta''_L + \frac{n+2}{r} \theta'_L - \frac{s}{a_0 r^{n-m}} \theta_L = 0,$$

that is, in this case the constants  $n$  and  $m$  are increased by two units.

A particular case of the problems considered is the case when the coefficients are linear functions of the spatial coordinate. By means of the proper substitutions, these can be reduced to particular instances of the solutions considered above. For instance, for  $c\gamma$  constant and  $\lambda = \lambda_0(1+ax)$ , a new variable  $\xi = 1+ax$  is introduced. Then, we obtain the equation

$$\frac{d^2 \theta_L}{d\xi^2} + \frac{1}{\xi} \frac{d\theta_L}{d\xi} - \frac{s}{a_0 \xi} \theta_L = 0,$$

which for  $n=1$  is identical to equation (10-3-5). Its solutions are therefore

$$I_0 \left[ 2 \frac{(1+ax)^{1/2}}{a} \sqrt{\frac{s}{a_0}} \right] \text{ and } K_0 \left[ 2 \frac{(1+ax)^{1/2}}{a} \sqrt{\frac{s}{a_0}} \right].$$

In the same way it can be shown that for  $c = c_0(1+ax)$ ,  $\lambda = \lambda_0(1+ax)$ , and  $\gamma$  constant the solutions of the resulting equation for the transforms are

$$I_0 \left[ \frac{1+ax}{a} \sqrt{\frac{s}{a_0}} \right] \text{ and } K_0 \left[ \frac{1+ax}{a} \sqrt{\frac{s}{a_0}} \right].$$

In the more general case, when

$$c = c_0(1+ax)^m \text{ and } \lambda = \lambda_0(1+ax)^m,$$

the introduction of the new variables

$$\begin{aligned}\xi &= (1 + lx)^{(m-n+2)/2}, \\ \eta &= \frac{l\lambda}{4c} (m-n+2)^2 \tau, \\ \zeta_0 &= \frac{1-n}{m-n+2}\end{aligned}$$

transforms equation (10-3-1) into

$$\gamma \xi^{1-\alpha} \frac{\partial \theta}{\partial \eta} = \frac{\partial}{\partial \xi} \left( \xi^{1-\alpha} \frac{\partial \theta}{\partial \xi} \right).$$

The subsequent steps of the solution do not differ from those of the solution of the first problem. Several solutions of specific problems of this type are given by Chudnovskii /5, 6/. In particular, in these references as well as in /7, 8/, the solutions of a number of problems for thermal waves are given. Thermal-wave problems are also studied in detail by Kornev\* /9/, as well as by other authors.

## 2. Solution for an infinite plate, when the conductivity is an exponential function of the coordinate

Let us solve equation (10-3-1) with boundary conditions of the first kind, when

$$\lambda = \lambda_0 \exp(-\kappa x), \text{ and } c\gamma = \text{const } (0 < x \leq R; \kappa > 0). \quad (10-3-12)$$

We assume that

$$\theta(x, 0) = 0; \quad (10-3-13)$$

$$\theta(0, \tau) = \varphi(\tau); \quad \theta(R, \tau) = \psi(\tau). \quad (10-3-14)$$

When the Laplace transformation is applied to equation (10-3-1) and conditions (10-3-14), then, taking into account (10-3-12) and (10-3-13), we obtain

$$\theta''_L(x, s) - \kappa \theta'_L(x, s) - \frac{s}{a_0} \exp(\kappa x) \theta_L(x, s) = 0; \quad (10-3-15)$$

$$\theta_L(0, s) = \varphi_L; \quad \theta_L(R, s) = \psi_L. \quad (10-3-16)$$

The introduction of the variable  $\xi = \frac{2}{\kappa} \sqrt{\frac{s}{a_0}} \exp\left(\frac{\kappa x}{2}\right)$  in (10-3-15) transforms this equation into

$$\theta''_L(\xi, s) - \frac{1}{\xi} \theta'_L(\xi, s) - \theta_L(\xi, s) = 0,$$

whose solution is

$$\theta_L(\xi, s) = A \xi I_1(\xi) + B \xi K_1(\xi). \quad (10-3-17)$$

If we return to the variable  $x$ , equation (10-3-17) becomes

$$\begin{aligned}\theta_L(x, s) &= A \frac{2}{\kappa} \sqrt{\frac{s}{a_0}} \exp\left(\frac{\kappa x}{2}\right) I_1\left[\frac{2}{\kappa} \sqrt{\frac{s}{a_0}} \exp\left(\frac{\kappa x}{2}\right)\right] + \\ &+ B \frac{2}{\kappa} \sqrt{\frac{s}{a_0}} \exp\left(\frac{\kappa x}{2}\right) K_1\left[\frac{2}{\kappa} \sqrt{\frac{s}{a_0}} \exp\left(\frac{\kappa x}{2}\right)\right].\end{aligned}$$

The quantities  $A$  and  $B$ , constant with respect to the spatial coordinates,

\* [This name is given as Kornev in the reference.]

are found using boundary conditions (10-3-16). After several transformations we obtain

$$\begin{aligned} \theta_L(x, s) = & \left\{ \left[ \frac{2\varphi_L}{x\sqrt{a_0}} K_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} \right) - \frac{2\varphi_L}{x\sqrt{a_0}} \exp \left( \frac{xR}{2} \right) K_1 \left( \frac{2}{x} \exp \left( \frac{xR}{2} \right) \sqrt{\frac{s}{a_0}} \right) \right] \times \right. \\ & \times \exp \left( \frac{xx}{2} \right) I_1 \left( \frac{2}{x} \exp \left( \frac{xx}{2} \right) \sqrt{\frac{s}{a_0}} \right) + \\ & + \left[ \frac{2\varphi_L}{x\sqrt{a_0}} \exp \left( \frac{xR}{2} \right) I_1 \left( \frac{2}{x} \exp \left( \frac{xR}{2} \right) \sqrt{\frac{s}{a_0}} \right) - \frac{2\varphi_L}{x\sqrt{a_0}} I_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} \right) \right] \times \\ & \times \exp \left( \frac{xx}{2} \right) K_1 \left( \frac{2}{x} \exp \left( \frac{xx}{2} \right) \sqrt{\frac{s}{a_0}} \right) \left. \right\} \left\{ \left[ K_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} \right) I_1 \left( \frac{2}{x} \exp \left( \frac{xR}{2} \right) \sqrt{\frac{s}{a_0}} \right) - \right. \right. \\ & \left. \left. - I_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} \right) K_1 \left( \frac{2}{x} \exp \left( \frac{xR}{2} \right) \sqrt{\frac{s}{a_0}} \right) \right] \frac{2}{x\sqrt{a_0}} \exp \left( \frac{xR}{2} \right) \right\}^{-1}. \end{aligned} \quad (10-3-18)$$

Equation (10-3-18) is the transform solution for the problem under consideration.

The inverse transform of (10-3-18) can be obtained using the following inversion formula:

$$\theta(x, \tau) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \theta_L(x, s) \exp(s\tau) ds, \quad (10-3-19)$$

where the integration is performed along any straight line situated parallel to the imaginary axis of the plane of the complex variable  $s$  and passing to the right of all discontinuities or singularities of the integrand expression. Let us calculate line integral (10-3-19) with conditions (10-3-16) in the following form:

$$\varphi_L = 0 \quad \text{and} \quad \psi_L = \frac{\theta_0}{s}, \quad (10-3-20)$$

which corresponds to the boundary conditions

$$\theta(0, \tau) = 0 \quad \text{and} \quad \theta(R, \tau) = \theta_0.$$

Taking into account (10-3-20), we obtain after some minor transformations equation (10-3-19) in the form

$$\begin{aligned} \theta(x, \tau) = & \frac{\theta_0}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp \left[ \frac{x}{2} (x-R) \right] \left[ K_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} \right) I_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} \frac{xx}{2} \right) - \right. \\ & \left. K_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} \right) I_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} e^{xR/2} \right) - \right. \\ & \left. - I_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} \right) K_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} \frac{xx}{2} \right) \right] \cdot \exp(s\tau)}{s \left[ K_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} \right) I_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} e^{xR/2} \right) - \right. \\ & \left. - I_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} \right) K_1 \left( \frac{2}{x} \sqrt{\frac{s}{a_0}} e^{xR/2} \right) \right]} ds. \end{aligned}$$

According to [10], this integral can be evaluated so as to give

$$\begin{aligned} \frac{\theta(x, \tau)}{\theta_0} = & \frac{\exp(xx) - 1}{\exp(xR) - 1} - \\ & - \pi \exp \left[ \frac{x}{2} (x-R) \right] \sum_{n=1}^{\infty} \frac{J_1(\mu_n) J_1 \left[ \mu_n \exp \left( \frac{xR}{2} \right) \right]}{J_1^2(\mu_n) - J_1^2 \left[ \mu_n \exp \left( \frac{xR}{2} \right) \right]} \left\{ J_1 \left[ \mu_n \exp \left( \frac{xx}{2} \right) \right] Y_1(\mu_n) - \right. \\ & \left. - J_1(\mu_n) Y_1 \left[ \mu_n \exp \left( \frac{xx}{2} \right) \right] \right\} \exp \left[ -\frac{a_0 x^2}{4} \mu_n^2 \tau \right], \end{aligned} \quad (10-3-21)$$

where  $\mu_n$  are the roots of the characteristic equation

$$\frac{J_1 \left[ \mu \exp \left( \frac{xR}{2} \right) \right]}{J_1(\mu)} = \frac{Y_1 \left[ \mu \exp \left( \frac{xR}{2} \right) \right]}{Y_1(\mu)}.$$

For the steady state, solution (10-3-21) simplifies to

$$\theta(x, \infty) = \theta_c \frac{\exp(\mu x) - 1}{\exp(\mu R) - 1}.$$

Several solutions of other problems in which the coefficients are exponential functions of the coordinate, especially for thermal waves, are given in /5, 6, 9/.

#### 10-4. Unsteady Potential Fields for Nonlinear Heat and Mass Transfer

During a high-intensity process the transfer potentials may undergo sizeable variations over short time intervals. This is the case during many processes of diffusion, gas seepage through a porous medium, thermal explosion, chemical transformations, etc. In the description of a transfer phenomenon which takes place with a wide range of potential variation, it is always necessary to take into account the dependence of the coefficients on the respective potentials. Under such conditions the mass and heat fluxes are nonlinear, and the determination of the transfer-potential fields involves the solution of the nonlinear differential transfer equation

$$c(\theta) \gamma(\theta) \frac{\partial \theta}{\partial x} = \text{div} [\lambda(\theta) \text{grad } \theta]. \quad (10-4-1)$$

The solution of equation (10-4-1) with the appropriate limiting conditions involves even more difficulty than the solution of problems in which the coefficients are functions of the spatial coordinates. Various approximation methods are thus widely used in this case. Very comprehensive reviews of the problems related to the solution of nonlinear transfer equations have been given by Friedmann /11/ and by Crank /12/.

Several methods are used at present to solve the problem of nonlinear transfer. For instance, in the linearization method, which is based on an approximation of the nonlinear coefficient, a new functional relation describing the coefficient is so chosen that equation (10-4-1) becomes linear /13, 14, 15/. In the method of substitutions, new variables are introduced which make it possible to transform nonlinear partial differential equation (10-4-1) into an ordinary nonlinear equation /12, 16, 17, 18, and others/, the solution of which is simpler. Some other methods for solving nonlinear transfer equations, for example, those in /19/ and /20/, also exist. Let us show how the various methods may be applied to problems of nonlinear transfer.

##### a) Linearization of the nonlinear differential transfer equation

The conductivity is usually plotted as a slowly rising curve  $\lambda = \lambda(\theta)$ , which can be represented quite accurately by a linear or exponential function. Charnyi /13/ has proposed for this case two methods for linearizing the transfer equation, with the condition that  $c\gamma$  is constant.

Equation (10-4-1) can be rewritten as

$$c\gamma \frac{d\theta}{d\lambda} \frac{\partial \lambda}{\partial \tau} = \operatorname{div} \left[ \lambda(\theta) \frac{d\theta}{d\lambda} \operatorname{grad} \lambda \right]. \quad (10-4-2)$$

If in equation (10-4-2) we set

$$\lambda \frac{d\theta}{d\lambda} = A = \text{const} \quad \text{and} \quad \frac{d\theta}{d\lambda} = B = \text{const}, \quad (10-4-3)$$

then we obtain a linear equation with constant coefficients. The first condition of (10-4-3) corresponds to the exponential  $\lambda(\theta)$  curve

$$A = \frac{\theta_1 - \theta_0}{\ln \frac{\lambda_1}{\lambda_0}},$$

while the second condition corresponds to its linear approximation

$$B = \frac{\theta_1 - \theta_0}{\lambda_1 - \lambda_0}.$$

Another linearization method consists in introducing the new function

$$G = \int_{\theta_0}^{\theta} \lambda(\theta') d\theta',$$

which transforms equation (10-4-1) into

$$c\gamma \frac{d\theta}{dG} \frac{\partial G}{\partial \tau} = \nabla^2 G. \quad (10-4-4)$$

Now, if we set

$$\frac{d\theta}{dG} = \text{const}$$

in equation (10-4-4), we also obtain a linear equation. This approximation assumes that some section of the  $G=G(\theta)$  curve is replaced by the corresponding chord.

The possibility of linearizing the one-dimensional transfer equation in the case when the rate of heat or mass transfer through the bounding surface ( $x=0$ ) is a known function  $\varphi(\tau)$  of time, while the conductivity is given by

$$\lambda^{-1/2} = \lambda_0^{-1/2} [1 + a(\theta - \theta_0)], \quad (10-4-5)$$

has been demonstrated by Storm /14/. In this case equation (10-4-5) serves as an approximation when considering problems in which the conductivity  $\lambda$  is an exponential function of the transfer potential:

$$\lambda = \lambda_0 \exp[-2a(\theta_0 - \theta)],$$

where  $a$  is constant and  $\lambda_0$  is the conductivity when the potential is  $\theta_0$ . By means of the substitutions

$$\xi = \int_0^x \lambda^{-1/2} dx' \quad \text{and} \quad \eta = \int_{\theta_0}^{\theta} \sqrt{\lambda(\theta')} d\theta'$$

the one-dimensional equation (10-4-1) is transformed into the linear equation

$$c\gamma \frac{\partial \omega}{\partial \tau} = \frac{\partial^2 \omega}{\partial \xi^2} - \frac{a}{\lambda_0^{1/2}} \varphi(\tau) \frac{\partial \omega}{\partial \xi},$$

where

$$\log \omega = -\frac{\pi}{\lambda_0^{1/2}} \eta.$$



Storm used this method to obtain the unsteady potential distribution in a semi-infinite medium, for a constant transfer rate through the surface  $x=0$ .

In conclusion let us mention the method proposed by Wiedenburg /15/ for linearizing the nonlinear transfer equation. For linear dependence of the conductivity on the transfer potential,  $\lambda=\lambda_0(1+\alpha\theta)$ , one-dimensional equation (10-4-1) becomes

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial}{\partial x}\left[a_0(1+\alpha\theta)\frac{\partial\theta}{\partial x}\right],$$

or, which is the same thing,

$$\frac{\partial\theta}{\partial\tau}=a_0\frac{\partial^2\theta}{\partial x^2}+\frac{a_0\alpha}{2}\cdot\frac{\partial^3(\theta^2)}{\partial x^2}, \quad (10-4-6)$$

where  $a_0=\frac{\lambda_0}{c\gamma}$  is constant. We write the limiting conditions as

$$\theta(x, 0)=\theta_0; \quad (0 < x < +\infty);$$

$$\theta(0, \tau)=0; \quad \frac{\partial\theta(\infty, \tau)}{\partial x}=0.$$

For a constant coefficient of potential conductivity (or diffusivity)  $\alpha(\theta)=\alpha_0$  ( $\alpha=0$ ), the solution of the equation is known to be

$$\{\theta\}_{\alpha=0}=\theta_0\left[1-\operatorname{erfc}\left(\frac{x}{2\sqrt{a_0\tau}}\right)\right]. \quad (10-4-7)$$

Wiedenburg has suggested that, in order to linearize the initial nonlinear transfer equation (10-4-6), we set in the  $\alpha$ -containing term of the equation some value of the potential which corresponds to the expression for this potential at  $\alpha=0$  (that is, to solution (10-4-7)). By means of this substitution, instead of nonlinear equation (10-4-6) we obtain the following differential equation with a source term, in which time and space are variable:

$$\frac{\partial\theta}{\partial\tau}=a_0\frac{\partial^2\theta}{\partial x^2}+a_0\frac{\alpha\theta_0^2}{2}\cdot\frac{\partial^2}{\partial x^2}\left[\operatorname{erf}\left(\frac{x}{2\sqrt{a_0\tau}}\right)\right]^2. \quad (10-4-8)$$

The argument of the Gaussian function is  $\xi=\frac{x}{2\sqrt{a_0\tau}}$ . If in equation (10-4-8) we introduce the new variable  $\xi$ , then we obtain the ordinary differential equation

$$\frac{d^2\theta}{d\xi^2}+2\xi\frac{d\theta}{d\xi}=-\frac{\alpha\theta_0^2}{2}\frac{d^2}{dz^2}[\operatorname{erf}\xi]^2$$

with the conditions

$$\theta=0 \text{ for } \xi=0 \text{ and } \theta=\theta_0 \text{ for } \xi=+\infty. \quad (10-4-9)$$

The solution of this equation will have the form

$$\theta=\varphi_1(\xi)+\varphi_2(\xi)\Phi(\xi),$$

where for brevity we have defined  $\Phi(\xi)=\operatorname{erfc}(\xi)$ . It is easy to show by substitution that the functions  $\varphi_1$  and  $\varphi_2$  are determined by the equations

$$\frac{d\varphi_1}{dz}=-\Phi\frac{d\varphi_2}{dz} \quad \text{and} \quad \frac{d\varphi_2}{dz}\frac{d\Phi}{dz}=-\frac{\alpha\theta_0^2}{2}\cdot\frac{d^2(1-\Phi)^2}{dz^2}. \quad (10-4-10)$$

The solution of equations (10-4-10) gives

$$\begin{aligned} \varphi_1(\xi) &= A + \alpha\theta_0^2\xi^2\Phi + \alpha\theta_0^2\frac{1-\Phi}{2} - \frac{\alpha\theta_0^2}{\sqrt{\pi}}\xi\exp(-\xi^2) - \alpha\theta_0^2\xi^2\Phi^2 - \\ &- \alpha\theta_0^2\frac{1-\Phi^2}{2} + 2\alpha\theta_0^2\frac{\Phi}{\sqrt{\pi}}\xi\exp(-\xi^2) + \frac{\alpha\theta_0^2}{\pi}[1-\exp(-2\xi^2)] + \frac{\alpha\theta_0^2}{2}\Phi^2; \end{aligned}$$

$$\varphi_2(\xi) = B - \alpha \theta_0^2 \xi^2 + \alpha \theta_0^2 \xi^2 \Phi + \alpha \theta_0^2 \frac{1-\Phi}{2} - \frac{\alpha \theta_0^2}{\sqrt{\pi}} \xi \exp(-\xi^2) - \alpha \theta_0^2 \Phi.$$

Consequently,

$$\theta = A + B\Phi - \frac{\alpha \theta_0^2}{\sqrt{\pi}} [1 - \Phi] \xi \exp(-\xi^2) + \frac{\alpha \theta_0^2}{\pi} [1 - \exp(-2\xi^2)] - \frac{\alpha \theta_0^2}{2} \Phi^2,$$

where  $A$  and  $B$  are constant with respect to  $\xi$ . In order for this solution to obey limiting conditions (10-4-9), it must be true that

$$A = \theta_0 - \frac{\alpha \theta_0^2}{\pi} \quad \text{and} \quad B = \frac{\alpha \theta_0^2}{2} + \frac{\alpha \theta_0^2}{\pi} - \theta_0.$$

In just the same way we can solve equation (10-4-6) for the conditions

$$\begin{aligned} \theta(x, 0) &= \theta_1, \quad 0 < x < +\infty; \\ \theta(x, 0) &= \theta_2, \quad -\infty < x < 0 \quad (\theta_2 > \theta_1); \\ \frac{\partial \theta(\pm \infty, \tau)}{\partial x} &= 0. \end{aligned}$$

Using the solution for  $\alpha=0$ ,

$$\{\theta\}_{\alpha=0} = \theta_2 + \left(\theta_1 - \frac{1}{2} \theta_2\right) \Phi,$$

we find that in this case

$$\begin{aligned} \theta &= A + B\Phi + \frac{\alpha \theta_1 (\theta_2 - \theta_1)}{2\sqrt{\pi}} \xi \exp(-\xi^2) + \frac{\alpha (\theta_2 - \theta_1)^2}{4\sqrt{\pi}} \Phi \xi \exp(-\xi^2) + \\ &+ \frac{\alpha (\theta_2 - \theta_1)^2}{4\pi} [1 - \exp(-2\xi^2)] - \frac{\alpha (\theta_2 - \theta_1)^2}{8} (1 - \Phi + \Phi^2) - \frac{\alpha \theta_1 (\theta_2 - \theta_1)}{4} (1 - \Phi), \end{aligned}$$

where

$$\begin{aligned} A &= \theta_1 + \frac{\alpha (\theta_2 - \theta_1)^2}{4} \left( \frac{1}{2} - \frac{1}{\pi} \right) + \frac{\alpha \theta_1 (\theta_2 - \theta_1)}{4}; \\ B &= \frac{\theta_2 - \theta_1}{2} + \frac{\alpha (\theta_2 - \theta_1)^2}{8} - \frac{\alpha \theta_1 (\theta_2 - \theta_1)}{4}. \end{aligned}$$

Some other linearization methods are discussed by Friedmann /11/.

## b) Some transformations used to solve nonlinear differential transfer equations

The exact or approximate solution (including numerical solutions) of a nonlinear differential equation is simplified considerably if the equation is transformed into an ordinary nonlinear differential equation. The effectiveness of some such transformations has been demonstrated by our study of the linearization of nonlinear equations. Let us now discuss this in more detail.

1. Consider the following one-dimensional form of differential equation (10-4-1):

$$c(\theta) \gamma(\theta) \frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x} \left[ \lambda(\theta) \frac{\partial \theta(x, \tau)}{\partial x} \right]. \quad (10-4-11)$$

To determine which substitutions are appropriate, it is possible to make use of the theory of generalized variables (see Chapter III).

A dimensional analysis of the quantities involved in (10-4-11) shows that this equation will remain unchanged if the length scale is varied by a factor of  $k$  and the time scale is varied by  $k^2$ . After such a variation in the scales, the limiting conditions, for the constant initial and boundary conditions

$$\theta(x, 0) = \theta_0, \quad (10-4-12)$$

$$\theta(0, \tau) = \theta_c, \quad (10-4-13)$$

also remain unchanged:

$$\theta(x, \tau) \equiv \theta(kx, k^2\tau)$$

for any values of  $x$ ,  $\tau$ , and  $k$ . Thus, setting  $k = \frac{1}{2\sqrt{\tau}}$ , we obtain

$$\theta(x, \tau) = \theta\left(\frac{x}{2\sqrt{\tau}}, 1/4\right) = \theta^* f(\xi),$$

where  $\theta^*$  is some assigned value of  $\theta$  and where

$$\xi = \frac{x}{2\sqrt{\tau}}. \quad (10-4-14)$$

Consequently,  $\theta$  is a function of  $\xi$  only. Therefore, the solution of the problem should be sought as a function of some combination of the variables  $\xi$  and  $\tau$  which will remain invariable under a similar transformation.

We have the relations

$$\begin{aligned} \frac{\partial \theta}{\partial \tau} &= -\frac{x}{4\sqrt{\tau^3}} \cdot \frac{\partial f}{\partial \xi} = -\theta^* \frac{\xi}{2\tau} \cdot \frac{df}{d\xi}; \\ \frac{\partial \theta}{\partial x} &= \frac{\theta^*}{2\sqrt{\tau}} \cdot \frac{df}{d\xi}; \quad \frac{\partial}{\partial x} \left( \lambda \frac{\partial \theta}{\partial x} \right) = \frac{\partial}{\partial x} \left( \lambda \frac{\theta^*}{2\sqrt{\tau}} \cdot \frac{df}{d\xi} \right) = \frac{\theta^*}{4\tau} \cdot \frac{d}{d\xi} \left( \lambda \frac{df}{d\xi} \right). \end{aligned}$$

Now, by substituting these expressions into equation (10-4-11), we obtain the ordinary differential equation

$$-2c\gamma\xi \frac{df}{d\xi} = \frac{d}{d\xi} \left[ \lambda(f) \frac{df}{d\xi} \right]. \quad (10-4-15)$$

Limiting conditions (10-4-12) and (10-4-13) have the form

$$f(\infty) = \theta_0 \quad \text{and} \quad f(0) = \theta_c.$$

In cases when it cannot be determined analytically, the function  $f$  can be found by means of numerical integration.

The substitution  $\xi = \frac{x}{2\sqrt{\tau}}$  is sometimes called the Boltzmann transformation, since Boltzmann used it in 1894 /21/ to solve equation (10-4-11). Substitution (10-4-14) can be applied to equation (10-4-11) when the transfer takes place in an infinite or semi-infinite medium. Only in these cases is it possible to separate the variables in such a way that the boundary conditions depend only on the variable  $\xi$ . In particular, substitution (10-4-14) cannot be used for a finite body of thickness  $R$  and with the boundary conditions

$$\theta(0, \tau) = \theta_0; \quad \theta(R, \tau) = \theta_0,$$

since the second condition is nonzero after transformation and depends on both  $\xi$  and  $\tau$ .

As an example of the use of this substitution, let us solve equation (10-4-11) for a semi-infinite body ( $\tau > 0$ ,  $0 < x < \infty$ ), for the following limiting conditions:

$$\theta(x, 0) = \theta_0; \quad \theta(0, \tau) = \theta_0; \quad \frac{\partial \theta(\infty, \tau)}{\partial x} = 0. \quad (10-4-16)$$

Let  $k=1$ ; in this case  $\xi = \frac{x}{\sqrt{\tau}}$  and we have

$$-\frac{\xi}{2} c\gamma \frac{\partial \theta}{\partial \xi} = \frac{\partial}{\partial \xi} \left( \lambda \frac{\partial \theta}{\partial \xi} \right). \quad (10-4-17)$$

If we write  $\frac{\partial \theta}{\partial \xi} = \theta'$ , then equation (10-4-17) can be written as

$$-\frac{\xi}{2} c\gamma \frac{d\xi}{\lambda} = \frac{d(\lambda \theta')}{\theta'}.$$

Thus, we obtain after integration

$$\ln(\lambda \theta') = -c\gamma \int_0^\xi \frac{\xi d\xi}{2\lambda} + \ln A$$

or

$$\lambda \theta' = A \exp \left[ -c\gamma \int_0^\xi \frac{1}{2\lambda} \xi d\xi \right].$$

A second integration gives

$$\theta = A \int_0^\xi \exp \left[ -c\gamma \int_0^\xi \frac{\xi}{2\lambda} d\xi \right] \frac{d\xi}{\lambda} + B.$$

Constants  $A$  and  $B$  may be determined from boundary conditions (10-4-16). The final form of the solution is

$$\theta = \frac{\theta_0 - \theta(x, \tau)}{\theta_0 - \theta_0} = \frac{\int_0^\xi \exp \left[ -c\gamma \int_0^\xi \frac{1}{2\lambda} \xi d\xi \right] \frac{d\xi}{\lambda}}{\int_0^\infty \exp \left[ -c\gamma \int_0^\xi \frac{\xi}{2\lambda} d\xi \right] \frac{d\xi}{\lambda}}. \quad (10-4-18)$$

Although equation (10-4-18) represents an exact solution of the nonlinear transfer equation for boundary conditions (10-4-16), a numerical result can be obtained from it only by carrying out the integration. The latter may be performed by numerical methods. This numerical integration imposes on the substitution an additional requirement concerning convenience in carrying out the subsequent calculations.

We will now demonstrate one of the possible means of calculating the unsteady field of the potential  $\theta$  using the iteration method. The iteration method states that, if a first approximate solution  $\theta_0$  of the equation

$$\theta = \varphi(\theta)$$

is known, then by substituting it into the original equation we can find a second approximation  $\theta_1 = \varphi(\theta_0)$ . Next, by substituting  $\theta_1$  into the equation, we find a third approximation  $\theta_2$ , and so on. By repeating this process several times, it is possible to obtain a solution to any preassigned accuracy. For simplicity we

assume that  $c\gamma$  is constant, so that (10-4-18) can be written as

$$\Theta = \frac{\theta_c - \theta(x, \tau)}{\theta_c - \theta_0} = \frac{\int_0^\eta \exp\left[-\int_0^\eta \frac{2\eta}{\lambda^*} d\eta\right] \frac{d\eta}{\lambda^*}}{\int_0^\infty \exp\left[-\int_0^\eta \frac{2\eta}{\lambda^*} d\eta\right] \frac{d\eta}{\lambda^*}}, \quad (10-4-19)$$

where

$$\lambda^* = \frac{\lambda}{\lambda_0} = \lambda^*(\Theta). \quad (10-4-20)$$

Here  $\eta = \frac{x}{2\sqrt{a_0\tau}}$  and  $a_0 = \frac{\lambda_0}{c\gamma}$  (where  $\lambda_0$  corresponds to the initial potential  $\theta_0$ ).

For a constant conductivity ( $\lambda = \lambda_0$  constant), equation (10-4-19) is transformed into

$$\Theta = \text{erf}(\eta), \quad (10-4-21)$$

which represents a first approximation for the beginning of the calculation.

By using (10-4-21) for equal intervals of variation of  $\eta$ , we find the corresponding values of  $\Theta_0$ . It should be noted here that for  $\eta = 3.0$ , the error function  $\text{erf } \eta$  is unity to the fifth significant place, so that a steady-state potential distribution is established in the material. At the same time, this determines the suitable variation interval for  $\eta$ .

Using formula (10-4-20), the values of  $\lambda_0^*$  are next found for each value obtained for  $\Theta_0$ . The substitution of these values into (10-4-19) then enables us to find the new values  $\Theta_1$ , for the specified values of  $\eta$ , using any convenient formulas for numerical integration (provided it is impossible to integrate directly). Finally, this calculation procedure is repeated for the approximations  $\Theta_2, \Theta_3, \dots$ , until the difference between the successive approximations satisfies the desired accuracy.

2. When electronic computers or analog models are used for numerical calculations of nonlinear transfer equation (10-4-1) it is convenient to put equation (10-4-1) into a different form. To do this the substitution

$$G = \int_{\theta_0}^{\theta} \frac{\lambda(\theta')}{\lambda_0(\theta_0)} d\theta' \quad (10-4-22)$$

is used. This substitution transforms equation (10-4-1) into

$$\frac{\partial G}{\partial \tau} = a(G) \nabla^2 G, \quad (10-4-23)$$

where  $a(G) = \lambda(G)/c(G)\gamma(G)$ . Let us note here, too, that equation (10-4-23) is also important in fluid dynamics. Its solution with the appropriate boundary conditions is obtained by specifying  $G$  as a function of the spatial coordinates and time. The correspondence between  $G$  and the potential  $\theta$  is established by (10-4-22), which can be expressed either in explicit form (when it is possible to integrate the expression) or in the form of a graph (in cases when numerical integration is necessary).

For a number of materials or media the integral expressions for the conductivity and the volume capacity, obtained as functions of the potential using formula (10-4-22), are quite well approximated by the power laws

$$\int_0^\theta \frac{\lambda(\theta')}{\lambda_0} d\theta' = \alpha \theta^n \quad \text{and} \quad \int_0^\theta \frac{\gamma c(\theta')}{\gamma_0 c_0} d\theta' = \beta \theta^m, \quad (10-4-24)$$

where  $\lambda_0, \gamma_0 c_0, \alpha, n, \beta$ , and  $m$  are constants. For example, the integral thermal conductivity and volume specific heat for pure aluminum and iron are represented, over a wide temperature range, by equations (10-4-24).

The coefficients have the following values:

$$\begin{array}{l} \text{aluminum: } \alpha = 1.57; \quad n = 0.905; \quad \beta = 0.871; \quad m = 1.03; \\ \text{iron: } \quad \alpha = 2.04; \quad n = 0.84; \quad \beta = 0.561; \quad m = 1.125. \end{array}$$

Transformation (10-4-22) can also be extended to include the case of a variable capacity coefficient. The transformed transfer equation (10-4-11) then becomes

$$\frac{\partial}{\partial \tau} \int_0^{\theta} \frac{\gamma c(\theta')}{\gamma c_0} d\theta' = a_0 \frac{\partial^2}{\partial x^2} \int_0^{\theta} \frac{\lambda(\theta')}{\lambda_0} d\theta'.$$

Friedmann has solved this equation /11, 22/ for the conditions

$$\theta(x, 0) = \theta(\infty, \tau) = 0 \quad \text{and} \quad \theta(0, \tau) = \theta_1$$

and taking into account (10-4-24). To a second-order approximation, the solution is

$$\begin{aligned} \left[ \frac{\theta(\xi)}{\theta_1} \right]^n = & \operatorname{erfc} \xi - 4s \left\{ \frac{1}{2} \left( \xi^2 - \frac{1}{2} \right) (\operatorname{erfc} \xi)^2 - \frac{1}{2\sqrt{\pi}} \exp(-\xi^2) \left[ 2\xi \operatorname{erfc} \xi - \right. \right. \\ & \left. \left. - \frac{1}{\sqrt{\pi}} \exp(-\xi^2) \right] + \frac{1}{2} \xi^2 \operatorname{erfc} \xi + (1 + \operatorname{erfc} \xi) \left[ \frac{1}{\sqrt{\pi}} \exp(-\xi^2) - \xi \operatorname{erfc} \xi + \right. \right. \\ & \left. \left. + \frac{1}{4} \left( (1 + 2\xi^2) \operatorname{erfc} \xi - \frac{2}{\sqrt{\pi}} \xi \exp(-\xi^2) \right) \right] - \left( \frac{1}{4} + \frac{1}{2\pi} \right) \operatorname{erfc} \xi \right\}, \end{aligned} \quad (10-4-25)$$

where

$$\xi = \frac{x}{2(k\theta_1^2 \tau)^{1/2}}; \quad s = 1 - m/n; \quad k = \frac{\lambda_0}{\gamma c_0} \cdot \frac{\alpha}{\beta} \cdot \frac{n}{m}.$$

The maximum discrepancy between the calculation result using formula (10-4-25) and the result taking into account the next approximation was less than 3% (5°C).

Similar solutions have been obtained for linear variations of coefficients  $\lambda$  and  $c$  with the potential  $\theta$ , for both semi-infinite and finite media /11, 16, 22, 23/. A comparison of the results obtained with the exact solutions showed good agreement in all cases.

3. Let us now consider several special substitutions which are used to solve nonlinear transfer equations when the conductivities are linear or exponential functions of the potentials. If we assume that  $c\gamma$  is constant, then equation (10-4-11) becomes

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x} \left[ a(\theta) \frac{\partial \theta}{\partial x} \right], \quad (10-4-26)$$

where

$$a(\theta) = \frac{1}{c\gamma} \lambda(\theta).$$

I. Let us assume that for an infinite medium the transfer potentials satisfy the initial conditions

$$\theta(x, 0) = \theta_1 \quad \text{for } x < 0; \quad (10-4-27)$$

$$\theta(x, 0) = \theta_2 \quad \text{for } x > 0, \quad (10-4-28)$$

while the potential conductivity  $a(\theta)$  is a linear function of the transfer potential:

$$a = \bar{a} \left[ 1 + \frac{1}{2} \alpha (\theta_1 + \theta_2) - \alpha \theta \right].$$

Here  $\bar{a}$  corresponds to the value of coefficient  $a$  for the average value of the potential,  $\bar{\theta} = \frac{1}{2}(\theta_1 + \theta_2)$ . By means of the substitutions

$$\psi = \left[ \frac{1 + \frac{1}{2}a(\theta_1 + \theta_2) - a\theta}{1 + \frac{1}{2}a(\theta_1 - \theta_2)} \right]^2;$$

$$\xi = \frac{x}{2 \left[ 1 + \frac{1}{2}a(\theta_1 - \theta_2) \right]^{1/2} (\bar{a}\tau)^{1/2}},$$

equation (10-4-26) can be transformed into

$$\frac{d^2\psi}{d\xi^2} = -\frac{2\xi}{\sqrt{\psi}} \cdot \frac{d\psi}{d\xi} \quad (10-4-29)$$

with the boundary conditions

$$\psi = 1 \text{ for } \xi = +\infty;$$

$$\psi = b^2 \text{ for } \xi = -\infty,$$

where

$$b = \frac{1 - \frac{1}{2}a(\theta_1 - \theta_2)}{1 + \frac{1}{2}a(\theta_1 - \theta_2)}.$$

A numerical solution of equation (10-4-29) has been obtained by Stokes /23/.

II. The case in which the potential conductivity is the following linear function of the potential:

$$a = a_0 \frac{\theta}{\theta_0}$$

was studied by Polubarinova-Kochina /17/ and by Wagner /24/. For a semi-infinite medium with the conditions

$$\theta(0, \tau) = \theta_0 \text{ and } \theta(x, 0) = 0$$

the substitution

$$\xi = \frac{x}{2(a_0\tau)^{1/2}}$$

transforms equation (10-4-26) into

$$2\xi \frac{d\theta}{d\xi} + \left( \frac{d\theta}{d\xi} \right)^2 + \theta \frac{d^2\theta}{d\xi^2} = 0. \quad (10-4-30)$$

Here  $\theta = \theta/\theta_0$ . This equation can be simplified further by substituting  $v = \theta^2$ , after which we have

$$\frac{d^2v}{d\xi^2} + \frac{2\xi}{\sqrt{v}} \cdot \frac{dv}{d\xi} = 0$$

with the boundary conditions

$$v = 1 \text{ for } \xi = 0;$$

$$v = 0 \text{ for } \xi = \infty.$$

III. Let us assume that, for a semi-infinite medium satisfying the conditions

$$\theta(x, 0) = \theta_0 \text{ and } \theta(0, \tau) = \theta_s, \quad (10-4-31)$$

the potential conductivity  $a$  is an exponential function of the potential:

$$a = a_s \exp [\beta (\theta - \theta_s)],$$

where the subscript  $s$  denotes the value of the parameter at the surface. In this case, after transformation we obtain the equation

$$\frac{d}{d\xi} \left( e^{\xi} \frac{d\zeta}{d\xi} \right) + 2\xi \frac{d\zeta}{d\xi} = 0,$$

where

$$\xi = \frac{x}{2(a_s \tau)^{1/2}} \quad \text{and} \quad \zeta = \beta (\theta - \theta_s). \quad (10-4-32)$$

In view of conditions (10-4-31) and (10-4-32), the integration should here be carried out beginning with  $\xi=0$  and  $\eta=0$ . More details regarding methods of numerical calculation and the calculation results are given in /12/. Some other transformations which may be used when the potential conductivity is an exponential function of  $\theta$  are also given in this reference.

### c) Some solutions of nonlinear differential transfer equations

Let us next examine what influence the nonlinearity of the transfer coefficients has on the distribution of the potentials, using several solutions of nonlinear differential transfer equations as examples.

1. Solution for a semi-infinite medium, with

$$a(\theta) = \frac{a_0}{1 - \lambda \theta} \quad (\text{where } a_0 \text{ and } \lambda \text{ are constant}).$$

The limiting conditions have the form

$$\theta(x, 0) = 0 \quad \text{and} \quad \theta(0, \tau) = \theta_0. \quad (10-4-33)$$

By introducing the new variables

$$\theta = \theta/\theta_0; \quad A(\theta) = a(\theta)/a_0; \quad \xi = \frac{x}{2\sqrt{a_0 \tau}};$$

we may transform equation (10-4-26) into the ordinary differential equation

$$-2\xi \frac{d\theta}{d\xi} = \frac{d}{d\xi} \left[ A(\theta) \frac{d\theta}{d\xi} \right],$$

whereas limiting conditions (10-4-33) become

$$\theta = 0 \quad \text{for} \quad \xi = \infty;$$

$$\theta = 1 \quad \text{for} \quad \xi = 0.$$

In addition,

$$A(\theta) = \frac{1}{1 - \alpha \theta} \quad (\alpha = \lambda \theta_0). \quad (10-4-34)$$

If we introduce still another new variable

$$g = \frac{\int_0^\theta A(\theta') d\theta'}{\int_0^1 A(\theta) d\theta} \quad (10-4-35)$$



then equation (10-4-34) gives the relation

$$\frac{\ln(1-\alpha\theta)}{\ln(1-\alpha)} = g. \quad (10-4-36)$$

The new variable (10-4-35) enables us to transform the previously obtained differential equation and limiting conditions into the equation

$$-2\xi e^{-\beta\xi} \frac{dg}{d\xi} = \frac{d^2g}{d\xi^2} \quad (10-4-37)$$

subject to the conditions

$$g=0 \text{ for } \xi=\infty; \quad (10-4-38)$$

$$g=1 \text{ for } \xi=0. \quad (10-4-39)$$

The quantity  $\beta$  is related to  $\alpha$  by the equation  $\beta = -\ln(1-\alpha)$ . In the present solution  $\lambda$  and  $\alpha$  are assumed positive, with  $0 < \alpha < 1$ . Consequently,  $\beta > 0$ .

If we make the additional substitutions

$$\frac{dg}{d\xi} = -\varphi \text{ and } \exp(-\beta g)/\beta = q, \quad (10-4-40)$$

then equation (10-4-37) simplifies to

$$q \frac{d^2\varphi}{dq^2} = -\frac{2}{\beta\varphi}.$$

The integration of this equation gives

$$\lg q + C_1 = - \int_0^{\varphi/\sqrt{q}} \left( C_2 + \frac{1}{4} z^2 - \frac{4}{\beta} \ln z \right)^{-1/2} dz, \quad (10-4-41)$$

where  $C_1$  and  $C_2$  are integration constants.

It follows from (10-4-38) and (10-4-40) that  $\varphi \rightarrow 0$  as  $q \rightarrow \frac{1}{\beta}$ . This condition makes it possible to determine constant  $C_1$  in equation (10-4-41). When the value  $C_1$  is introduced, equation (10-4-41) becomes

$$\ln q\beta = - \int_0^{\varphi/\sqrt{q}} \left( C_2 + \frac{1}{4} z^2 - \frac{4}{\beta} \ln z \right)^{-1/2} dz. \quad (10-4-42)$$

From condition (10-4-39) it follows that

$$\frac{d\varphi}{dq} = 0 \text{ and } q = \exp(-\beta)/\beta.$$

By substituting these conditions into equation (10-4-42) and performing some simple transformations, we find that

$$C_2 = \frac{4}{\beta} \ln s \text{ and } s = \left( \frac{\varphi}{\sqrt{q}} \right)_{q = \frac{e^{-\beta}}{\beta}},$$

so that equation (10-4-42) may be written as

$$\ln r = - \int_0^u (u_1^2 - \mu \ln u_1^2)^{-1/2} du_1, \quad (10-4-43)$$

where

$$r = (q\beta)^{1/2}; \quad u = \varphi/(\beta\sqrt{q}); \quad \mu = 8/(\beta s^2)$$

are new variables. The last equation is used to determine  $\mu$ , and consequently the unknown parameter  $s$ , as a function of the parameter  $\beta$ , defined as

$$\beta = 2 \int_0^1 (u^2 - \mu \ln u)^{-1/2} du. \quad (10-4-44)$$

Relation (10-4-44) is obtained from (10-4-43) by taking into account that  $\varphi/\sqrt{q} = s$  for  $q = \frac{\exp(-\beta)}{\beta}$ . The integral (10-4-44) cannot be calculated analytically (in a finite form), so that we must consider  $\mu$  as a given parameter.

From equation (10-4-37) and the conditions following it, we determine  $\xi$  as a function of  $u$  and  $r$ :

$$\xi = -\frac{1}{(2\mu)^{1/2}} \left( \frac{u}{r} + \frac{1}{r^2} \cdot \frac{du}{dr} \right).$$

Equation (10-4-43) may be used to eliminate  $r$ , so that we have

$$\xi = \frac{1}{(2\mu)^{1/2}} [(u^2 - \mu \ln u^2)^{1/2} - u] \exp \left[ \int_0^u (u_1^2 - \mu \ln u_1^2)^{-1/2} du_1 \right]. \quad (10-4-45)$$

On the other hand, by combining the second relation of (10-4-40), equation (10-4-43), and the definition  $r = (q\beta)^{1/2}$ , we find an expression for  $g$ :

$$g = \frac{2}{\beta} \int_0^u (u_1^2 - \mu \ln u_1^2)^{-1/2} du_1.$$

Then, taking into account (10-4-35), we obtain the final expression for  $\theta$ :

$$\theta = \frac{1}{1 - \exp(-\beta)} \left[ 1 - \exp \left\{ -2 \int_0^u (u_1^2 - \mu \ln u_1^2)^{-1/2} du_1 \right\} \right]. \quad (10-4-46)$$

It should be noted that the parameter  $\mu$  in the above equations is related to the given parameter  $\beta$  by (10-4-44).

The above solution was derived by Fujita /25/. He shows that for constant coefficients ( $\alpha = \beta = \lambda = 0$ ) the solution is transformed into the familiar solution involving the probability function. The solution method for a given  $\mu$  is as follows:

a) by numerical integration of equation (10-4-44) the value of  $\beta$  is estimated, and thus  $\alpha$  is found using the relation  $\beta = -\ln(1-\alpha)$ ;

b) then,  $\xi$  and  $\theta$  are determined by numerical integration of equations (10-4-45) and (10-4-46).

A relation between  $\alpha$  and  $\lg \mu$  may be plotted on the basis of equation (10-4-44) and the equation  $\beta = -\ln(1-\alpha)$ . The curve is shown in Figure 10-2; if the value of  $\alpha$  is found, then  $\mu$  can be determined directly from the graph. The calculation results are given in Figure 10-3.

2. Solution for a semi-infinite medium, with

$$a(\theta) = \frac{a_0}{(1 - \lambda\theta)^2}.$$

The limiting conditions remain as before, and once again we make the substitutions

$$\theta = \frac{\theta}{\theta_0}; \quad \xi = \frac{x}{2\sqrt{a_0\tau}}; \quad \alpha = \theta_0\lambda.$$

This makes it possible to obtain an ordinary differential equation in terms of  $\xi$  and  $\theta$ :

$$-2\xi \frac{d\theta}{d\xi} = \frac{d}{d\xi} \left[ \frac{1}{(1-\alpha\theta)^2} \cdot \frac{d\theta}{d\xi} \right] \quad (10-4-47)$$

subject to the limiting conditions

$$\theta = 0 \text{ for } \xi = \infty; \quad (10-4-48)$$

$$\theta = 1 \text{ for } \xi = 0. \quad (10-4-49)$$

The solution of equation (10-4-47) with conditions (10-4-48) and (10-4-49) gives a relation between  $\theta$  and  $\xi$ .

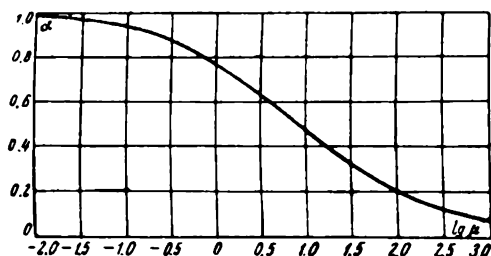


FIGURE 10-2. The parameter  $\alpha$  as a function of  $\lg \mu$

We will give here only the final results obtained by Fujita. Detailed calculations and an analysis of the results obtained are given in /26/. The solution can be written as

$$\theta = \frac{\psi(u, \beta)}{\alpha [1 - \alpha + \psi(u, \beta)]}; \quad (10-4-50)$$

$$\xi = \frac{\beta}{1-\alpha} \{ [1 - \alpha + \psi(u, \beta)] u - \exp[\beta^2(1-u^2)] \}, \quad (10-4-51)$$

where  $\psi(u, \beta)$  is determined by the relation

$$\psi(u, \beta) = \sqrt{\pi} \beta [1 - \operatorname{erf}(\beta u)] \exp(\beta^2);$$

and  $\beta$  is a constant determined as a function of the given parameter  $\alpha$  by means of the equation

$$\psi(1, \beta) = \alpha \quad (0 < \alpha < 1).$$

Equations (10-4-50) and (10-4-51) give a solution  $\theta = \theta(\xi)$  which involves a parameter  $u$  whose values lie in the range  $1 \leq u < \infty$ . The results of the solution of this problem are plotted in Figure 10-4.

3. Solution for a semi-infinite medium, with

$$a(\theta) = \frac{a_0}{1 + 2\kappa\theta + \nu\theta^2} \quad (\text{where } \kappa \text{ and } \nu \text{ are constant}).$$

The limiting conditions are as before.

This expression for the coefficient of potential conductivity [diffusivity] as a function of the potential includes the previous problems as particular

cases. Moreover, this equation makes it possible to investigate problems in which the potential conductivity (diffusivity) passes through a minimum or maximum as the potential varies. Solutions of this problem were also obtained by Fujita.

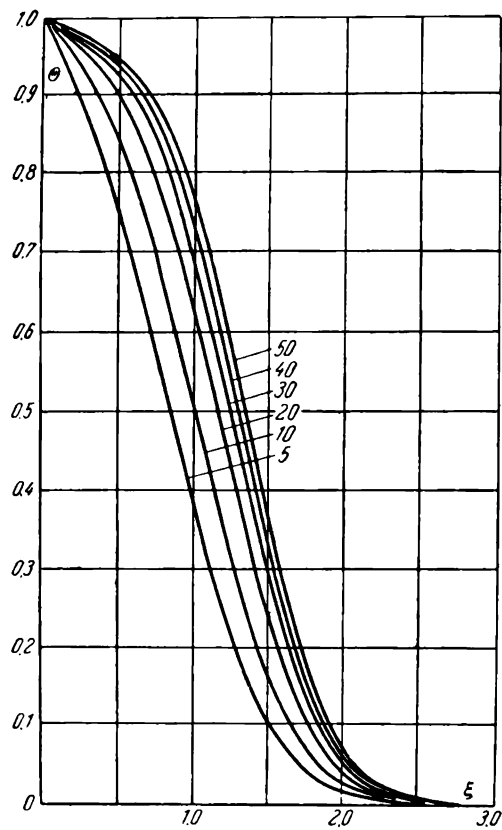


FIGURE 10-3. The dimensionless potential  $\theta = \frac{\theta}{\theta_0}$  as a function of  $\xi = \frac{x}{(4a_0\tau)^{1/2}}$ , for  $a(\theta) = \frac{a_0}{1 - \alpha\theta}$  (the number on the curve is equal to  $1/(1 - \alpha)$ , and represents the ratio between  $a\theta$  at  $\theta = \theta_0$  and  $a_0$  at  $\theta = 0$ )

The substitutions

$$\theta = \theta/\theta_0; \xi = \frac{x}{2\sqrt{a_0\tau}}; \alpha = \kappa\theta_0; \beta = \nu\theta_0^2$$

transform (10-4-1) into the ordinary differential equation

$$-2\xi \frac{d\theta}{d\xi} = \frac{d}{d\xi} \left[ \frac{1}{1 + 2\alpha\theta + \beta\theta^2} \cdot \frac{d\theta}{d\xi} \right]$$

with the limiting conditions

$$\begin{aligned} \theta &= 0 \text{ for } \xi = \infty; \\ \theta &= 1 \text{ for } \xi = 0. \end{aligned}$$

Following /12/, we now consider the case when  $a(\theta)$  goes through a maximum between  $\theta=0$  and  $\theta=\theta_*$ . This situation is equivalent to saying that a function  $f(\theta)$ , defined as

$$f(\theta) = 1 + 2a\theta + \beta\theta^2,$$

goes through a minimum between  $\theta=0$  and  $\theta=1$ . It is easy to show that this condition is determined by the inequalities

$$0 < -\frac{a}{\beta} < 1; \quad (10-4-52)$$

$$a < 0. \quad (10-4-53)$$

It follows from (10-4-52) and (10-4-53) that

$$\beta > 0. \quad (10-4-54)$$

From the physical point of view, any maximum of  $a(\theta)$  in the range of variation of the potential must remain finite. This means that the corresponding minimum of  $f(\theta)$  must be positive, which is equivalent to the condition

$$0 < \frac{a^2}{\beta} < 1. \quad (10-4-55)$$

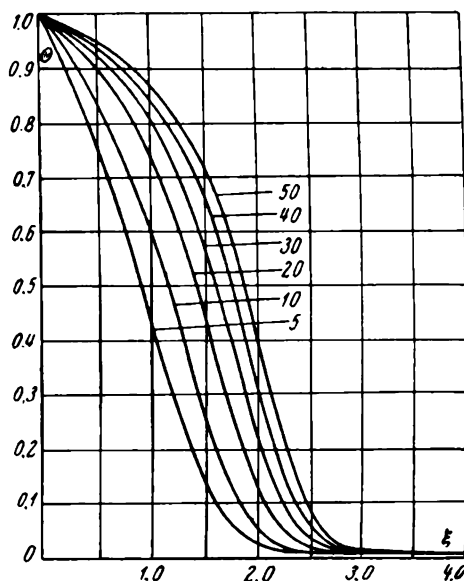


FIGURE 10-4. The dimensionless potential  $\theta$  as a function of  $\xi$ , for  $a(\theta) = \frac{a_*}{(1 - a\theta)^2}$  (the number on the curve is equal to  $1/(1 - a)^2$ , and represents the ratio between  $a$  at  $\theta = \theta_*$  and  $a_*$  at  $\theta = 0$ )

Relations (10-4-52) through (10-4-55) express the conditions under which the coefficient  $a$  goes through a maximum.

For convenience let us denote  $\alpha = -\gamma$ , so that the above inequalities may be written as

$$0 < \gamma/\beta < 1 \quad \text{and} \quad 0 < \gamma^2/\beta < 1.$$

The final solution of the problem thus becomes

$$\beta\theta = (\beta - \gamma^2)^{1/2} \tan[F(u, \epsilon) - \tan^{-1}k] + \gamma, \quad 0 < \theta < \theta_1; \quad (10-4-56)$$

$$\beta\theta = (\beta - \gamma^2)^{1/2} \tan[\tan^{-1}m - F(u, \epsilon) - F(u_m, \epsilon)] + \gamma, \quad \theta_1 < \theta < 1, \quad (10-4-57)$$

where

$$\beta\theta_1 = (\beta - \gamma^2)^{1/2} \tan[F(1, \epsilon) - \tan^{-1}k] + \gamma.$$

In addition,

$$\xi = \left[ \frac{\beta}{\epsilon(\beta - \gamma^2)(1 + z^2)} \right]^{1/2} [zu + (1 - u^2 - \epsilon \ln u)^{1/2}], \quad -k < z < z_1; \quad (10-4-58)$$

$$\xi = \left[ \frac{\beta}{\epsilon(\beta - \gamma^2)(1 + z^2)} \right]^{1/2} [zu - (1 - u^2 - \epsilon \ln u)^{1/2}], \quad z_1 < z < m, \quad (10-4-59)$$

where

$$\tan^{-1}z_1 = F(1, \epsilon) - \tan^{-1}k;$$

$$\tan^{-1}z = F(u, \epsilon) - \tan^{-1}k, \quad -k < z < z_1$$

and

$$\tan^{-1}z = \tan^{-1}m - F(u, \epsilon) + F(u_m, \epsilon), \quad z_1 < z < m.$$

The auxiliary variables in these equations are defined as

$$k = \frac{\gamma}{(\beta - \gamma^2)^{1/2}}; \quad m = \frac{\beta}{(\beta - \gamma^2)^{1/2}} \left( 1 - \frac{\gamma}{\beta} \right), \quad (10-4-60)$$

where  $\gamma = -\alpha$  and  $\beta$  are constants appearing in the expressions for the potential conductivity (diffusivity). In addition,  $\epsilon$  and  $u_m$  are related to  $k$  and  $m$  as follows:

$$\epsilon \ln u_m + (m^2 + 1) u_m^2 = 1 \quad (10-4-61)$$

and

$$\tan^{-1}k + \tan^{-1}m = 2F(1, \epsilon) - F(u_m, \epsilon), \quad (10-4-62)$$

where

$$F(u, \epsilon) = \int_0^u (1 - u_1^2 - \epsilon \ln u_1)^{-1/2} du_1.$$

In evaluating (10-4-56) and (10-4-57) with respect to  $u$ , it should be noted that in (10-4-56)  $u$  ranges from 0 to 1, whereas in (10-4-57)  $u_m < u < 1$ .

As an example, let us consider a diffusion process. We assume that in the expression for the diffusion coefficient  $\alpha = -\gamma = -1.646$  and  $\beta = 2.877$ . Figure 10-5 shows the graphical anamorphosis of the potential diffusivity (diffusion coefficient)  $a_m(\theta)/a_{m0}$  as a function of the dimensionless potential  $\theta_m$  (in this case, the concentration). For  $\theta = 1$ , we have  $a_m/a_{m0} = 1.710$ . The ratio  $a_m/a_{m0}$  is a maximum, equal to 17.20, for  $\theta = 0.572$ ; the quantities  $k$  and  $m$ , calculated using equation (10-4-60), are 4.025 and 3.010 respectively.

Values of  $\varepsilon$  calculated using (10-4-61) and (10-4-62) for various values of  $u_m$  are plotted in Figure 10-6. The point at which the  $\varepsilon(u_m)$  curves intersect gives the solution of these equations:  $\varepsilon = 0.2036$  and  $u_m = 0.3475$ . The final result, the relation between  $\Theta_m$  and  $\xi$  calculated according to equations (10-4-56) through (10-4-59), is given in Figure 10-7.

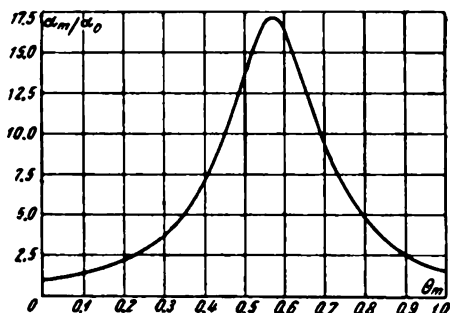


FIGURE 10-5. Diffusion coefficient as a function of concentration for  $a_m = \frac{a_0}{1 + 2\alpha\Theta_m + \beta\Theta_m^2}$  (with  $\alpha = -1.646$  and  $\beta = 2.877$ )

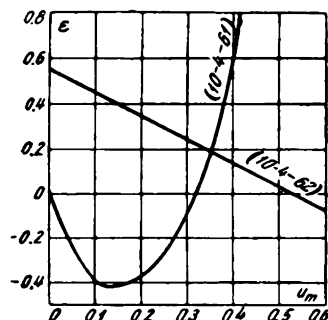


FIGURE 10-6. Graph for the determination of the common solutions of equations (10-4-61) and (10-4-62)

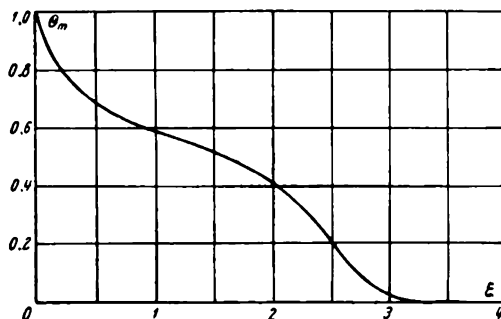


FIGURE 10-7. Distribution of concentration, for the diffusion coefficient represented in Figure 10-5

We will conclude this section with a problem which is of particular interest for the study and calculation of heat transfer in metals.

#### 4. Transfer phenomena in a medium with a moving phase-transition boundary. Coefficients $\lambda$ and $c$ linear functions of the potentials

The propagation velocity of a crystallization front in a metal bar or of the thawing front in frozen soil is usually calculated assuming constant values of all the thermophysical characteristics of the material. The

accuracy of the results so obtained can be estimated only on the basis of more general solutions. Such a solution, for the case of linear dependence of the coefficients on the potential, has been obtained by Lyubov /27/. His method for finding the solution by means of series is a quite simple one and can be used to solve both similar transfer problems involving moving boundaries and also problems with more general boundary conditions or a more complicated dependence of the coefficients on the potential.

It will be assumed that the zone of the material which has undergone phase transition is not heated, and that the temperature of the material is a function only of the position of the phase-transition surface and of time. In this case the equation representing the variation of transfer potential in the zone which has undergone phase transition (for example, in the solidified crust of a metal) has the form

$$\gamma(c_0 + v\theta) \frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x} \left[ (\lambda_0 + \kappa\theta) \frac{\partial \theta}{\partial x} \right]; \quad (10-4-63)$$

$$\theta(x, 0) = \theta_{ph} \text{ for } x > 0; \quad \theta(0, \tau) = \theta_0; \quad (10-4-64)$$

where  $\theta_{ph}$  is the temperature corresponding to the phase transition (in particular, for the solidification of a bar this is the crystallization temperature of the metal), and  $\theta_0$  is the temperature at the surface of the material.

At the phase-transition front  $x = y(\tau)$ , where  $y(\tau)$  is the coordinate of the phase-transition front, so that

$$\theta[y(\tau), \tau] = \theta_{ph}. \quad (10-4-65)$$

The condition for thermal balance determines the law of variation of the phase-transition front:

$$\frac{\partial y}{\partial \tau} = \frac{\lambda_0 + \kappa\theta_{ph}}{\rho\gamma} \left( \frac{\partial \theta}{\partial x} \right)_{x=y(\tau)}, \quad (10-4-66)$$

where  $\rho$  is the phase-transition heat.

If we denote

$$\begin{aligned} \bar{\lambda} &= \lambda_0 + \kappa \frac{\theta_{ph} + \theta_0}{2}; \quad \bar{c} = c_0 + v \frac{\theta_{ph} + \theta_0}{2}; \quad \bar{a} = \frac{\bar{\lambda}}{\gamma \bar{c}}; \\ a_1 &= \frac{\kappa(\theta_{ph} - \theta_0)}{2\bar{\lambda}}; \quad a_2 = \frac{v(\theta_{ph} - \theta_0)}{2\bar{c}}; \quad j = \frac{2\rho}{\bar{c}(\theta_{ph} - \theta_0)}; \\ \theta &= \frac{2}{\theta_{ph} - \theta_0} \left( \theta - \frac{\theta_0 + \theta_{ph}}{2} \right), \end{aligned}$$

then equation (10-4-63) and conditions (10-4-64) through (10-4-66) may be rewritten as follows:

$$(1 + a_1\theta) \frac{\partial \theta}{\partial \tau} = \bar{a} \frac{\partial}{\partial x} \left[ (1 + a_1\theta) \frac{\partial \theta}{\partial x} \right];$$

$$\theta(0, \tau) = -1; \quad \theta[y(\tau), \tau] = 1; \quad \frac{dy}{d\tau} = \frac{\bar{a}}{j} (1 + a_1) \left( \frac{\partial \theta}{\partial x} \right)_{x=y(\tau)}$$

The substitutions

$$\xi = \frac{x}{\sqrt{2a\tau}} \quad \text{and} \quad y(\tau) = \beta \sqrt{2a\tau}$$



enable us to transform the last equation into the ordinary differential equation

$$\frac{d}{d\xi} \left[ (1 + \alpha_1 \theta) \frac{d\theta}{d\xi} \right] + \xi (1 + \alpha_2 \theta) \frac{d\theta}{d\xi} = 0 \quad (10-4-67)$$

with the conditions

$$\theta(0) = -1; \quad \theta(\beta) = 1; \quad (10-4-68)$$

$$\beta = \frac{1 + \alpha_1}{j} \left( \frac{d\theta}{d\xi} \right)_{\xi=\beta}. \quad (10-4-69)$$

If we set  $\alpha_1 = \alpha_2 = 0$  in (10-4-67) and (10-4-69), then we obtain the problem with constant coefficients, in which these coefficients may be assumed to equal the arithmetic means of their values for  $\theta_{ph}$  and  $\theta_0$ . The solution of the equation is then

$$\theta = 2 \frac{\operatorname{erf}(\xi/\sqrt{2})}{\operatorname{erf}(\beta/\sqrt{2})} - 1.$$

For  $\alpha_1 = \alpha_2 = 0$ , condition (10-4-69) leads to the equation

$$\operatorname{erf}(\beta/\sqrt{2}) \exp \left[ -\frac{(\beta)^2}{2} \right] \beta j = \frac{2\sqrt{2}}{\sqrt{\pi}}, \quad (10-4-70)$$

determining the value of  $\beta$ .

The solution of equation (10-4-67) is sought in the form of the series

$$\theta(\xi) = \sum_{n=0}^{\infty} \left( \frac{d^n \theta}{d\xi^n} \right)_{\xi=\beta} \frac{(\xi - \beta)^n}{n!}. \quad (10-4-71)$$

Values of  $\frac{d^n \theta}{d\xi^n}$  for  $\xi = \beta$  and for  $n > 1$  may be found by successive differentiations of equation (10-4-67), whereas for  $n = 1$  they are found from condition (10-4-69):

$$\left( \frac{d\theta}{d\xi} \right)_{\xi=\beta} = \frac{\beta}{1 + \alpha_1} j;$$

$$\left( \frac{d^2 \theta}{d\xi^2} \right)_{\xi=\beta} = -\beta^2 \left[ \frac{1 + \alpha_2}{(1 + \alpha_1)^2} j + \frac{\alpha_1}{(1 + \alpha_1)^3} j^2 \right];$$

$$\left( \frac{d^3 \theta}{d\xi^3} \right)_{\xi=\beta} = -\beta \frac{1 + \alpha_2}{(1 + \alpha_1)^3} j + \beta^3 \left[ \frac{(1 + \alpha_2)^2}{(1 + \alpha_1)^3} j + \right.$$

$$\left. + \frac{4\alpha_1 + 3\alpha_1\alpha_2 - \alpha_2}{(1 + \alpha_1)^4} j^2 + \frac{3\alpha_1^2}{(1 + \alpha_1)^5} j^3 \right];$$

$$\left( \frac{d^4 \theta}{d\xi^4} \right)_{\xi=\beta} = \beta^4 \left[ \frac{3(1 + \alpha_2)^3}{(1 + \alpha_1)^4} j + \frac{6\alpha_1 + 4\alpha_1\alpha_2 - 2\alpha_2}{(1 + \alpha_1)^5} j^2 \right] - \beta^6 \left[ \frac{(1 + \alpha_2)^4}{(1 + \alpha_1)^5} j + \right.$$

$$\left. + \frac{(1 + \alpha_2)(11\alpha_1 + 7\alpha_1\alpha_2 - 4\alpha_2)}{(1 + \alpha_1)^6} j^2 + \frac{\alpha_1(25\alpha_1 + 18\alpha_1\alpha_2 - 7\alpha_2)}{(1 + \alpha_1)^7} j^3 + \frac{15\alpha_1^3}{(1 + \alpha_1)^8} j^4 \right];$$

etc.

According to the first condition of (10-4-68),

$$-1 = \sum_{n=0}^{\infty} (-1)^n \left( \frac{d^n \theta}{d\xi^n} \right)_{\xi=\beta} \frac{\beta^n}{n!}. \quad (10-4-72)$$

By substituting into (10-4-72) the expressions obtained for  $\left( \frac{d^n \theta}{d\xi^n} \right)_{\xi=\beta}$ ,

we obtain, after certain transformations, the following series in  $\beta$ :

$$\begin{aligned} \frac{2}{q}(1 + \alpha_1) = & \beta^2 + \beta^4 \left[ \frac{1 + \alpha_2}{3(1 + \alpha_1)} + \frac{\alpha_1}{2(1 + \alpha_1)^2} j \right] + \beta^6 \left[ \frac{(1 + \alpha_2)^2}{15(1 + \alpha_1)^2} + \right. \\ & \left. + \frac{5\alpha_1 + 4\alpha_1\alpha_2 - \alpha_2}{12(1 + \alpha_1)^3} j + \frac{\alpha_1^2}{2(1 + \alpha_1)^4} j^2 \right] + \beta^8 \left[ \frac{(1 + \alpha_2)^3}{105(1 + \alpha_1)^3} + \right. \\ & \left. + \frac{(1 + \alpha_2)(31\alpha_1 + 22\alpha_1\alpha_2 - 9\alpha_2)}{180(1 + \alpha_1)^4} j + \frac{\alpha_1(19\alpha_1 + 15\alpha_1\alpha_2 - 4\alpha_2)}{30(1 + \alpha_1)^5} j^2 + \frac{5\alpha_1^3}{8(1 + \alpha_1)^6} j^3 \right] + \dots \quad (10-4-73) \end{aligned}$$

It can be shown that for  $\alpha_1 = \alpha_2 = 0$  series (10-4-73) is transformed into the series which can be derived from (10-4-69) / 27/. If  $\beta$  is known, it is possible to use (10-4-71) to find values of  $\Theta(\xi)$  for various  $\xi$  and then to determine, from the corresponding graph, the time variation of the temperature at different coordinate points. The velocity of propagation of the phase-transition front is

$$\frac{dy}{d\tau} = \beta \sqrt{\frac{a}{2\tau}}.$$

Another method for solving nonlinear heat-conduction and mass-conduction equations has recently become widely used. This is the so-called integral method, which is similar to the Karman-Polhausen method used in boundary-layer theory. Some variants of this method are considered by Goodman /28/, Bakaleev /29/, and others.

## Chapter XI

### HEAT AND MASS TRANSFER IN A LAYERED MEDIUM

#### 11-1. The System of Heat-Transfer and Mass-Transfer Equations for a Layered Medium

The transfer of heat and matter often takes place in complex coupled systems, and the thermal and mass coefficients of each system may obey different laws and may be discontinuous at the contact boundaries. Such problems are often encountered during the study of heat and mass transfer under various conditions, such as transfer in shielding devices, the transfer which accompanies variation in the chemical state or the state of aggregation, and also during calculations of transfer in electrical insulation, during the application of various methods of determining the transfer coefficients, etc.

With the passing of time, a surface delimiting two systems may remain unchanged or else may move into the interior of one of the systems. The latter problem (the so-called Stefan problem) is of great practical importance in metallurgy (ingot crystallization), in construction heat-engineering and agrophysics (the freezing of soil and walls), etc.

The problem of maintaining normal working conditions for men and equipment in modern flying devices is a typical example of the problems considered in this section. To obtain such conditions, the shielding structures in the working section of the device are made multilayered. Each layer is constructed of a material which has properties qualitatively different from the others and which has its own specific function. For example, the surface layers may be made of materials possessing a high phase-transition heat, while layers situated further from the surface may be of materials with a low thermal conductivity. As a result, in spite of the large thermal fluxes which may be set up by friction between the device and its surroundings, these thermal fluxes do not penetrate to the working sections.

For a composite multilayered medium the system of differential heat-transfer and mass-transfer equations can be written as

$$c_{qk}\gamma_{0k}\frac{\partial t_k}{\partial \tau} = \operatorname{div}(\lambda_{qk}\nabla t_k) + \epsilon_{kp}k\gamma_{0k}c_{mk}\frac{\partial \theta_k}{\partial \tau}; \quad (11-1-1)$$

$$c_{mk}\gamma_{0k}\frac{\partial \theta_k}{\partial \tau} = \operatorname{div}[\lambda_{mk}\nabla \theta_k + \lambda_{mk}\delta_k\nabla t_k], \quad (11-1-2)$$

where the subscript  $k$  denotes the number of the given layer in the multilayered medium ( $k=1, 2, 3, \dots, n$ ).

The solution of system of equations (11-1-1) and (11-1-2) is determined according to certain conditions, which for the one-dimensional problem

have the following general form:

$$\text{for } \tau=0, t=f_1(x) \text{ and } \theta=f_2(x). \quad (11-1-3)$$

At the boundaries between the layers (that is, at  $x=l_k$ , where  $k=1, 2, \dots, n-1$ ) the heat-transfer and mass-transfer potentials and the heat and mass fluxes [in adjoining layers] are equal (Figure 11-1):

$$t_k=t_{k+1}; \quad \theta_k=\theta_{k+1}; \quad (11-1-4)$$

$$\lambda_{qk} \frac{\partial t_k}{\partial x} + (1-\epsilon_k) \rho_k j_{mk}(\tau) = \lambda_{q(k+1)} \frac{\partial t_{k+1}}{\partial x}; \quad j_{mk}(\tau) = j_{m(k+1)}(\tau). \quad (11-1-5)$$

At the external surfaces ( $x=0$  and  $x=l_n=l$ ), where heat and mass exchange with the surroundings takes place (see Figure 11-1), the boundary conditions may be written as

$$-\lambda_{q1} \frac{\partial t_1(0, \tau)}{\partial x} + j_{q0}(\tau) + (1-\epsilon_1) \rho_1 j_{m0}(\tau) = 0; \quad (11-1-6)$$

$$\lambda_{m1} \frac{\partial \theta_1(0, \tau)}{\partial x} + \lambda_{m1} \delta_1 \frac{\partial t_1(0, \tau)}{\partial x} + j_{m0}(\tau) = 0; \quad (11-1-7)$$

$$-\lambda_{qn} \frac{\partial t_n(l, \tau)}{\partial x} + j_{ql}(\tau) - (1-\epsilon_n) \rho_n j_{ml}(\tau) = 0; \quad (11-1-8)$$

$$\lambda_{mn} \frac{\partial \theta_n(l, \tau)}{\partial x} + \lambda_{mn} \delta_n \frac{\partial t_n(l, \tau)}{\partial x} + j_{ml}(\tau) = 0, \quad (11-1-9)$$

where  $j_q(\tau)$  and  $j_m(\tau)$  are respectively the heat and mass fluxes from the surface of the multilayered body. A zero subscript denotes the external surface at the left ( $x=0$ ), while the subscript  $l$  denotes the internal surface at the right ( $x=l$ ).

Heat exchange between the surfaces and the surroundings takes place by convection and by radiation, so that the heat fluxes  $j_{q0}(\tau)$  and  $j_{ql}(\tau)$  are functions of time. If the contribution of the radiation energy is small, then the heat fluxes can be expressed as

$$j_{q0}(\tau) = \alpha'_q (t_{s0} - t'_{s0});$$

$$j_{ql}(\tau) = \alpha''_q (t''_{s0} - t_{ls}),$$

where  $t'_{s0}$  and  $t''_{s0}$  are the temperatures of the surroundings, and  $\alpha'_q$  and  $\alpha''_q$  are the overall coefficients of heat exchange between the surroundings and the surfaces of the body (Figure 11-1).

In the moist-state range of the external surfaces ( $\theta_k > 100^\circ \text{M}$ ), the mass (moisture) fluxes  $j_{m0}(\tau)$  and  $j_{ml}(\tau)$  are constant provided the parameters of the surroundings are constant. In the hygroscopic range ( $\theta_k < 100^\circ \text{M}$ ), the mass fluxes are

$$j_{m0}(\tau) = \alpha'_m (\theta_s - \theta'_{s0}) \quad \text{and} \quad j_{ml}(\tau) = \alpha''_m (\theta_s - \theta''_{s0}). \quad (11-1-10)$$

The solution of system of equations (11-1-1) and (11-1-2) with limiting conditions (11-1-3) through (11-1-9) is quite difficult. However, several simplifications can be introduced, since the heat and mass exchange at the external surfaces of the body takes place within a limited range of the regime parameters of the surroundings.

If the external surfaces of the body consist of materials which have low potential conductivities [diffusivities]  $a_m$ , then the mass-exchange

number  $Bi_m = \frac{a_m l}{\lambda_m}$  can be taken as infinity (actually,  $Bi_m > 100$ ). The mass-transfer potential  $\theta$  at the surface of the body will then equal that of the surroundings (that is,  $\theta(0, \tau) = \theta'_e$  and  $\theta(l, \tau) = \theta''_e$ ).

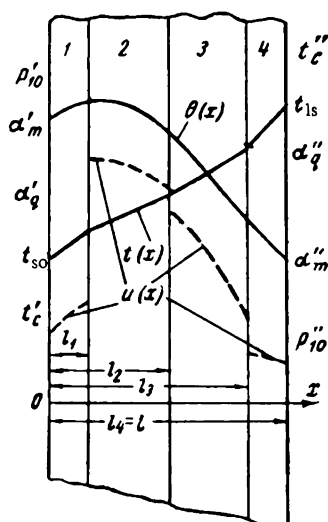


FIGURE 11-1. The calculation of heat and mass transfer in a multilayered medium

A second simplification can be introduced into the differential mass-transfer equation. If the dimensionless parameter describing the relative rate of propagation of the field of mass-transfer potential is much lower than unity ( $Lu \ll 1$ ), then the field of the potential  $\theta$  lags behind the temperature field considerably. Therefore, for constant regime parameters in the surroundings, unsteady heat transfer takes place almost with a constant mass content of the body. When the field of the potential  $\theta$  begins to develop, a steady-state temperature distribution is established, and the differential mass-transfer equation can be solved for this steady-state condition ( $\partial t / \partial \tau = 0$ ). This simplification can be made, for example, for many building materials. For example, for wood  $Lu$  is approximately from 0.007 to 0.040 and for a diatomic brick material it is from 0.01 to 0.07. However, for moist quartz sand the parameter  $Lu$  is approximately unity, so that in sand an unsteady field of the mass-transfer potential  $\theta$  is

accompanied by an unsteady temperature field.

With the above simplifications, the system of differential heat-transfer and mass-transfer equations can be written as

$$\text{div}(\lambda_{qh} \nabla t_h) + \epsilon_h \rho_h \gamma_{oh} c_{mh} \frac{\partial \theta_h}{\partial \tau} = 0; \quad (11-1-11)$$

$$c_{mh} \gamma_{oh} \frac{\partial \theta_h}{\partial \tau} = \text{div}(\lambda_{mh} \nabla \theta_h) + \text{div}(\lambda_{mh} \delta_h \nabla t_h). \quad (11-1-12)$$

For the zonal method of calculation, the coefficients  $\lambda_{qh}$ ,  $\lambda_{mh}$ , and  $\lambda_{mh} \delta_h$  can be assumed to be independent of the spatial coordinates. Thus, (11-1-11) and (11-1-12) combine into the single equation

$$\frac{\partial \theta_h}{\partial \tau} = \frac{a_{mh}}{1 + b_h} \nabla^2 \theta_h,$$

where  $b_h = \epsilon_h \rho_h \delta_h \lambda_{mh} / \lambda_{qh}$ . In most cases the coefficient  $b_h$  is considerably less than unity, so that it may be neglected. Consequently, we obtain the familiar equation describing uncoupled mass transfer.

System (11-1-1) and (11-1-2) can be simplified in another way when  $Lu \gg 1$ . A quasisteady distribution of the mass-transfer potential ( $\partial \theta_h / \partial \tau$  constant) is established very rapidly in the material, and the problem reduces to the solution of a system of uncoupled heat-transfer equations with source terms.

Thus, in both of the limiting cases we end up with a system of differential equations describing uncoupled transfer:

$$\frac{\partial \theta_k}{\partial \tau} = a_k \nabla^2 \theta_k + w_k(x, y, z, \tau) \quad (k=1, 2, \dots, n), \quad (11-1-13)$$

and the solution of such a system constitutes the subject of the present section. Here, for generality, the potential  $\theta$  may refer to either the temperature or the mass-transfer potential.

## 11-2. A System Consisting of Two Bodies

Let us now solve system (11-1-13) for various two-layered one-dimensional bodies.

a) Solution for two semi-infinite bodies, with  $\theta_1$  at  $0 \leq x < \infty$  and with  $\theta_2$  at  $-\infty < x \leq 0$

We have here the system of equations

$$\frac{\partial \theta_1}{\partial \tau} = a_1 \frac{\partial^2 \theta_1}{\partial x^2}; \quad (11-2-1)$$

$$\frac{\partial \theta_2}{\partial \tau} = a_2 \frac{\partial^2 \theta_2}{\partial x^2} \quad (11-2-2)$$

with the following limiting conditions:

$$\theta_1(x, 0) = f_1(x) \quad (x \geq 0); \quad \theta_2(x, 0) = f_2(x) \quad (x \leq 0); \quad (11-2-3)$$

$$\theta_1(+0, \tau) = \theta_2(-0, \tau); \quad \lambda_1 \frac{\partial \theta_1(0, \tau)}{\partial x} = \lambda_2 \frac{\partial \theta_2(0, \tau)}{\partial x}. \quad (11-2-4)$$

Following Tsoi [1], let us assume that  $\theta_1$  and  $\theta_2$  are functions to which Fourier transformations with respect to the variable  $x$  are applicable. The quantity

$$\lambda_1 \frac{\partial \theta_1(0, \tau)}{\partial x} = \lambda_2 \frac{\partial \theta_2(0, \tau)}{\partial x} = j(\tau),$$

representing an unknown function describing the mass flux occurring at the boundary between the two media, will be determined in the following. If we apply to equation (11-2-1) the infinite Fourier cosine transformation

$$\{\theta_k(\xi, \tau)\}_c = \sqrt{\frac{2}{\pi}} \int_0^\infty \theta_k(x, \tau) \cos x\xi dx,$$

then, taking into account (11-2-3), we obtain

$$\frac{d\{\theta_1\}_c}{d\tau} = -\xi^2 a_1 \{\theta_1\}_c - \frac{a_1}{\lambda_1} \sqrt{\frac{2}{\pi}} j(\tau).$$

Next, by using transformed initial conditions (11-2-3) to solve this equation and then applying an inverse cosine transformation, we find that

$$\begin{aligned} \theta_1(x, \tau) = & \frac{1}{2\sqrt{\pi a_1 \tau}} \int_0^\infty f_1(\alpha) \left\{ \exp\left[-\frac{(x+\alpha)^2}{4a_1 \tau}\right] + \exp\left[-\frac{(x-\alpha)^2}{4a_1 \tau}\right] \right\} d\alpha - \\ & - \frac{1}{\lambda_1} \sqrt{\frac{a_1}{\pi}} \int_0^\tau \frac{j(\tau^*)}{\sqrt{\tau-\tau^*}} \exp\left[-\frac{x^2}{4a_1(\tau-\tau^*)}\right] d\tau^*. \end{aligned} \quad (11-2-5)$$

To determine the function  $\theta_2(x, \tau)$  we first transform the domain  $\Omega (-\infty < x \leq 0)$  into the domain  $\Omega' (0 \leq z < \infty)$  by means of the substitution  $x = -z$ . The solution for  $\theta_2(-z, \tau)$  can be obtained using the same operations which were used to obtain (11-2-5), so that we have

$$\begin{aligned} \theta_2(-z, \tau) = & \frac{1}{2\sqrt{\pi a_2 \tau}} \int_0^\infty f_2(-a) \left\{ \exp \left[ -\frac{(z+a)^2}{4a_2 \tau} \right] + \exp \left[ -\frac{(z-a)^2}{4a_2 \tau} \right] \right\} da + \\ & + \frac{1}{\lambda_2} \sqrt{\frac{a_2}{\pi}} \int_0^\tau \frac{j(\tau^*)}{\sqrt{\tau - \tau^*}} \exp \left[ -\frac{z^2}{4a_2(\tau - \tau^*)} \right] d\tau^*. \end{aligned} \quad (11-2-6)$$

To determine  $j(\tau)$  we use the first of the coupling conditions (11-2-4). We have

$$\int_0^\tau \frac{j(\tau^*)}{\sqrt{\tau - \tau^*}} d\tau^* = F(\tau), \quad (11-2-7)$$

where

$$\begin{aligned} F(\tau) = & \frac{\lambda_1 \lambda_2}{(\lambda_1 \sqrt{a_2} + \lambda_2 \sqrt{a_1}) \sqrt{\tau}} \left[ \frac{1}{\sqrt{a_1}} \int_0^\infty f_1(a) \exp \left( -\frac{a^2}{4a_1 \tau} \right) da - \right. \\ & \left. - \frac{1}{\sqrt{a_2}} \int_0^\infty f_2(-a) \exp \left( -\frac{a^2}{4a_2 \tau} \right) da \right]. \end{aligned}$$

Equation (11-2-7) is an Abel integral equation with respect to the function  $j(\tau)$ ; its solution is

$$j(\tau) = \frac{1}{\pi} \cdot \frac{d}{d\tau} \int_0^\tau \frac{F(\tau^*)}{\sqrt{\tau - \tau^*}} d\tau^*. \quad (11-2-8)$$

Let us note that the presence of the kernel  $\exp \left[ -\frac{(x \pm a)^2}{4a\tau} \right]$  in formulas (11-2-5) and (11-2-6) makes it possible to speak of the convergence of the improper integrals for a wide range of functions  $f_1(x)$  and  $f_2(x)$ , that is, for a wide range of initial conditions. The latter fact is very significant during a zonal calculation of the system of heat-transfer and mass-transfer equations. In particular, it can be shown that the following equalities hold true:

$$\lim_{x \rightarrow 0+} \frac{\partial \theta_1}{\partial x} = \frac{j(\tau)}{\lambda_1} \quad \text{and} \quad \lim_{x \rightarrow 0-} \frac{\partial \theta_2}{\partial x} = \frac{j(\tau)}{\lambda_2}.$$

Let us consider a particular case of the problem at hand, namely the case when

$$f_1(x) = \theta_1^0 = \text{const} \quad \text{and} \quad f_2(x) = \theta_2^0 = \text{const}. \quad (11-2-9)$$

For constant initial potential distributions, equations (11-2-5) and (11-2-6) represent "generalized solutions" of the system (11-2-1) and (11-2-2), since it is obvious that Fourier transformations are inapplicable to these functions. After some simple transformations, we obtain

$$F(\tau) = \frac{(\theta_1^0 - \theta_2^0) \lambda_1 \lambda_2 \sqrt{\pi}}{\lambda_1 \sqrt{a_2} + \lambda_2 \sqrt{a_1}} = \text{const}.$$

The substitution of the value of the constant  $F(\tau)$  into equation (11-2-8)

gives

$$j(\tau) = \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \cdot \frac{\theta_1 - \theta_2}{\sqrt{\pi \tau}},$$

where for problems of uncoupled transfer we define  $\epsilon = \frac{\lambda}{\sqrt{a}} = \sqrt{\lambda c \gamma}$  as the coefficient of penetration or activity (either thermal or mass).

Subject to conditions (11-2-9), solutions (11-2-5) and (11-2-6) become

$$\frac{\theta_1(x, \tau) - \theta_1^0}{\theta_1^0 - \theta_2^0} = -\frac{1}{\pi(1+K_*)} \int_0^\tau \frac{\exp\left[-\frac{x^2}{4a_1(\tau-\tau^*)}\right]}{\sqrt{\tau^*(\tau-\tau^*)}} d\tau^*; \quad (11-2-10)$$

$$\frac{\theta_2(x, \tau) - \theta_2^0}{\theta_1^0 - \theta_2^0} = -\frac{K_*}{\pi(1+K_*)} \int_0^\tau \frac{\exp\left[-\frac{x^2}{4a_2(\tau-\tau^*)}\right]}{\sqrt{\tau^*(\tau-\tau^*)}} d\tau^*, \quad (11-2-11)$$

where  $K_* = \frac{\lambda_1}{\lambda_2} \sqrt{\frac{a_2}{a_1}} = \sqrt{\frac{\lambda_1 c_1 \gamma_1}{\lambda_2 c_2 \gamma_2}} = \frac{\epsilon_1}{\epsilon_2}$  is a dimensionless parameter characterizing the activity of the first medium with respect to the second. We may also write  $K_* = \frac{K_\lambda}{\sqrt{K_a}}$ , where  $K_\lambda$  is a parameter characterizing the relative conductivity of the medium ( $K_\lambda = \frac{\lambda_1}{\lambda_2}$ ), and  $K_a$  is a parameter characterizing the inertial properties of the first medium with respect to the second ( $K_a = \frac{a_1}{a_2}$ ).

In solutions (11-2-10) and (11-2-11) the integrals are time integrals. In some applications, however, it is desirable to integrate with respect to the spatial coordinate. Then, equations (11-2-10) and (11-2-11) have the form

$$\theta_1(x, \tau) = \frac{\theta_2^0 + K_* \theta_1^0}{1+K_*} + \frac{\theta_1^0 - \theta_2^0}{1+K_*} \operatorname{erf}\left(\frac{x}{2\sqrt{a_1 \tau}}\right) \quad (x > 0); \quad (11-2-12)$$

$$\theta_2(x, \tau) = \frac{\theta_2^0 + K_* \theta_1^0}{1+K_*} + \frac{K_* (\theta_2^0 - \theta_1^0)}{1+K_*} \operatorname{erf}\left(\frac{|x|}{2\sqrt{a_2 \tau}}\right) \quad (x < 0). \quad (11-2-13)$$

If we set  $\theta_2^0 = 0$ , then we obtain the solution given in /2/. The more general system of equations

$$\frac{\partial \theta_1}{\partial \tau} = a_1 \frac{\partial^2 \theta_1}{\partial x^2} + w_1(x, \tau); \quad (11-2-14)$$

$$\frac{\partial \theta_2}{\partial \tau} = a_2 \frac{\partial^2 \theta_2}{\partial x^2} + w_2(x, \tau) \quad (11-2-15)$$

can be solved similarly or else by means of Green's function. For limiting conditions (11-2-3) and (11-2-4) the solution may be written as

$$\begin{aligned} \theta_1(x, \tau) = & \frac{1}{2\sqrt{\pi a_1 \tau}} \int_0^\infty f_1(a) \left\{ -h \exp\left[-\frac{(x+a)^2}{4a_1 \tau}\right] + \exp\left[-\frac{(x-a)^2}{4a_1 \tau}\right] \right\} da + \\ & + \frac{1}{(1+K_*)\sqrt{\pi a_2 \tau}} \int_{-\infty}^0 f_2(a) \exp\left[-\frac{(x-a\sqrt{K_a})^2}{4a_1 \tau}\right] da + \\ & + \frac{1}{2\sqrt{\pi a_1}} \int_0^\tau d\alpha \int_0^\tau \frac{w_1(\alpha, \tau)}{\sqrt{\tau-\tau^*}} \left\{ -h \exp\left[-\frac{(x+\alpha)^2}{4a_1(\tau-\tau^*)}\right] + \exp\left[-\frac{(x-\alpha)^2}{4a_1(\tau-\tau^*)}\right] \right\} d\tau^* + \\ & + \frac{1}{(1+K_*)\sqrt{\pi a_2}} \int_{-\infty}^0 d\alpha \int_0^\tau \frac{w_2(\alpha, \tau)}{\sqrt{\tau-\tau^*}} \exp\left[-\frac{(x-a\sqrt{K_a})^2}{4a_1(\tau-\tau^*)}\right] d\tau^*; \end{aligned} \quad (11-2-16)$$



$$\begin{aligned}
\theta_2(x, \tau) = & \frac{1}{2\sqrt{\pi a_1 \tau}} \int_{-\infty}^0 f_2(a) \left\{ h \exp \left[ -\frac{(x+a)^2}{4a_1 \tau} \right] + \exp \left[ -\frac{(x-a)^2}{4a_1 \tau} \right] \right\} da + \\
& + \frac{K_s}{(1+K_s)\sqrt{\pi a_1 \tau}} \int_0^{\infty} f_1(a) \exp \left[ -\frac{(a-x\sqrt{K_s})^2}{4a_1 \tau} \right] da + \\
; & + \frac{1}{2\sqrt{\pi a_2}} \int_{-\infty}^0 d\alpha \int_0^{\tau} \frac{w_2(a, \tau)}{\sqrt{\tau-\tau^*}} \left\{ h \exp \left[ -\frac{(x+a)^2}{4a_2(\tau-\tau^*)} \right] + \exp \left[ -\frac{(x-a)^2}{4a_2(\tau-\tau^*)} \right] \right\} d\alpha + \\
& + \frac{K_s}{(1+K_s)\sqrt{\pi a_2}} \int_0^{\infty} d\alpha \int_0^{\tau} \frac{w_1(a, \tau)}{\sqrt{\tau-\tau^*}} \exp \left[ -\frac{(a-x\sqrt{K_s})^2}{4a_2(\tau-\tau^*)} \right] d\alpha, \quad (11-2-17)
\end{aligned}$$

where

$$h = \frac{1-K_s}{1+K_s}.$$

If in equations (11-2-16) and (11-2-17) we set  $w_k(x, \tau) = 0$ , for  $(k=1, 2)$ , then we obtain solutions (11-2-5) and (11-2-6), provided the expression for  $F(\tau)$  is taken into account.

If the contact between the media is not perfect, then boundary conditions (11-2-4) are somewhat different, namely

$$\theta_2(0, \tau) - \theta_1(0, \tau) = \chi_1(\tau) \quad \text{and} \quad \lambda_2 \frac{\partial \theta_2(0, \tau)}{\partial x} - \lambda_1 \frac{\partial \theta_1(0, \tau)}{\partial x} = \chi_2(\tau).$$

The solution of this problem has the form

$$\begin{aligned}
\theta_1(x, \tau) &= \theta_1^{(1)} + \theta_1^{(2)}; \\
\theta_2(x, \tau) &= \theta_2^{(1)} + \theta_2^{(2)};
\end{aligned}$$

where  $\theta_1^{(1)}$  and  $\theta_2^{(1)}$  are determined respectively by equations (11-2-16) and (11-2-17), and where  $\theta_1^{(2)}$  and  $\theta_2^{(2)}$  are determined by the expressions

$$\begin{aligned}
\theta_1^{(2)} = & -\frac{1}{1+K_s} \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{a_1 \tau}}}^{\infty} \exp(-\alpha^2) \chi_1 \left( \tau - \frac{x^2}{4a_1 \alpha^2} \right) d\alpha + \\
& + \frac{\sqrt{a_2}/\lambda_2}{1+K_s} \cdot \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{a_1}}^{\infty} d\beta \int_{\beta/2\sqrt{\tau}}^{\infty} \exp(-\alpha^2) \chi_2 \left( \tau - \frac{\beta^2}{4\alpha^2} \right) d\alpha; \\
\theta_2^{(2)} = & \frac{K_s}{1+K_s} \cdot \frac{2}{\sqrt{\pi}} \int_{-x/2\sqrt{a_1 \tau}}^{\infty} \exp(-\alpha^2) \chi_1 \left( \tau - \frac{x^2}{4a_1 \alpha^2} \right) d\alpha + \\
& + \frac{\sqrt{a_1}/\lambda_1}{1+K_s} \cdot \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{a_2}} d\beta \int_{-\beta/2\sqrt{\tau}}^{\infty} \exp(-\alpha^2) \chi_2 \left( \tau - \frac{\beta^2}{4\alpha^2} \right) d\alpha.
\end{aligned}$$

From the above solutions it is possible to obtain several particular solutions.

- b) Solution for a system consisting of a finite body and a semi-infinite body, with  $\theta_1$  at  $0 \leq x \leq l$  and with  $\theta_2$  at  $l \leq x < \infty$**

The solution for a layered system consisting of a finite body and a semi-infinite body can be obtained using various integral transformations, but most often Laplace integral transformations are used. The detailed solution

of a simple problem by means of this method will now be given; for other problems only a summary of the final results will be given.

Let us consider system of equations (11-2-1) and (11-2-2), subject to the following limiting conditions:

$$\theta_1(x, 0) = \theta_2(x, 0) = 0; \quad (11-2-18)$$

$$\theta_1(l, \tau) = \theta_2(l, \tau); \quad (11-2-19)$$

$$\lambda_1 \frac{\partial \theta_1(l, \tau)}{\partial x} = \lambda_2 \frac{\partial \theta_2(l, \tau)}{\partial x}; \quad (11-2-20)$$

$$\theta_1(0, \tau) = \theta_0 = \text{const}; \quad (11-2-21)$$

$$\theta_2(\infty, \tau) = 0. \quad (11-2-22)$$

The solutions of equations (11-2-1) and (11-2-2), in terms of the transforms of the functions  $\theta(x, \tau)$ , with conditions (11-2-18) and (11-2-22), are

$$\{\theta_1\}_L = A \exp\left(\sqrt{\frac{s}{a_1}} x\right) + B \exp\left(-\sqrt{\frac{s}{a_1}} x\right); \quad (11-2-23)$$

$$\{\theta_2\}_L = C \exp\left(-\sqrt{\frac{s}{a_2}} x\right). \quad (11-2-24)$$

Constants  $A$ ,  $B$ , and  $C$  can be found from boundary conditions (11-2-19) through (11-2-21), which for the transforms can be written as

$$\{\theta_1(l, s)\}_L = \{\theta_2(l, s)\}_L; \quad K_\lambda \{\theta'_1(l, s)\}_L = \{\theta'_2(l, s)\}_L;$$

$$\{\theta_1(0, s)\}_L = \frac{\theta_0}{s}.$$

When the values of the constants are inserted, solutions (11-2-23) and (11-2-24) become

$$\begin{aligned} \{\theta_1(x, s)\}_L &= \frac{\theta_0}{s} \exp\left(-\sqrt{\frac{s}{a_1}} x\right) - \\ &- \frac{h\theta_0}{s} \exp\left(-\sqrt{\frac{s}{a_1}} l\right) \left[ \frac{\exp\left(\sqrt{\frac{s}{a_1}} x\right) - \exp\left(-\sqrt{\frac{s}{a_1}} x\right)}{\exp\left(\sqrt{\frac{s}{a_1}} l\right) - h \exp\left(-\sqrt{\frac{s}{a_1}} l\right)} \right]; \end{aligned} \quad (11-2-25)$$

$$\begin{aligned} \{\theta_2(x, s)\}_L &= \left\{ \frac{\theta_0}{s} \exp\left(-\sqrt{\frac{s}{a_1}} l\right) - \left[ \frac{\exp\left(\sqrt{\frac{s}{a_1}} l\right) - \exp\left(-\sqrt{\frac{s}{a_1}} l\right)}{\exp\left(\sqrt{\frac{s}{a_1}} l\right) - h \exp\left(-\sqrt{\frac{s}{a_1}} l\right)} \right] \right\} \times \\ &\times \frac{h\theta_0}{s} \exp\left(-\sqrt{\frac{s}{a_1}} l\right) \exp\left[-\sqrt{\frac{s}{a_2}} (x-l)\right]. \end{aligned} \quad (11-2-26)$$

Since  $|h| = \left| \frac{1-K_2}{1+K_2} \right| < 1$ , therefore on the basis of the expansion

$$\frac{1}{1-x} = 1 + x + x^2 + \dots \quad (|x| < 1),$$

it is possible to use the transformation

$$\begin{aligned} \frac{1}{\exp\left(\sqrt{\frac{s}{a_1}} l\right) - h \exp\left(-\sqrt{\frac{s}{a_1}} l\right)} &= \frac{\exp\left(-\sqrt{\frac{s}{a_1}} l\right)}{1 - h \exp\left(-2\sqrt{\frac{s}{a_1}} l\right)} = \\ &= \sum_{n=1}^{\infty} h^{n-1} \exp\left[-(2n-1)\sqrt{\frac{s}{a_1}} l\right]. \end{aligned} \quad (11-2-27)$$

Moreover, since  $K_2 > 0$  and  $|h| < 1$ , the infinite sum converges rapidly. Thus, taking into account (11-2-27),

we may rewrite solutions (11-2-25) and (11-2-26) as

$$\begin{aligned} \frac{\{\theta_1(x, s)\}_L}{\theta_0} &= \frac{1}{s} \exp\left(-\sqrt{\frac{s}{a_1}} x\right) - \\ &- \frac{h}{s} \sum_{n=1}^{\infty} h^{n-1} \left\{ \exp\left[-(2n-1)x\sqrt{\frac{s}{a_1}}\right] - \exp\left[-(2n+1)x\sqrt{\frac{s}{a_1}}\right] \right\}; \\ \frac{\{\theta_2(x, s)\}_L}{\theta_0} &= \frac{1-h}{s} \sum_{n=1}^{\infty} h^{n-1} \exp\left\{-\left[(x-l)+(2n-1)l\sqrt{\frac{a_1}{a_2}}\right]\sqrt{\frac{s}{a_1}}\right\}. \end{aligned}$$

Finally, in terms of the inverse transforms, we obtain

$$\frac{\theta_1(x, \tau)}{\theta_0} = \operatorname{erfc} \frac{X}{2\sqrt{Fo_1}} - h \sum_{n=1}^{\infty} h^{n-1} \left[ \operatorname{erfc} \frac{2n-X}{2\sqrt{Fo_1}} - \operatorname{erfc} \frac{2n+X}{2\sqrt{Fo_1}} \right]; \quad (11-2-28)$$

$$\frac{\theta_2(x, \tau)}{\theta_0} = \frac{2K_e}{1+K_e} \sum_{n=1}^{\infty} h^{n-1} \operatorname{erfc} \left[ \frac{X-1+(2n-1)K_a^{-1/2}}{2\sqrt{Fo_2}} \right], \quad (11-2-29)$$

where

$$X = x/l; \quad Fo_1 = \frac{a_1 \tau}{l^2}; \quad Fo_2 = \frac{a_2 \tau}{l^2}.$$

Let us now give several solutions for other limiting conditions.

I. When conditions (11-2-18) and (11-2-22) are

$$\theta_1(x, 0) = \theta_1^0 = \text{const}; \quad \theta_2(x, 0) = \theta_2^0 = \text{const}; \quad (11-2-30)$$

$$\theta_2(\infty, \tau) = \theta_2^0 = \text{const}; \quad \frac{\partial \theta_2(\infty, \tau)}{\partial x} = 0, \quad (11-2-31)$$

then the solution of system of equations (11-2-1) and (11-2-2) with conditions (11-2-30) and (11-2-31) and (11-2-19) through (11-2-21) is

$$\begin{aligned} \frac{\theta_1(x, \tau) - \theta_1^0}{\theta_0 - \theta_1^0} &= \operatorname{erfc} \frac{X}{2\sqrt{Fo_1}} - \sum_{n=1}^{\infty} h^n \left[ \operatorname{erfc} \frac{2n-X}{2\sqrt{Fo_1}} - \operatorname{erfc} \frac{2n+X}{2\sqrt{Fo_1}} \right] - \\ &- \frac{1}{1+K_e} \cdot \frac{\theta_1^0 - \theta_2^0}{\theta_0 - \theta_1^0} \sum_{n=1}^{\infty} h^{n-1} \left[ \operatorname{erfc} \frac{(2n-1)-X}{2\sqrt{Fo_1}} - \operatorname{erfc} \frac{(2n-1)+X}{2\sqrt{Fo_1}} \right]; \\ \frac{\theta_2(x, \tau) - \theta_2^0}{\theta_0 - \theta_1^0} &= \frac{2K_e}{1+K_e} \sum_{n=1}^{\infty} h^{n-1} \operatorname{erfc} \frac{X-1+(2n-1)K_a^{-1/2}}{2\sqrt{Fo_2}} + \\ &+ \frac{K_e}{1+K_e} \frac{\theta_1^0 - \theta_2^0}{\theta_0 - \theta_1^0} \operatorname{erfc} \frac{X-1}{2\sqrt{Fo_2}} + \\ &+ \frac{2K_e}{(1+K_e)^2(\theta_0 - \theta_1^0)} \sum_{n=1}^{\infty} h^{n-1} \operatorname{erfc} \frac{X-1+2nK_a^{-1/2}}{2\sqrt{Fo_2}}. \end{aligned}$$

II. When condition (11-2-21) is

$$\theta_1(0, \tau) = b\tau \quad (b = \text{const}), \quad (11-2-32)$$

then the solution of system (11-2-1) and (11-2-2) with conditions (11-2-18) through (11-2-20), (11-2-22), and (11-2-32) is /3/

$$\begin{aligned} \theta_1(x, \tau) &= 4b\tau \left\{ i^3 \operatorname{erfc} \frac{X}{2\sqrt{Fo_1}} + \sum_{n=2}^{\infty} h^{n-1} i^3 \operatorname{erfc} \frac{2(n-1)+X}{2\sqrt{Fo_1}} - \right. \\ &\left. - h \left[ i^3 \operatorname{erfc} \frac{2-X}{2\sqrt{Fo_1}} + \sum_{n=2}^{\infty} h^{n-1} i^3 \operatorname{erfc} \frac{2n-X}{2\sqrt{Fo_1}} \right] \right\}; \end{aligned}$$

$$\theta_2(x, \tau) = 4b\tau \left\{ (1-h)I^2 \operatorname{erfc} \left[ \frac{1}{2\sqrt{Fo_1}} - \frac{1-X}{2\sqrt{Fo_2}} \right] + \right. \\ \left. + (1-h) \sum_{n=2}^{\infty} h^{n-1} I^2 \operatorname{erfc} \left[ \frac{2n-1}{2\sqrt{Fo_1}} - \frac{1-X}{2\sqrt{Fo_2}} \right] \right\},$$

where

$$i^2 \operatorname{erfc} z = \int_z^{\infty} i \operatorname{erfc} \xi d\xi \quad \text{and} \quad i \operatorname{erfc} z = \int_z^{\infty} \operatorname{erfc} \xi d\xi.$$

III. When condition (11-2-21) is

$$\theta_1(0, \tau) = \theta_m \sin \omega \tau,$$

then the solution /4/ can be written as

$$\begin{aligned} \frac{\theta_1(x, \tau)}{\theta_m} = & \frac{1}{M} \left[ \exp \left( -\sqrt{\frac{1}{2} Pd_1 X} \right) \sin \left( Pd_1 Fo_1 - \sqrt{\frac{1}{2} Pd_1 X} \right) - \right. \\ & - h \exp \left[ -\sqrt{\frac{1}{2} Pd_1 (2-X)} \right] \sin \left[ Pd_1 Fo_1 + \sqrt{\frac{1}{2} Pd_1 (2-X)} \right] - \\ & - h \left\{ \exp \left[ -\sqrt{\frac{1}{2} Pd_1 (4-X)} \right] \sin \left( Pd_1 Fo_1 + \sqrt{\frac{1}{2} Pd_1 X} \right) + \right. \\ & \left. + \exp \left[ -\sqrt{\frac{1}{2} Pd_1 (2-X)} \right] \sin \left[ Pd_1 Fo_1 - \sqrt{\frac{1}{2} Pd_1 (2-X)} \right] \right\} \Bigg] + \\ & + \frac{Pd^*}{\pi} \left[ \int_0^{\infty} \exp(-Pd^* Fo_1) \sin \left( X \sqrt{Pd^*} - \arctan \frac{-h \sin 2 \sqrt{Pd^*}}{1-h \cos 2 \sqrt{Pd^*}} \right) \frac{d Pd^*}{(Pd^*)^2 + Pd_1^2} - \right. \\ & \left. - h \int_0^{\infty} \exp(-Pd^* Fo_1) \sin \left( -X \sqrt{Pd^*} \arctan \frac{\sin 2 \sqrt{Pd^*}}{\cos 2 \sqrt{Pd^*} - h} \right) \frac{d Pd^*}{(Pd^*)^2 + Pd_1^2} \right]; \\ \frac{\theta_2(x, \tau)}{\theta_m} = & \frac{1-h}{M} \left\{ \exp \left[ -\sqrt{\frac{1}{2} Pd_1} - \sqrt{\frac{1}{2} Pd_2 (X-1)} \right] \sin \left[ Pd_1 Fo_1 - \right. \right. \\ & \left. - \sqrt{\frac{1}{2} Pd_1} - \sqrt{\frac{1}{2} Pd_2 (X-1)} \right] - h \exp \left[ -3 \sqrt{\frac{1}{2} Pd_1} - \sqrt{\frac{1}{2} Pd_2 (X-1)} \right] \times \\ & \times \sin \left[ Pd_1 Fo_1 + \sqrt{\frac{1}{2} Pd_1} - \sqrt{\frac{1}{2} Pd_2 (X-1)} \right] \Bigg\} + \\ & + (1-h) Pd_1 \frac{1}{\pi} \int_0^{\infty} \exp(-Pd^* Fo_1) \sin \left[ (X-1) \sqrt{Pd^*} K_a^{1/2} + \arctan \frac{\tan \sqrt{Pd^*}}{K_a} \right] \times \\ & \times \frac{d Pd^*}{(Pd^*)^2 + Pd_1^2}, \end{aligned}$$

where

$$M = 1 - 2h \exp \left( -2 \sqrt{\frac{1}{2} Pd_1} \right) \cos 2 \sqrt{\frac{1}{2} Pd_1} + h^2 \exp \left( -4 \sqrt{\frac{1}{2} Pd_1} \right);$$

$Pd_h = \frac{\omega}{a_h} I^2$ ; and  $Pd^* = \frac{\omega^*}{a_1} I^2$  is the Predvoditelev number.

### c) Solution for a system of two finite bodies

A number of solutions obtained using Laplace integral transformations will be given.

**I. Two infinite plates.** Let us consider the transfer of heat or matter in a double-layered wall of thickness  $l$ . The first layer has a thickness  $l_1$  ( $0 \leq x \leq l_1$ ), while the second has a thickness  $l_2$  ( $l_1 \leq x \leq l$ ). The following

limiting conditions are assumed:

$$\theta_1(x, 0) = \theta_2(x, 0) = \theta_0 = \text{const}; \quad (11-2-33)$$

$$\theta_1(l, \tau) = \theta_2(l, \tau); \quad \lambda_1 \frac{\partial \theta_1(l_1, \tau)}{\partial x} = \lambda_2 \frac{\partial \theta_2(l_1, \tau)}{\partial x}; \quad (11-2-34)$$

$$\theta_1(0, \tau) = \theta_0 = \text{const}; \quad \theta_2(l, \tau) = \theta_0 = \text{const}. \quad (11-2-35)$$

The solution of system of equations (11-2-1) and (11-2-2), subject to conditions (11-2-33) through (11-2-35), can be written as

$$\begin{aligned} \frac{\theta_1(x, \tau) - \theta_0}{\theta_0 - \theta_0} = & \frac{K_\lambda - \frac{x l_1}{l_2}}{K_\lambda + l_1/l_2} - \sum_{n=1}^{\infty} \frac{2}{\mu_n \psi_n} \left( K_0 \cos \mu_n \frac{x - l_1}{l_1} \sin K_0^{1/2} \mu_n K_l - \right. \\ & \left. - \sin \mu_n \frac{x - l_1}{l_1} \cos K_0^{1/2} \mu_n K_l \right) \exp(-\mu_n^2 Fo_1); \end{aligned} \quad (11-2-36)$$

$$\frac{\theta_2(x, \tau) - \theta_0}{\theta_0 - \theta_0} = \frac{l - x}{l_2 - l_1 K_\lambda^{-1}} - \sum_{n=1}^{\infty} \frac{2}{\mu_n \psi_n} K_0 \sin K_0^{1/2} \mu_n \frac{l - x}{l_2} \exp(-K_0^2 \mu_n^2 Fo_1), \quad (11-2-37)$$

where

$$\psi_n = (K_0 + K_0^{1/2} K_l) \sin \mu_n \sin K_0^{1/2} K_l \mu_n - (1 + K_0 K_0^{1/2} K_l) \cos \mu_n \cos K_0^{1/2} K_l \mu_n.$$

Here  $\mu_n$  are the roots of the equation

$$\tan \mu + K_0 \tan (K_0^{1/2} K_l \mu) = 0 \quad (11-2-38)$$

or of the two equations

$$\sin \mu = 0 \quad \text{and} \quad \sin K_0^{1/2} K_l \mu = 0; \quad (11-2-38')$$

and also

$$Fo_1 = \frac{a \tau}{l_1^2} \quad \text{and} \quad K_l = l_2/l_1.$$

Solutions (11-2-36) and (11-2-37) correspond to the case when  $K_0^{1/2} K_l$  is not a proper fraction. If  $K_0^{1/2} K_l$  is a proper fraction, which we will write as the ratio

$$K_0^{1/2} K_l = \frac{b}{a},$$

then  $K_0^{1/2} K_l \mu_n = \frac{b}{a} \mu_n = n\pi$  and  $\mu_n = \frac{an\pi}{b}$ . In this case it is necessary to add to solutions (11-2-26) and (11-2-37), respectively, the following quantities:

$$\begin{aligned} & - \frac{2}{\pi(\beta K_0 + b)} \sum_{m=1}^{\infty} \frac{\cos mb\pi}{m} \sin \frac{mb\pi x}{l} \exp(-m^2 b^2 \pi^2 Fo_1); \\ & - \frac{2}{\pi(\beta K_0 + b)} \sum_{m=1}^{\infty} \frac{\cos mb\pi}{m} \sin \frac{bm\pi x}{l} \exp(-b^2 m^2 \pi^2 K_0 K_l^2 Fo_1). \end{aligned}$$

In the steady state ( $Fo_1 = \infty$ ), the distribution of the dimensionless potential is given by a linear law. The dimensionless potential  $\frac{\theta_1(0, \tau) - \theta_0}{\theta_0 - \theta_0}$  at the left surface ( $x=0$ ) is equal to unity, while that at the opposite surface ( $x=l_1+l_2$ ) is equal to zero. At the contact surface ( $x=l_1$ ), the dimensionless potential is  $\frac{K_\lambda K_l}{K_\lambda K_l + 1}$ . If the conductivities  $\lambda$  are equal ( $K_\lambda = 1$ ), then for  $l_1 = l_2$  ( $K_l = 1$ ) the dimensionless potential is  $1/2$ .

Solution (11-2-36) and (11-2-37) is convenient for calculations when  $Fo_1$  is relatively large, in which case all the series terms except the first can be neglected. For low values of  $Fo_1$ , the transfer phenomena in this system are analogous to those in a double-layered system in which the thickness of the second layer is infinite ( $l_2 = \infty$ ).

II. Two spherical bodies (sphere within a sphere). For this problem system of equations (11-2-1) and (11-2-2) may be written as

$$\frac{\partial [r\theta_1(r, \tau)]}{\partial \tau} = a_1 \frac{\partial^2 [r\theta_1(r, \tau)]}{\partial r^2} \quad (\tau > 0; 0 < r < R_1); \quad (11-2-39)$$

$$\frac{\partial [r\theta_2(r, \tau)]}{\partial \tau} = a_2 \frac{\partial^2 [r\theta_2(r, \tau)]}{\partial r^2} \quad (\tau > 0; R_1 < r < R_2). \quad (11-2-40)$$

The solution of system of equations (11-2-39) and (11-2-40) will now be given, for the following limiting conditions:

$$\begin{aligned} \theta_1(r, 0) &= \theta_0; \quad \theta_2(r, 0) = 0; \\ \theta_1(R_1, \tau) &= \theta_2(R_1, \tau); \quad \lambda_1 \frac{\partial \theta_1(R_1, \tau)}{\partial r} = \lambda_2 \frac{\partial \theta_2(R_1, \tau)}{\partial r}; \\ \theta_2(R_2, \tau) &= 0. \end{aligned}$$

The solutions are

$$\frac{\theta_1(r, \tau)}{\theta_0} = \sum_{n=1}^{\infty} \frac{2}{\psi(\mu_n)} \sin \mu_n \sin K_a^{1/2} (K_R - 1) \mu_n \frac{\sin \mu_n r / R_1}{r / R_2} \exp(-\mu_n^2 Fo_1); \quad (11-2-41)$$

$$\frac{\theta_2(r, \tau)}{\theta_0} = \sum_{n=1}^{\infty} \frac{2}{\psi(\mu_n)} \sin^2 \mu_n \frac{\sin K_a^{1/2} (K_R - r/R_1) \mu_n}{r/R_2} \exp(-\mu_n^2 K_a K_R^2 Fo_2), \quad (11-2-42)$$

where

$$\begin{aligned} \psi(\mu_n) &= K_a \mu_n \sin^2 K_a^{1/2} (K_R - 1) \mu_n + K_a^{1/2} (K_R - 1) \mu_n \sin^2 \mu_n + \\ &+ \frac{1 - K_a^{1/2} K_R}{K_a^{1/2} \mu_n} \sin^2 \mu_n \sin^2 K_a^{1/2} (K_R - 1) \mu_n. \end{aligned}$$

The roots  $\mu_n$  obey the equation

$$[K_a^{1/2} \mu \cot K_a^{1/2} (K_R - 1) \mu + 1] + K_a [\mu \cot \mu - 1] = 0 \quad (11-2-43)$$

or else the two equations

$$\sin \mu = 0 \quad \text{and} \quad \sin K_a^{1/2} (K_R - 1) \mu = 0. \quad (11-2-43')$$

It should be kept in mind that solutions (11-2-41) and (11-2-42) refer to the case when  $K_a^{1/2} (K_R - 1)$  is not equal to the ratio of two integers. If  $K_a^{1/2} (K_R - 1)$  is equal to the ratio  $\beta/b$  of two integers, then  $\frac{\beta}{b} \mu = n\pi$  and  $\mu = \frac{bn\pi}{\beta}$ . In this case it is necessary to add to the first solution (11-2-41) the quantity

$$-\frac{2}{\pi(\beta K_a + b)} \sum_{m=1}^{\infty} (-1)^{(b+\beta)m} \frac{\sin b\pi m r / R_1}{m r / R_2} \exp(-m^2 b^2 \pi^2 Fo_1),$$

and to add to the second solution (11-2-42) the quantity

$$+\frac{2K_a}{\pi(\beta K_a + b)} \sum_{m=1}^{\infty} \frac{\sin m\beta\pi (K_R - r/R_1)}{m(K_a - 1)r/R_2} \exp(-K_a K_R^2 m^2 b^2 \pi^2 Fo_2),$$

where

$$Fo_1 = \frac{a_1 \tau}{R_1^2}; \quad Fo_2 = \frac{a_2 \tau}{R_2^2}; \quad K_R = \frac{R_2}{R_1}.$$

**III. Two cylindrical bodies (cylinder within a shell).** Let us consider an infinite cylinder surrounded by a thin shell. The shell will be considered, to a first approximation, to be a plane. The problem is formulated as follows:

$$\begin{aligned} \frac{\partial \theta_1}{\partial \tau} &= a_1 \left( \frac{\partial^2 \theta_1}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \theta_1}{\partial r} \right); \quad (r > 0; 0 \leq r \leq R_1); \\ \frac{\partial \theta_2}{\partial \tau} &= a_2 \frac{\partial^2 \theta_2}{\partial r^2} \quad (\tau > 0; R_1 \leq r \leq R_2); \\ \theta_1(r, 0) &= \theta_2(r, 0) = \theta_0 = \text{const}; \\ \theta_1(R_1, \tau) &= \theta_2(R_1, \tau); \quad \lambda_1 \frac{\partial \theta_1(R_1, \tau)}{\partial r} = \lambda_2 \frac{\partial \theta_2(R_1, \tau)}{\partial r}; \\ \theta_1(0, \tau) &\leq \infty; \quad -\lambda_2 \frac{\partial \theta_2(R_2, \tau)}{\partial r} + \alpha [\theta_0 - \theta_2(R_2, \tau)] = 0. \end{aligned} \quad (11-2-44)$$

The solution has the form/5/:

$$\frac{\theta_1(r, \tau) - \theta_0}{\theta_0 - \theta_0} = 1 - \sum_{n=1}^{\infty} A_n \cdot J_0(\mu_n r / R_1) \exp(-\mu_n^2 Fo); \quad (11-2-45)$$

$$\begin{aligned} \frac{\theta_2(r, \tau) - \theta_0}{\theta_0 - \theta_0} &= 1 - \sum_{n=1}^{\infty} A_n \{ J_0(\mu_n) \cos[\mu_n K_a^{1/2} (r/R_1 - 1)] - \\ &\quad - K_a J_1(\mu_n) \sin[\mu_n K_a^{1/2} (r/R_1 - 1)] \} \exp(-\mu_n^2 Fo), \end{aligned} \quad (11-2-46)$$

where

$$\begin{aligned} A_n &= \frac{2Bi K_a [K_a^{1/2} (K_R - 1) \mu_n + Bi \tan K_a^{1/2} (K_R - 1) \mu_n]}{\mu_n J_0(\mu_n) \sin K_a^{1/2} (K_R - 1) \mu_n} \left/ \left\{ \left[ K_a^2 K_a (K_R - 1)^2 \mu_n^2 + \right. \right. \right. \\ &\quad \left. \left. + Bi^2 \right] \cot K_a^{1/2} (K_R - 1) \mu_n + \frac{2K_a K_a^{1/2} (K_R - 1)}{\sin 2K_a^{1/2} (K_R - 1) \mu_n} [Bi^2 + K_a (K_R - 1)^2 \mu_n^2] + \right. \\ &\quad \left. + [K_a (K_R - 1)^2 \mu_n^2 + 2K_a K_a^{1/2} (K_R - 1) Bi + K_a^2 Bi^2] \tan K_a^{1/2} (K_R - 1) \mu_n + \right. \\ &\quad \left. + K_a K_a (K_R - 1)^2 \mu_n^2 + 2K_a K_a^{1/2} (K_R - 1) \mu_n Bi - 2K_a^{1/2} (K_R - 1) \mu_n Bi - \frac{K_a Bi^2}{\mu_n} \right\}. \end{aligned}$$

When we take the limit as  $Bi \rightarrow \infty$ , the second condition of (11-2-44) becomes

$$\theta_2(R_2, \tau) = \theta_0.$$

Thus, equation (11-2-45) remains the same, while equation (11-2-46) becomes

$$\frac{\theta_2(r, \tau) - \theta_0}{\theta_0 - \theta_0} = 1 - \sum_{n=1}^{\infty} \frac{2 \sin [K_a^{1/2} (K_R - r/R_1) \mu_n] \exp(-\mu_n^2 Fo)}{\mu_n \left[ \frac{K_a^2 - 1}{K_a} \sin^2 K_a^{1/2} (K_R - 1) \mu_n - \frac{1}{2\mu_n} \sin 2K_a^{1/2} (K_R - 1) \mu_n + b \right]},$$

where

$$A_n = \frac{2 \sin K_a^{1/2} (K_R - 1) \mu_n}{\mu_n J_0(\mu_n) \left[ \frac{K_a^2 - 1}{K_a} \sin^2 K_a^{1/2} (K_R - 1) \mu_n - \frac{1}{2\mu_n} \sin 2K_a^{1/2} (K_R - 1) \mu_n + b \right]};$$

$$b = K_a^{1/2} (K_R - 1) + \frac{1}{K_a};$$

and  $\mu_n$  are the roots of the equation

$$J_0(\mu) - K_a J_1(\mu) \tan K_a^{1/2} (K_R - 1) \mu = 0.$$

More general solutions for a system consisting of two finite bodies can be obtained by means of integral transformations using Green's function.

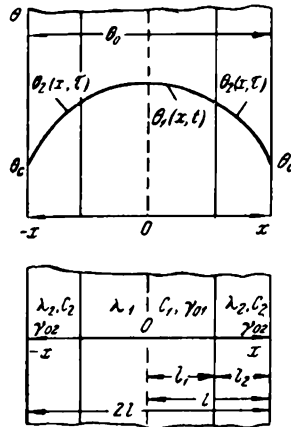


FIGURE 11-2. Symmetric transfer in a three-layered wall

The study of heat and mass transfer in layered media also includes problems in which unsteady potential fields are encountered not only in the body but in the infinite surroundings as well. The methods for solving such problems do not differ in principle from those of the previously considered problems. Several solutions are given in /2/ and in /6/ through /9/.

### 11-3. A System Consisting of Three Bodies

At the present time there exist a number of solutions describing transfer phenomena in a layered system consisting of three bodies. We will give here only two of the most typical solutions, since these have practical interest.

1. A symmetrical system of bodies, consisting of three infinite plates. It is assumed that the middle plate, of thickness  $2l$ , contacts in its planes the other two plates, each of which has a thickness  $l$ . The properties of the outer plates are identical but differ from those of the middle plate (Figure 11-2).

The limiting conditions of the problem (in accordance with Figure 11-2) may be written as

$$\begin{aligned} \theta_1(x, 0) &= \theta_2(x, 0) = \theta_0 = \text{const}; \\ \theta_1(l, \tau) &= \theta_2(l, \tau); \quad \lambda_1 \frac{\partial \theta_1(l, \tau)}{\partial x} = \lambda_2 \frac{\partial \theta_2(l, \tau)}{\partial x}; \quad \frac{\partial \theta_1(0, \tau)}{\partial x} = 0; \end{aligned}$$



$$\lambda_s \frac{\partial \theta_s(l, \tau)}{\partial x} + \alpha [\theta_s(l, \tau) - \theta_c] = 0. \quad (11-3-1)$$

The solution, which was obtained by Smirnov /10/, is

$$\frac{\theta_s - \theta_1(x, \tau)}{\theta_s - \theta_c} = 1 - \sum_{n=1}^{\infty} \frac{2}{\mu_n \psi_n} \cos(\mu_n K_s^{-1/2} x / l_s) \exp(-\mu_n^2 K_l^2 K_s^{-1} F o_1);$$

$$\begin{aligned} \frac{\theta_s - \theta_1(x, \tau)}{\theta_s - \theta_c} = 1 - \sum_{n=1}^{\infty} \frac{2}{\mu_n \psi_n} & \left[ \cos \mu_n \frac{x - l_1}{l_1} \cos \mu_n K_s^{-1/2} K_l - \right. \\ & \left. - K_s \sin \mu_n \frac{x - l_1}{l_1} \sin \mu_n K_s^{-1/2} K_l \right] \exp(-\mu_n^2 K_s^{-1} K_l^2 F o_1), \end{aligned}$$

where

$$\begin{aligned} \psi_n = & \left[ \left( 1 + K_l K_l K_s^{-1/2} + \frac{1 + K_l}{Bi} \right) \sin \mu_n + \mu_n \frac{1 + K_l}{Bi} (1 + K_l K_l K_s^{-1/2}) \cos \mu_n \right] \times \\ & \times \cos \mu_n K_l K_s^{-1/2} + \left[ \left( 1 + K_s^{-1} K_l K_s^{-1/2} + \frac{1 + K_l}{Bi} \right) \cos \mu_n - \right. \\ & \left. - \mu_n \frac{1 + K_l}{Bi} (1 + K_s^{-1} K_l K_s^{-1/2}) \sin \mu_n \right] K_s \sin \mu_n K_l K_s^{-1/2}; \end{aligned}$$

and  $\mu_n$  are the roots of the characteristic equation

$$K_s \frac{\mu}{Bi} (1 + K_l) \tan(\mu K_l K_s^{-1/2}) = 1 - \frac{\mu}{Bi} (1 + K_l) \tan \mu - K_s \tan \mu \tan(\mu K_l K_s^{-1/2});$$

with

$$Bi = \frac{a l}{\lambda_s}; \quad K_l = \frac{l_1}{l_s}; \quad F o_1 = \frac{a_1 \tau}{l_1^2}; \quad K_s = \frac{a_1}{a_s}.$$

If  $Bi = \infty$ , the second condition of (11-3-1) becomes

$$\theta_s(l, \tau) = \theta_c = \text{const.}$$

In this case the general form of the solution is the same, but

$$\begin{aligned} \psi_n = & (1 + K_l K_l K_s^{-1/2}) \sin \mu_n \cos \mu_n K_l K_s^{-1/2} + \\ & + K_s (1 + K_s^{-1} K_l K_s^{-1/2}) \cos \mu_n \sin \mu_n K_l K_s^{-1/2}, \end{aligned}$$

and  $\mu_n$  are the roots of the equation

$$K_s \tan \mu \tan(\mu K_l K_s^{-1/2}) = 1.$$

II. The general case of a ternary system with a source. Let us consider the system of equations

$$c_1 \gamma_1 \frac{\partial \theta_1}{\partial \tau} = \lambda_1 \frac{\partial^2 \theta_1}{\partial x^2} + w_1(x, \tau) \quad (\tau > 0; 0 < x < l_1); \quad (11-3-2)$$

$$c_2 \gamma_2 \frac{\partial \theta_2}{\partial \tau} = \lambda_2 \frac{\partial^2 \theta_2}{\partial x^2} + w_2(x, \tau) \quad (\tau > 0; l_1 < x < l_2); \quad (11-3-3)$$

$$c_3 \gamma_3 \frac{\partial \theta_3}{\partial \tau} = \lambda_3 \frac{\partial^2 \theta_3}{\partial x^2} + w_3(x, \tau) \quad (\tau > 0; l_2 < x < l) \quad (11-3-4)$$

with the limiting conditions

$$\theta_1(x, 0) = \theta_2(x, 0) = \theta_3(x, 0) = f(x); \quad (11-3-5)$$

$$\theta_1(0, \tau) = \psi_1(\tau); \quad \theta_2(l, \tau) = \psi_2(\tau); \quad (11-3-6)$$

$$\theta_1(l_1, \tau) = \theta_2(l_1, \tau); \quad \lambda_1 \frac{\partial \theta_1(l_1, \tau)}{\partial x} - \lambda_2 \frac{\partial \theta_2(l_1, \tau)}{\partial x} = \varphi_1(\tau); \quad (11-3-7)$$

$$\theta_2(l_2, \tau) = \theta_1(l_2, \tau); \quad \lambda_2 \frac{\partial \theta_2(l_2, \tau)}{\partial x} - \lambda_1 \frac{\partial \theta_1(l_2, \tau)}{\partial x} = \varphi_2(\tau). \quad (11-3-8)$$

To solve differential equations (11-3-2) through (11-3-4), we apply the following finite integral transformations with Green functions [11/:

$$\theta_{1G} = \{\theta_1(\mu_n, \tau)\}_G = \int_{l_1}^{l_2} \theta_1(x, \tau) G_1(\mu_n, x) dx; \quad (11-3-9)$$

$$\theta_{2G} = \{\theta_2(\mu_n, \tau)\}_G = \int_{l_1}^{l_2} \theta_2(x, \tau) G_2(\mu_n, x) dx; \quad (11-3-10)$$

$$\theta_{3G} = \{\theta_3(\mu_n, \tau)\}_G = \int_{l_1}^{l_2} \theta_3(x, \tau) G_3(\mu_n, x) dx, \quad (11-3-11)$$

where the Green functions  $G_k (k=1, 2, 3)$ , taking into account conditions (11-3-5) through (11-3-8), are

$$G_1(\mu_n, x) = P(\mu_n) \sin\left(\frac{\mu_n}{a_1} x\right);$$

$$G_2(\mu_n, x) = M(\mu_n) \sin\left[\frac{\mu_n}{a_2} (l_2 - x)\right] + N(\mu_n) \cos\left[\frac{\mu_n}{a_2} (l_2 - x)\right];$$

$$G_3(\mu_n, x) = \sin\left[\frac{\mu_n}{a_3} (l - x)\right].$$

Here

$$P(\mu_n) = \left\{ M(\mu_n) \sin\left[\frac{\mu_n}{a_2} (l_2 - l_1)\right] + N(\mu_n) \left[\frac{\mu_n}{a_2} (l_2 - l_1)\right] \right\} \frac{1}{\sin\left(\frac{\mu_n}{a_1} l_1\right)};$$

$$M(\mu_n) = \frac{a_2}{a_3} \cdot \frac{\lambda_2}{\lambda_3} \cos\left[\frac{\mu_n}{a_3} (l - l_2)\right];$$

$$N(\mu_n) = \sin\left[\frac{\mu_n}{a_3} (l - l_2)\right];$$

and  $\mu_n$  are the roots of the characteristic equation

$$1 - \frac{\lambda_2}{\lambda_3} \cdot \frac{a_2}{a_3} \cot\left[\frac{\mu_n}{a_2} (l_2 - l_1)\right] \cot\left[\frac{\mu_n}{a_3} (l - l_2)\right] = \\ = \frac{\lambda_1}{\lambda_2} \cdot \frac{a_2}{a_1} \cot\left(\frac{\mu_n}{a_1} l_1\right) \left\{ \cot\left[\frac{\mu_n}{a_2} (l_2 - l_1)\right] + \frac{\lambda_2}{\lambda_3} \cdot \frac{a_2}{a_3} \cot\left[\frac{\mu_n}{a_3} (l - l_2)\right] \right\}. \quad (11-3-12)$$

The inversion formulas for transformations (11-3-9) through (11-3-11) are

$$\theta_k(x, \tau) = \sum_{n=1}^{\infty} \xi^2 \left[ \sum_{j=1}^3 c_j \gamma_j \{\theta_j(\mu_n, \tau)\}_G \right] G_k(\mu_n, x) \quad (k=1, 2, 3), \quad (11-3-13)$$

the summation being performed over all the positive roots  $\mu_n$  which satisfy equation (11-3-12). The values of  $\xi$  are given by the expression

$$\xi^2 = \frac{1}{c_j \gamma_j \sum_{j=1}^3 \varphi_j(\mu_n)},$$

where

$$\varphi_1(\mu_n) = P^2(\mu_n) \left[ \frac{l_1}{2} - \frac{a_1}{4\mu_n} \sin\left(2 \frac{\mu_n}{a_1} l_1\right) \right];$$

$$\begin{aligned}\varphi_1(\mu_n) &= [M^2(\mu_n) + N^2(\mu_n)] \frac{l_2 - l_1}{2} + M(\mu_n) N(\mu_n) \frac{a_2}{\mu_n} \sin \left[ 2 \frac{\mu_n}{a_2} (l_2 - l_1) \right] + \\ &+ \frac{a_2}{4\mu_n} [N^2(\mu_n) - M^2(\mu_n)] \sin \left[ 2 \frac{\mu_n}{a_2} (l_2 - l_1) \right]; \\ \varphi_2(\mu_n) &= \frac{l - l_2}{2} - \frac{a_2}{4\mu_n} \sin \left[ 2 \frac{\mu_n}{a_2} (l - l_2) \right].\end{aligned}$$

If we multiply equations (11-3-2), (11-3-3), and (11-3-4) respectively by Green functions  $G_1$ ,  $G_2$ , and  $G_3$ , and then integrate these equations over the intervals for these functions, then we obtain

$$c_1 \gamma_1 \frac{d\theta_{10}}{d\tau} + \mu^2 c_1 \gamma_1 \theta_{10} = \left[ \lambda_1 \frac{\partial \theta_1}{\partial x} G_1 \right]_0^{l_1} - [\lambda_1 \theta_1 G_1]_0^{l_1} + \{w_1\}_0; \quad (11-3-14)$$

$$c_2 \gamma_2 \frac{d\theta_{20}}{d\tau} + \mu^2 c_2 \gamma_2 \theta_{20} = \left[ \lambda_2 \frac{\partial \theta_2}{\partial x} G_2 \right]_{l_1}^{l_2} - [\lambda_2 \theta_2 G_2]_{l_1}^{l_2} + \{w_2\}_0; \quad (11-3-15)$$

$$c_3 \gamma_3 \frac{d\theta_{30}}{d\tau} + \mu^2 c_3 \gamma_3 \theta_{30} = \left[ \lambda_3 \frac{\partial \theta_3}{\partial x} G_3 \right]_{l_2}^l - [\lambda_3 \theta_3 G_3]_{l_2}^l + \{w_3\}_0, \quad (11-3-16)$$

where

$$\{w_k(\mu_n, \tau)\}_0 = \int_{x_1}^{x_2} w_k(x, \tau) G_k dx \quad (k=1, 2, 3).$$

For  $k=1$ , we have  $x_1=0$  and  $x_2=l_1$ ; for  $k=2$ , we have  $x_1=l_1$  and  $x_2=l_2$ ; for  $k=3$ , we have  $x_1=l_2$  and  $x_2=l$ .

If we define  $\theta_0 = \theta_{10} + \theta_{20} + \theta_{30}$ , then we may add equations (11-3-14) through (11-3-16) to obtain

$$\begin{aligned}\frac{d\theta_0}{d\tau} + \mu^2 \theta_0 &= \sum_{i=1}^3 \{w_i\}_0 + \left[ \lambda_1 \frac{\partial \theta_1}{\partial x} G_1 - \lambda_2 \frac{\partial \theta_2}{\partial x} G_2 \right]_{x=l_1} + \\ &+ \left[ \lambda_2 \frac{\partial \theta_2}{\partial x} G_2 - \lambda_3 \frac{\partial \theta_3}{\partial x} G_3 \right]_{x=l_2} + [\lambda_2 \theta_2 G_2 - \lambda_1 \theta_1 G_1]_{x=l_1} + \\ &+ [\lambda_2 \theta_2 G_2 - \lambda_3 \theta_3 G_3]_{x=l_2} - \left[ \lambda_1 \frac{\partial \theta_1}{\partial x} G_1 \right]_{x=0} + \\ &+ [\lambda_1 \theta_1 G_1]_{x=0} + \left[ \lambda_3 \frac{\partial \theta_3}{\partial x} G_3 \right]_{x=l} - [\lambda_3 \theta_3 G_3]_{x=l}. \quad (11-3-17)\end{aligned}$$

Then, taking into account conditions (11-3-6) through (11-3-8), we can show that

$$\left[ \lambda_1 \frac{\partial \theta_1}{\partial x} G_1 - \lambda_2 \frac{\partial \theta_2}{\partial x} G_2 \right]_{x=l_1} = P(\mu_n) \sin \left( \frac{\mu_n}{a_1} l_1 \right) \varphi_1(\tau);$$

$$\left[ \lambda_2 \frac{\partial \theta_2}{\partial x} G_2 - \lambda_3 \frac{\partial \theta_3}{\partial x} G_3 \right]_{x=l_2} = N(\mu_n) \varphi_2(\tau);$$

$$[\lambda_1 \theta_1 G_1]_{x=0} = \lambda_1 P(\mu_n) \frac{\mu_n}{a_1} \psi_1(\tau);$$

$$[\lambda_3 \theta_3 G_3]_{x=l} = -\lambda_3 \frac{\mu_n}{a_3} \psi_3(\tau),$$

whereas the other quantities in brackets are equal to zero. Therefore,

equation (11-3-17) may be written as

$$\begin{aligned} \frac{d\theta_0}{d\tau} + \mu^2 \theta_0 = & \sum_{i=1}^3 \{w_i\}_0 + P(\mu_n) \sin\left(\frac{\mu_n}{a_1} l_1\right) \varphi_1(\tau) + \\ & + N(\mu_n) \varphi_2(\tau) + P(\mu_n) \frac{\lambda_1 \mu_n}{a_1} \psi_1(\tau) + \frac{\lambda_2 \mu_n}{a_2} \psi_2(\tau). \end{aligned} \quad (11-3-18)$$

The right side of equation (11-3-18) may be denoted by  $Q(\tau)$ , so that

$$\frac{d\theta_0}{d\tau} + \mu^2 \theta_0 = Q(\tau). \quad (11-3-19)$$

The solution of this equation is

$$\theta_0 = \exp(-\mu_n^2 \tau) \left\{ A + \int_0^{\tau} Q(\tau^*) \exp(\mu_n^2 \tau^*) d\tau^* \right\}.$$

The integration constant is determined from initial conditions (11-3-5) as

$$A = \{\theta(\mu_n, 0)\}_0 = c_1 \gamma_1 \int_0^{l_1} f(x) G_1 dx + c_2 \gamma_2 \int_{l_1}^{l_2} f(x) G_2 dx + c_3 \gamma_3 \int_{l_2}^l f(x) G_3 dx.$$

The final solution of the problem is found by applying inversion formula (11-3-13) to  $\theta_0$ , that is, by finding

$$\theta_h(x, \tau) = \sum_{n=1}^{\infty} \xi^n \theta_0 G_h.$$

The above system of equations has been used to describe an unsteady temperature field in a colloidal body. During the heating (baking) of dough, an upper and lower crust are formed, as well as crumbs; thus, the dough

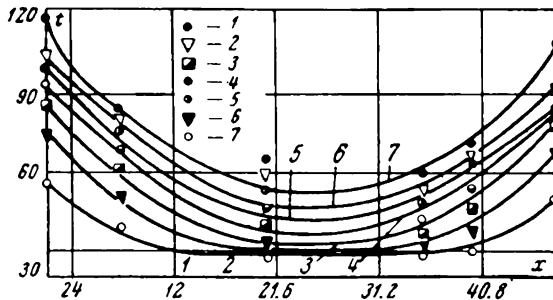


FIGURE 11-3. Temperature field in a material during baking, calculated using a hydrointegrator (solid curves) and using the experimental data recorded by a potentiometer. The curves and points 1 through 7 correspond to times  $\tau = 3, 6, 9, 12, 15, 18$ , and 21 minutes after the beginning of baking;  $x$  is in millimeters

acquires a layered structure. Junction conditions (11-3-7) and (11-3-8) also take into account the fact that the heat flux supplied from the crust is not completely transferred to the crumbs, because of the partial heat loss

due to evaporation ( $\varphi_1$  and  $\varphi_2$ ). In addition, the sources  $w_k$  in equations (11-3-2) through (11-3-4), which are functions of the spatial coordinates and of time, may be interpreted as follows:  $w_1$  and  $w_3$  are sources related to vapor transfer from the evaporation zone through the crust, while  $w_2$  is a heat source related to the transfer of liquid from the evaporation zone to the center of the crumbs, due to the temperature gradient (thermal moisture conduction). The solution, obtained using a hydrointegrator, shows good agreement with experimental results /12/ (see Figure 11-3).

In conclusion, let us note that interesting solutions for a ternary system with a transient source have been derived by Shteinberg /13/, and that solutions for the unsteady temperature fields in two bodies separated by a gap have been given by Mikulin /4/. Finally, several solutions for unsteady transfer in layered media, for the case of variable coefficients, have been given by Chudnovskii and his co-workers /15, 16/.

#### 11-4. Multilayered Systems

The analytical determination of an unsteady field of transfer potential in a multilayered system is very difficult mathematically, so that very few solutions have been obtained so far. The most general results have been obtained by means of transformations with Green functions, the form of which is, as usual, determined in accordance with the boundary conditions. In principle, methods for applying Green functions to a layered system do not differ from those used to determine the potential fields in two-layer and three-layer systems (see, for example § 11-3). However, the amount of work involved in the necessary transformations increases sharply with the number of layers comprising the body.

Let us consider here a very general solution proposed by Kulik and Shapovalov /17/. This solution also serves as an illustration of how the Dirac function may be applied to the solution of differential mass-conduction and heat-conduction equations. A solution which is very similar but which does not introduce the Dirac function can be found in /18/.

Consider the system of equations

$$\frac{\partial \theta_k}{\partial \tau} = a_k \frac{\partial^2 \theta_k}{\partial x^2} + w_k(x, \tau) \quad (k=1, 2, \dots, n). \quad (11-4-1)$$

Let us assume that the source term  $w_k$  can be represented as the product of two factors, one of which is a function of the spatial coordinates alone while the other is a function of the time alone:

$$w_k(x, \tau) = P_k(x) Q_k(\tau).$$

The boundary conditions at the free surfaces may be written in general form. By means of an appropriate choice of the coefficients it is possible to obtain boundary conditions of the first, second, or third kinds, including conditions of the third kind with a variable potential in the surroundings and with an arbitrarily varying heat or mass flux:

$$\text{for } x=l_0, \text{ we have } \alpha_1 \frac{\partial \theta_1}{\partial x} + \beta_1 [\theta_1 - \theta_{c1}(\tau)] + \gamma_1 j_1(\tau) = 0; \quad (11-4-2)$$

$$\text{for } x=l_n=l, \text{ we have } \alpha_n \frac{\partial \theta_n}{\partial x} + \beta_n (\theta_n - \theta_{cn}(\tau)) + \gamma_n j_n(\tau) = 0. \quad (11-4-3)$$

The initial conditions have the following general form:

$$\theta(x, 0) = f(x). \quad (11-4-4)$$

In addition, we assume the following contact conditions between the individual layers:

$$\begin{aligned} \theta_k(l_k, \tau) &= \theta_{k+1}(l_k, \tau); \\ \lambda_k \frac{\partial \theta_k(l_k, \tau)}{\partial x} &= \lambda_{k+1} \frac{\partial \theta_{k+1}(l_k, \tau)}{\partial x} \quad (k=1, 2, \dots, n-1). \end{aligned} \quad (11-4-5)$$

Let us convert the nonhomogeneous limiting conditions (11-4-2) and (11-4-3) into homogeneous conditions. To do this, we introduce the new variable

$$u(x, \tau) = \theta(x, \tau) - \psi(x, \tau), \quad (11-4-6)$$

where the function  $\psi(x, \tau)$  satisfies the conditions

$$\left. \begin{aligned} \psi(x, \tau) &= \psi_k(x, \tau) \\ \psi_1(0, \tau) &= \theta_{c1}(\tau); \\ \lambda_1 \frac{\partial \psi_1(l_1, \tau)}{\partial x} &= -x_1 j_1(\tau); \\ &\dots \dots \dots \end{aligned} \right\} (l_{k-1} \leq x \leq l_k; \tau > 0; 1 \leq k \leq n); \quad (11-4-7)$$

$$\left. \begin{aligned} \psi_k(l_k, \tau) &= \psi_{k+1}(l_k, \tau); \\ \lambda_k \frac{\partial \psi_k(l_k, \tau)}{\partial x} &= \lambda_{k+1} \frac{\partial \psi_{k+1}(l_k, \tau)}{\partial x}; \\ &\dots \dots \dots \end{aligned} \right\} \quad (11-4-8)$$

$$\lambda_n \frac{\partial \psi_n(l_n, \tau)}{\partial x} = -x_2 j_2(\tau);$$

$$\psi_n(l_n, \tau) = \theta_{c2}(\tau).$$

When system of equations (11-4-7) and (11-4-8) is solved, we obtain the following expression for the function  $\psi(x, \tau)$ :

$$\psi_k(x, \tau) = \frac{1}{6\lambda_k} A(\tau)(2x - 3l_n)x^2 + \frac{x_1 j_1(\tau)}{2\lambda_k l_n} (x - 2l_n)x - \frac{x_2 j_2(\tau)}{2\lambda_k l_n} x^2 + A_k(\tau), \quad (11-4-9)$$

where

$$A(\tau) = \frac{6B}{\sum_{k=2}^n \frac{l_{k-1}^2 (2l_{k-1} - 3l_n)(\lambda_k - \lambda_{k-1})}{\lambda_k \lambda_{k-1}} - \frac{l_n^3}{\lambda_n}};$$

$$\begin{aligned} B = \theta_{c2}(\tau) - \theta_{c1}(\tau) &+ \frac{x_1 j_1(\tau)}{2l_n} \left[ \frac{l_n^2}{\lambda_n} - \sum_{k=2}^n \frac{l_{k-1} (l_{k-1} - 2l_n)(\lambda_k - \lambda_{k-1})}{\lambda_k \lambda_{k-1}} \right] + \\ &+ \frac{x_2 j_2(\tau)}{2l_n} \left[ \frac{l_n^2}{\lambda_n} - \sum_{k=2}^n \frac{l_{k-1} (\lambda_{k-1} - \lambda_k)}{\lambda_k \lambda_{k-1}} \right]; \end{aligned}$$

$$A_1(\tau) = \theta_{c1}(\tau);$$

$$\begin{aligned} A_k(\tau) = \frac{A(\tau)}{6} \sum_{j=2}^k \frac{l_{j-1} (2l_{j-1} - 3l_n)(\lambda_j - \lambda_{j-1})}{\lambda_j \lambda_{j-1}} &+ \theta_{c1}(\tau) + \\ + \frac{x_1 j_1(\tau)}{2l_n} \sum_{j=2}^k \frac{l_{j-1} (l_{j-1} - 2l_n)(\lambda_j - \lambda_{j-1})}{\lambda_j \lambda_{j-1}} &+ \frac{x_2 j_2(\tau)}{2l_n} \sum_{j=2}^k \frac{l_{j-1}^2 (\lambda_{j-1} - \lambda_j)}{\lambda_j \lambda_{j-1}}. \end{aligned}$$

Now, taking (11-4-7) through (11-4-9) into account, we find that system of equations (11-4-1) and conditions (11-4-2) through (11-4-4) have the form:

$$\frac{\partial u_k}{\partial \tau} = a_k \frac{\partial^2 u_k}{\partial x^2} - \frac{\partial^2 \psi_k(x, \tau)}{\partial \tau} + a_k \frac{\partial^2 \psi_k(x, \tau)}{\partial x^2} + P_k(x) Q_k(\tau); \quad (11-4-10)$$

$$u(x, 0) = f(x) - \psi(x, 0); \quad (11-4-11)$$

[illegible]

A Laplace transformation may now be applied to equation (11-4-10). Taking into account initial condition (11-4-11), we find that this equation, in terms of transforms, is

$$su_L - [f(x) - \phi(x, 0)] = a(x) \frac{\partial^2 u_L}{\partial x^2} -$$

$$- \int_0^\infty \left[ \frac{\partial \psi(x, \tau)}{\partial \tau} - a(x) \frac{\partial^2 \psi(x, \tau)}{\partial x^2} - P(x) Q(\tau) \right] \exp(-s\tau) d\tau. \quad (11-4-13)$$

After a term-by-term inversion of equation (11-4-13), we obtain the differential equation

$$\begin{aligned} \frac{\partial u(x, \tau)}{\partial \tau} = & a(x) \frac{\partial^2 u(x, \tau)}{\partial x^2} - \frac{\partial \phi(x, \tau)}{\partial \tau} + a(x) \frac{\partial^2 \phi(x, \tau)}{\partial x^2} + \\ & + P(x)Q(\tau) + \delta(\tau)[f(x) - \phi(x, 0)], \end{aligned} \quad (11-4-14)$$

where  $\delta(\tau)$  is the Dirac delta function, a quantity whose Laplace transform is unity:

$$\int_0^{\infty} \delta(\tau) \exp(-s\tau) d\tau = 1.$$

The Dirac  $\delta$  function  $\delta(\tau - \tau')$  is zero at all points lying outside a quite small region (with coordinates  $-\epsilon$  and  $+\epsilon$ ) surrounding the point  $\tau'$ . Within this region, however, the function is so large that its integral over the region is unity:

$$\int_{\tau^*-\varepsilon}^{\tau^*+\varepsilon} \delta(\tau - \tau^*) d\tau = 1, \quad \text{but} \quad \delta(\tau - \tau^*) = 0 \quad \text{for} \quad \tau \neq \tau^*.$$

The initial conditions for equation (11-4-14) are

$$u(x, 0) = 0. \quad (11-4-15)$$

Since equation (11-4-14) with initial condition (11-4-15) and equation (11-4-10) with initial condition (11-4-11) are represented by the same equations in terms of their Laplace transforms, therefore from the uniqueness conditions for the inverse transform and for the transform (with an accuracy up to the zero function) the solutions for  $u(x, \tau)$  of equations (11-4-14) and (11-4-10) are the same. Thus, we obtain nonhomogeneous differential transfer equations with homogeneous limiting conditions. In the case of condition (11-4-15), equation (11-4-14) has the following simple physical interpretation: the presence of the term  $\delta(\tau)[\varphi(x) - \psi(x, 0)]$  indicates that the initial potential distribution (11-4-11) can be replaced by the result of the action which instantaneous point sources have at the initial moment.

Using operator notation, we may write equation (11-4-14) as

where

$$L(u) = \frac{\partial u(x, \tau)}{\partial \tau} - a(x) \frac{\partial^2 u(x, \tau)}{\partial x^2} = F(x, \tau), \quad (11-4-16)$$

$$F(x, \tau) = a(x) \frac{\partial^2 \phi(x, \tau)}{\partial x^2} + P(x) Q(\tau) + \delta(\tau) [f(x) - \phi(x, 0)].$$

Let us seek the solution by defining a Green function which corresponds to the problem at hand. This function





Now, after integration, we find the following expression for the Green function:

$$G(x, x^*, \tau, \tau^*) = \sum_j C_j \exp[-P_j(\tau - \tau^*)] \varphi_j(x). \quad (11-4-22)$$

We must still determine the coefficients  $C_j$ . These can be determined from the condition that the eigenfunctions  $\varphi_j(x)$  must be orthogonal. The system of functions  $\varphi_j(x)$  defined by equations (11-4-20) and (11-4-21) is complete but not orthogonal. Thus, let us construct a new system of orthogonal functions in the interval  $(l_0, l_n)$ , where the new functions differ from  $\varphi_j(x)$  by the factor  $m_j(x)$ . We then find from (11-4-19) that

$$C_j = \frac{\varphi_j(x^*) m_j^2(x)}{\int_{l_0}^{l_n} \varphi_j^2(x) m_j^2(x) dx}.$$

Consequently, the solution of the problem at hand has the form

$$\theta(x, \tau) = u(x, \tau) + \psi(x, \tau),$$

where  $u(x, \tau)$  is defined by equations (11-4-17) and (11-4-22), while  $\psi(x, \tau)$  is defined by (11-4-9).

## APPENDIX I

### A SUMMARY OF VECTOR AND TENSOR NOTATION

During an analytical investigation of transfer phenomena, the gradients and divergences of tensor quantities of various ranks are often encountered. Thus, it will be useful to list the basic relationships for these quantities.

#### VECTORS

For a vector  $\vec{A}$  with components  $A_x$ ,  $A_y$ , and  $A_z$  we may write

$$\vec{A} = \vec{i}_x A_x + \vec{i}_y A_y + \vec{i}_z A_z, \quad (1)$$

where  $\vec{i}_x$ ,  $\vec{i}_y$ , and  $\vec{i}_z$  are mutually orthogonal vectors of unit length lying along the Cartesian coordinate axes (unit vectors).

The scalar product of two vectors  $\vec{A}$  and  $\vec{B}$  is a scalar quantity, written as

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z. \quad (2)$$

The vector product of two vectors is a vector quantity, written as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i}_x (A_y B_z - A_z B_y) + \vec{i}_y (A_z B_x - A_x B_z) + \vec{i}_z (A_x B_y - A_y B_x). \quad (3)$$

The vector product of three vectors  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$  is

$$[\vec{A} \times (\vec{B} \times \vec{C})] = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}. \quad (4)$$

#### TENSORS

A tensor  $\vec{\vec{M}}$  has the following nine components:

$$\begin{vmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{vmatrix}. \quad (5)$$

The transposed tensor  $\vec{\vec{M}}^T$  is obtained from the tensor  $\vec{\vec{M}}$  by interchanging the rows with the columns, so that we have

$$\begin{vmatrix} T_{xx} & T_{yx} & T_{zx} \\ T_{xy} & T_{yy} & T_{zy} \\ T_{xz} & T_{yz} & T_{zz} \end{vmatrix}. \quad (6)$$

#### DYADS

The dyad  $\vec{A} \vec{B}$  of two vectors has the nine components

$$\begin{vmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{vmatrix}. \quad (7)$$

This quantity should not be confused with the scalar product

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z.$$

The product of a vector  $\vec{C}$  times a dyad  $\vec{A}\vec{B}$  is

$$(\vec{A} \vec{B} \cdot \vec{C}) = \vec{A} (\vec{B} \cdot \vec{C}) \quad (8)$$

or

$$(\vec{C} \cdot \vec{A} \vec{B}) = (\vec{C} \cdot \vec{A}) \vec{B}. \quad (9)$$

## PRODUCTS OF TENSORS

All quantities (scalar, vector, or tensor) can be regarded as tensors of various ranks. A scalar is a zero-rank tensor, a vector is a first-rank tensor, and the quantity  $\vec{\mathcal{M}}$  is a second-rank tensor. This may be denoted arbitrarily as follows:

$$a \rightarrow a_0 \quad (10)$$

indicating that  $a$  is independent of the spatial coordinates (is a scalar quantity);

$$\vec{W} \rightarrow w_i (i = x, y, z) \quad \text{or} \quad \vec{W} \rightarrow w_x, w_y, w_z; \quad (11)$$

$$\vec{\mathcal{M}} \rightarrow T_{ik} (i, k = x, y, z), \quad (12)$$

or see formula (5).

The exterior or ordered product of two tensors of ranks  $m$  and  $n$  is a tensor of rank  $m + n$ . For example, the dyad  $\vec{A}\vec{B}$  is the exterior product of the two first-rank tensors  $\vec{A}$  and  $\vec{B}$  and is a tensor of the second rank:

$$\vec{A} \vec{B} \rightarrow (\vec{A} \vec{B})_{ik} = A_i B_k. \quad (13)$$

The exterior product of a first-rank tensor (a vector) times a second-rank tensor is a tensor of the third rank:

$$(\vec{A} \vec{\mathcal{M}}) \rightarrow (\vec{A} \vec{\mathcal{M}})_{ikh} = A_i T_{kh} \quad (i, k, h = x, y, z); \quad (14)$$

$$(\vec{\mathcal{M}} \vec{A}) \rightarrow (\vec{\mathcal{M}} \vec{A})_{ikh} = T_{ik} A_h. \quad (15)$$

The interior or contracted product of two tensors is indicated by the following examples:

$$\vec{A} \vec{B} = A_x B_x + A_y B_y + A_z B_z = \sum_i A_i B_i \quad (i = x, y, z); \quad (16)$$

$$\vec{B} \vec{\mathcal{M}} \rightarrow (\vec{B} \vec{\mathcal{M}})_i = B_x T_{xi} + B_y T_{yi} + B_z T_{zi} = \sum_k B_k T_{ki}; \quad (17)$$

$$(\vec{\mathcal{M}} \cdot \vec{B}) \rightarrow (\vec{\mathcal{M}} \cdot \vec{B})_i = T_{ix} B_x + T_{iy} B_y + T_{iz} B_z = \sum_k T_{ik} B_k; \quad (18)$$

$$(\vec{\mathcal{J}} \cdot \vec{\mathcal{M}}) \rightarrow (\vec{\mathcal{J}} \cdot \vec{\mathcal{M}})_i = S_{ix} T_{xh} + S_{iy} T_{yh} + S_{iz} T_{zh} = \sum_l S_{il} T_{lh}. \quad (19)$$

Similarly, the scalar product of two second-rank tensors can be written as

$$\vec{\mathcal{J}} : \vec{\mathcal{M}} = \sum_{i,k} S_{ik} T_{ki} = \vec{\mathcal{M}} : \vec{\mathcal{J}}. \quad (20)$$

Let us denote a unit tensor with components  $\delta_{ik}$  (where  $\delta_{ik} = 1$  if  $i = k$ , and  $\delta_{ik} = 0$  if  $i \neq k$ ) by  $\vec{1}_T$ . Thus,

$$\vec{1}_T = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}. \quad (21)$$

The gradient of a scalar quantity  $\varphi$  is a vector, since the product  $\nabla\varphi$  of a first-rank tensor times a zero-rank tensor is a first-rank tensor (a vector):

$$\nabla\varphi = \vec{1}_x \frac{\partial\varphi}{\partial x} + \vec{1}_y \frac{\partial\varphi}{\partial y} + \vec{1}_z \frac{\partial\varphi}{\partial z} = \text{grad } \varphi. \quad (34)$$

The divergence of a vector is the scalar product of two vectors  $\nabla$  and  $\vec{A}$ :

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z. \quad (35)$$

The vector product  $\nabla \times \vec{A}$ , known as  $\text{rot } \vec{A}$  or  $\text{curl } \vec{A}$ , is the vector quantity

$$\begin{aligned} \text{curl } \vec{A} = \nabla \times \vec{A} &= \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \vec{1}_x \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \\ &+ \vec{1}_y \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \vec{1}_z \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) = \vec{1}_x \xi + \vec{1}_y \eta + \vec{1}_z \zeta. \end{aligned} \quad (36)$$

The divergence of the product  $(\varphi \vec{A})$  of a scalar times a vector may be written as

$$\text{div } \varphi \vec{A} = \nabla \cdot \varphi \vec{A} = \varphi \nabla \cdot \vec{A} + \vec{A} \nabla \varphi = \varphi \text{div } \vec{A} + \vec{A} \text{grad } \varphi. \quad (37)$$

The following operations are also often encountered:

$$\text{curl } (\varphi \vec{A}) = \nabla \times \varphi \vec{A} = \varphi \nabla \times \vec{A} + \nabla \varphi \times \vec{A} = \varphi \text{curl } \vec{A} + (\text{grad } \varphi) \times \vec{A}; \quad (38)$$

$$\text{div } (\text{grad } \varphi) = \nabla \cdot \nabla \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \nabla^2 \varphi, \quad (39)$$

where  $\nabla^2$  is the Laplace operator,

The total or substantial derivative of a scalar is

$$\frac{d\varphi}{d\tau} = \frac{\partial\varphi}{\partial\tau} + \vec{w} \cdot \text{grad } \varphi = \frac{\partial\varphi}{\partial\tau} + \vec{w} \cdot \nabla \varphi, \quad (40)$$

where  $\vec{w} \cdot \nabla \varphi$  is a scalar quantity, since it is a scalar product of the two vectors  $\vec{w}$  and  $\nabla \varphi$ :

$$\vec{w} \cdot \nabla \varphi = w_x \frac{\partial\varphi}{\partial x} + w_y \frac{\partial\varphi}{\partial y} + w_z \frac{\partial\varphi}{\partial z}. \quad (41)$$

The gradient of a vector  $\vec{w}$  is

$$\begin{aligned} \text{grad } \vec{w} &= \nabla \vec{w} \rightarrow (\nabla \vec{w})_{ik} = \frac{\partial w_k}{\partial \xi_i} \\ (k, i &= x, y, z; \xi_x \equiv x; \xi_y \equiv y; \xi_z \equiv z). \end{aligned} \quad (42)$$

An example of such a gradient is found in the total derivative of the vector  $\vec{w}$ :

$$\frac{d\vec{w}}{d\tau} = \frac{\partial \vec{w}}{\partial \tau} + \vec{w} \cdot \text{grad } \vec{w} = \frac{\partial \vec{w}}{\partial \tau} + \vec{w} \cdot \nabla \vec{w} = \frac{\partial \vec{w}}{\partial \tau} + (\vec{w} \cdot \nabla) \vec{w}, \quad (43)$$

where the quantity  $(\vec{w} \cdot \nabla) \vec{w}$  is equal to

$$\left. \begin{aligned} w_x \frac{\partial}{\partial x} w_x + w_y \frac{\partial}{\partial y} w_x + w_z \frac{\partial}{\partial z} w_x; \\ w_x \frac{\partial}{\partial x} w_y + w_y \frac{\partial}{\partial y} w_y + w_z \frac{\partial}{\partial z} w_y; \\ w_x \frac{\partial}{\partial x} w_z + w_y \frac{\partial}{\partial y} w_z + w_z \frac{\partial}{\partial z} w_z. \end{aligned} \right\} \quad (44)$$

The divergence of a second-rank tensor is

$$\text{div } \vec{\mathcal{H}} = \nabla \vec{\mathcal{H}} \rightarrow (\nabla \cdot \vec{\mathcal{H}})_i = \sum_k \frac{\partial T_{ki}}{\partial \xi_k} \quad (i, k = x, y, z).$$

The interior product of this unit tensor and a vector is

$$\vec{\vec{1}}_T \cdot \vec{A} = \vec{1}_x A_x + \vec{1}_y A_y + \vec{1}_z A_z = \vec{A} \quad (22)$$

By analogy with (20), the interior product of the tensor  $\vec{\vec{M}}$  and the unit tensor  $\vec{\vec{1}}_T$  is

$$\vec{\vec{M}} : \vec{\vec{1}}_T = \sum_i T_{ii} = T_{xx} + T_{yy} + T_{zz} \quad (23)$$

a quantity called the trace of the tensor.

The product of two unit tensors is

$$\vec{\vec{1}}_T : \vec{\vec{1}}_T = \sum_i 1_{ii} = 1 + 1 + 1 = 3.$$

## SYMMETRIC AND ANTISYMMETRIC TENSORS

A tensor is symmetric if

$$\vec{\vec{M}}^s = \vec{\vec{M}}, \quad (24)$$

and it is antisymmetric if

$$\vec{\vec{M}}^a = -\vec{\vec{M}}. \quad (25)$$

A transposed tensor may be written as  $\vec{\vec{M}}_{ih} = \vec{\vec{M}}_{hi}$ , while for a dyad we have

$$\vec{\vec{A}} \vec{\vec{B}} = \vec{\vec{B}} \vec{\vec{A}}. \quad (26)$$

A similar relationship exists for the interior product:

$$\vec{\vec{S}} \cdot \vec{\vec{M}} = \vec{\vec{M}} \cdot \vec{\vec{S}}. \quad (27)$$

A tensor  $\vec{\vec{M}}$  can be represented as the sum of a symmetric tensor  $\vec{\vec{M}}^s$  plus an antisymmetric tensor  $\vec{\vec{M}}^a$ :

$$\vec{\vec{M}} = \vec{\vec{M}}^s + \vec{\vec{M}}^a, \quad (28)$$

where

$$\vec{\vec{M}}^s = \frac{1}{2} (\vec{\vec{M}} + \vec{\vec{M}}^t); \quad (29)$$

$$\vec{\vec{M}}^a = \frac{1}{2} (\vec{\vec{M}} - \vec{\vec{M}}^t). \quad (30)$$

Similarly, for the dyad  $\vec{\vec{A}} \vec{\vec{B}} \equiv \vec{\vec{D}}$  we have

$$\vec{\vec{D}}^s = \frac{1}{2} (\vec{\vec{A}} \vec{\vec{B}} + \vec{\vec{B}} \vec{\vec{A}}); \quad (31)$$

$$\vec{\vec{D}}^a = \frac{1}{2} (\vec{\vec{A}} \vec{\vec{B}} - \vec{\vec{B}} \vec{\vec{A}}). \quad (32)$$

## DIFFERENTIATION OPERATIONS

The generalized differentiation operation is denoted by the symbol  $\nabla$  (nabla), known as the Hamiltonian operator. This quantity is a vector (first-rank tensor) and is equal to

$$\nabla = \vec{1}_x \frac{\partial}{\partial x} + \vec{1}_y \frac{\partial}{\partial y} + \vec{1}_z \frac{\partial}{\partial z} = \text{grad}. \quad (33)$$

\* [The transpose of the dyad  $\vec{\vec{A}} \vec{\vec{B}}$  is actually  $\vec{\vec{A}} \vec{\vec{B}} = \vec{\vec{B}} \vec{\vec{A}}$  .]

The gradient of a vector, as well as that of any tensor, can be represented as the sum of a symmetric tensor plus an antisymmetric tensor:

$$\vec{\nabla} \vec{w} = (\vec{\nabla} \vec{w})^s + (\vec{\nabla} \vec{w})^a; \quad (44)$$

$$(\vec{\nabla} \vec{w})_{ik}^s = \frac{1}{2} \left( \frac{\partial w_k}{\partial \xi_i} + \frac{\partial w_i}{\partial \xi_k} \right); \quad (45)$$

$$(\vec{\nabla} \vec{w})_{ik}^a = \frac{1}{2} \left( \frac{\partial w_k}{\partial \xi_i} - \frac{\partial w_i}{\partial \xi_k} \right). \quad (46)$$

For a three-dimensional space, the tensor  $(\vec{\nabla} \vec{w})^a$  is related to  $\text{curl } \vec{w}$  as follows:

$$(\vec{\nabla} \vec{w})^a = \frac{1}{2} \text{curl } \vec{w} = \frac{1}{2} (\vec{\nabla} \times \vec{w}). \quad (47)$$

The scalar product of the unit tensor  $\vec{1}_F$  times the tensor  $\vec{\nabla} \vec{w}$  is equal to the divergence of the vector  $\vec{w}$ :

$$\vec{1}_F : \vec{\nabla} \vec{w} = \vec{\nabla} \cdot \vec{w} = \text{div } \vec{w}. \quad (48)$$

#### INTEGRATION OPERATIONS

If  $\vec{1}_n$  is the unit vector in the direction of the normal  $n$  to a surface  $F$ , then we may write

$$\int_F (\vec{A} \cdot d\vec{F}) = \int_F (\vec{A} \cdot \vec{1}_n) dF, \quad (49)$$

where the integration is performed over the surface  $F$ .

A similar relation is obtained in the case of integration along a curve  $l$ , namely

$$\int_l (\vec{A} \cdot d\vec{l}) = \int_l (\vec{A} \cdot \vec{1}_l) dl, \quad (50)$$

where  $\vec{1}_l$  is a unit vector tangential to the curve. The Gauss-Ostrogradskii theorem states that

$$\int_F (\vec{A} \cdot \vec{1}_n) dF = \int_v (\vec{\nabla} \cdot \vec{A}) dv, \quad (51)$$

where in the first integral integration is over the closed surface  $F$ , while in the second integral it is over the volume  $v$  enclosed by this surface. Stokes' theorem states that

$$\int_l (\vec{A} \cdot \vec{1}_l) dl = \int_F (\vec{\nabla} \times \vec{A}) \cdot \vec{1}_n dF.$$

The symbol  $l$  indicates integration over the closed curve  $l$  enclosing the surface  $F$ .

## APPENDIX II

### Dimensionless Numbers\* Introduced in the Text

Symbol	Definition	Defined on page	Name
Pe	$\frac{wl}{a}$	94	Péclet number
Nu	$\frac{al}{\lambda}$	94	Nusselt number
Bi	$\frac{al}{\lambda'}$	95	Biot number
Fo	$\frac{a^2}{l^2}$	96	Fourier number
Ho	$\frac{w^2}{l}$	99	Homochronicity (Strouhal) number
Fr	$\frac{gl}{w^2}$	99	Froude number
Eu	$\frac{p}{\rho w^2}$	99	Euler number
Re	$\frac{wl}{\nu}$	99	Reynolds number
Ga	$\frac{gl^3}{\nu^2}$	100	Galileo number
Ar	$\frac{gl^3(\rho - \rho_0)}{\nu^2 \rho}$	100	Archimedes number
Gr	$\beta \frac{gl^3}{\nu^2} \Delta t$	100	Grashof number
Le	$\frac{D}{a}$	100	Lewis number
Pr	$\frac{\nu}{a}$	100	Prandtl number
Gu	$\frac{l}{\lambda T}$	101	Gukhman number
So	$c_m Q^*$	102	Soret number
Du	$Q^*/c_p T_c$	102	Dufour number
Lu	$\frac{a_m}{a_q}$	103	Coupling (Lukomskii) number

\* Also referred to as similarity criteria, generalized variables, and dimensionless parameters.

Symbol	Definition	Defined on page	Name
Ko	$\frac{rc_m \Delta \theta}{c_q \Delta t}$	103	Kossovich number
Pn	$\frac{\delta \Delta t}{\Delta \theta}$	103	Posnov number
Fe	$\frac{e \delta r}{c_q}$	103	Fedorov number
Ra	$\frac{c_p \Delta p}{c_m \Delta \theta}$	103	Ramzin number
Bu	$\frac{rc_p \Delta p}{c_q \Delta T}$	103	Bulygin number
Ki <sub>m</sub>	$\frac{j_m l}{\lambda_m \Delta \theta}$	104	Kirpichev mass-exchange number
Ki <sub>q</sub>	$\frac{j_q l}{\lambda \Delta t}$	104	Kirpichev heat-exchange number
Bi <sub>m</sub>	$\frac{a_m l}{\lambda_m}$	104	Biot mass-exchange number
Bi <sub>q</sub>	$\frac{a l}{\lambda}$	104	Biot heat-exchange number
Sc	$v/a_m$	106	Schmidt number
St	$\frac{a}{c_p \rho w}$	106	Stanton (Margulis) number
Po	$\frac{l^2 w}{\lambda z_*}$	180	Pomerantsev number
Pd	$\frac{\Delta t l^2}{\Delta \tau a}$	182	Predvoditelev number
W	$\frac{t_s - t_{ce}}{t_o - t_s}$	192	Parameter for temperature-field nonuniformity
V	$\frac{\theta_{ce} - \theta_s}{\theta_s - \theta_{eq}}$	192	Parameter for mass-transfer potential-field nonuniformity
Mn	$\frac{F l}{m}$	336	Miniovich number
Kn	$\frac{m l}{a_q}$	343	Kondrat'ev number
Le	$\frac{m \Delta t \beta}{\gamma c_p p}$	445	Lebedev number



### Notation Used in the Text

$\text{ch}$	= cosh	$\beta$	= bulk expansion coefficient
$\text{sh}$	= sinh	$t$	= temperature
$\text{erfc}$	= error function	$D$	= diffusion coefficient
$w$	= reference velocity	$I$	= source strength
$l$	= characteristic dimension	$c_m$	= specific mass
$a$	= diffusivity	$c_p$	= specific heat
$a_m$	= potential conductivity (diffusivity) for mass transfer	$c_q$	= specific heat
$a_t$	= thermal diffusivity	$Q^*$	= transfer heat
$\alpha$	= transfer coefficient	$\theta$	= mass-transfer potential
$\lambda$	= conductivity	$\delta$	= thermal-gradient (Soret) coefficient
$\tau$	= time	$\epsilon$	= phase-transition ratio
$g$	= acceleration	$j$	= flux
$p$	= pressure	$z_s$	= fixed value of potential
$\rho$	= density	$F$	= area (of layer)
$\nu$	= kinematic viscosity	$\gamma$	= adiabatic index
		$r$	= specific vaporization heat

### Subscripts

$m$	= mass transfer	$s$	= surface
$q$	= heat transfer	$ce$	= center
$p$	= seepage	$eq$	= equilibrium
$c$	= surroundings		

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#### LIST OF ABBREVIATIONS

Abbreviation	Full name (transliterated)	Translation
AN BSSR	Akademiya Nauk Belorusskoi SSR	Academy of Sciences of the Belorussian SSR
AN Latv. SSR	Akademiya Nauk Latviiskoi SSR	Academy of Sciences of the Latvian SSR
AN SSSR	Akademiya Nauk SSSR	Academy of Sciences of the USSR
IFZh	Inzhenerno-Fizicheskii Zhurnal	Journal of Engineering Physics
IL	Izdatel'stvo Inostranoi Literatury	Foreign Literature Publishing House
MTIPP	Moskovskii Tekhnologicheskii Institut Pishchevoi Promyshlennosti	Moscow Technological Institute of the Food Industry
NIKFI	Nauchno-Issledovatel'skii Kino-Foto Institut	Motion Picture and Photography Scientific Research Institute
ONTI	Ob"edinenie Nauchno-Tekhnicheskikh Izdatel'stv	United Scientific and Technical Presses
OTN	Otdelenie Tekhnicheskikh Nauk	Division of Technical Sciences

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